ON THE SEPARATELY OPEN TOPOLOGY

Zbigniew Piotrowski — Robert W. Vallin — Eric Wingler

ABSTRACT. We consider the relationship between separately continuous functions and separately open sets, and we study the properties of the separately open topology on \( \mathbb{R}^2 \) and on \( \mathbb{Q}^2 \). We show that \( \mathbb{R}^2 \) with this topology (denoted \( \mathbb{R} \oplus \mathbb{R} \)) is completely and strongly Hausdorff and that \( \mathbb{Q} \oplus \mathbb{Q} \) is normal but not a \( p \)-space. In addition, we show that each point of \( \mathbb{Q} \oplus \mathbb{Q} \) has an uncountable neighborhood base.

1. Introduction

This paper deals with two topologies on the space \( \mathbb{R}^2 \), the usual Euclidean topology and the separately open (or plus) topology. In this paper we will compare and contrast these topologies and the \( G_\delta \) sets formed by each.

Let \( f \) be a function from \( \mathbb{R}^2 \) into \( \mathbb{R} \). We say that \( f \) is continuous with respect to \( x \) (with respect to \( y \)) if the restricted function \( f_y(x) = f(x, y) \), where \( y \) is fixed \( (f_x(y) = f(x, y), \text{where } x \text{ is fixed}) \) is a continuous function from \( \mathbb{R} \) into \( \mathbb{R} \). If \( f \) is continuous with respect to both \( x \) and \( y \), then \( f \) is called a separately continuous function. The canonical example of a function that is separately continuous at a point where it is not continuous, is

\[
\begin{align*}
  f(x, y) &= \begin{cases} 
  2xy & \text{if } (x, y) \neq (0, 0), \\
  0 & \text{if } (x, y) = (0, 0). 
  \end{cases}
\end{align*}
\]

Since \( f \) is not continuous at \((0, 0)\), we know that there exist open intervals \( I = (-a, a) \) such that \( f^{-1}(I) \) is not an open Euclidean set in the plane. It is natural to ask what such a set \( f^{-1}(I) \) looks like. The answer is a separately open set containing the origin.

**Definition 1.** The \( \varepsilon \)-plus at \((a, b)\) of radius \( \varepsilon > 0 \) is

\[
P_{\varepsilon}(a, b) = \{(x, b) \in \mathbb{R}^2 : |x - a| < \varepsilon\} \cup \{(a, y) \in \mathbb{R}^2 : |y - b| < \varepsilon\}.
\]

(Note: We shall use \( B_{\varepsilon}(a, b) \) to denote a Euclidean open ball about \((a, b)\).)

2000 Mathematics Subject Classification: Primary 05C38, 15A15; Secondary 05A15, 15A18.

Keywords: separately open topology, separate continuity.
More generally, if \( X \) and \( Y \) are topological spaces, \((p,q) \in X \times Y\), and \( U \) and \( V \) are open neighborhoods of \( p \) and \( q \), respectively, we define the \((U,V)\)-plus at \((p,q)\) by

\[
P_{U,V}(p,q) = \{(x,q) : x \in U\} \cup \{(p,y) : y \in V\}.
\]

**Definition 2.** A set \( G \subset \mathbb{R}^2 \) is said to be separately open if for each point \((a,b) \in G\) there exists \( \varepsilon > 0 \) such that \( P_\varepsilon (a,b) \subset G \).

In general, the separately open topology is formed as follows: Let \( X_1, X_2, \ldots, X_n \) be a finite collection of topological spaces and let \( X = \prod_{i=1}^{n} X_i \). We say that \( S \subset X \) is separately open provided that for each \( x = (x_1, x_2, \ldots, x_n) \in S \) and each \( i = 1, 2, \ldots, n \) there is a neighborhood \( N_i \) of \( x_i \) in \( X_i \) such that \( \prod_{i=1}^{n} A_i \subset S \) where \( A_j = \{x_j\} \) when \( j \neq i \) and \( A_i = N_i \). For more information, see \([6]\) and \([7]\).

2. **Structure of separately open sets**

It is obvious that Euclidean open sets are separately open. The following example shows that the converse is not true.

**Example 1.** The Maltese Cross

\[
A = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : |y| > 3|x|\} \cup \left\{(x,y) \in \mathbb{R}^2 : |y| < \left|\frac{x}{3}\right|\right\}
\]

is a separately open, but not Euclidean open set.

The Maltese Cross has only one point \((0,0)\) where it is not open in the usual sense; that is, it is the union of an open set with a singleton. Obviously, one can quickly come up with a set with an infinite number of such points. For example, let

\[
A_{(0,0)} = A \cap \left[-\left(\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right],
\]

and let \( A_{(i,j)} = (i,j) + A_{(0,0)} \) for each \((i,j) \in \mathbb{Z}^2\). Then \( \cup \{A_{(i,j)} : (i,j) \in \mathbb{Z}^2\} \) is separately open, but each point \((i,j) \in \mathbb{Z}^2\) lies outside of the (Euclidean) interior.

**Example 2.** Another example of a separately open set that is not Euclidean open was given by Popvassilev \([12]\). Remove any circle from the plane letting one point \( P \) of this circle remain. The remaining set is separately open, but \( P \) is not in the (Euclidean) interior.
ON THE SEPARATELY OPEN TOPOLOGY

These routine examples motivate us to ask the following question: Where can these points of “essential” separate openness occur; that is, can a nonempty separately open set be constructed in a way different from adding points to an existing nonempty open set?

The answer to this question is yes. We mention here a few ways to show this. One of the easiest examples to construct is the following: Let \( \alpha \) and \( \beta \) be real numbers such that \( \alpha^2 + \beta^2 = 1 \) and \( \alpha \beta \notin \mathbb{Q} \), and let \( f \) be the rotation defined by

\[
f(x, y) = (\alpha x + \beta y, -\beta x + \alpha y).
\]

Then it can be easily seen that the set \( G = f(\mathbb{Q}^2) \) has the property that every horizontal or vertical line intersects it in at most one point. Hence \( \mathbb{R}^2 \setminus G \) is separately open. Since \( G \) is dense in \( \mathbb{R}^2 \) under the usual topology, \( \mathbb{R}^2 \setminus G \) cannot be obtained by adding points to an existing nonempty open set.

The following is a construction that can be generalized to other topological spaces. In the unit square \( I \times I \), where \( I = (0, 1) \), pick a countable base \( \mathcal{B} = \{B_1, B_2, \ldots \} \). Using induction, we shall first construct a dense countable set \( D \) that has at most one point in common with every horizontal and every vertical segment. (Such a set \( D \) is called a dense thin subset of \( I \times I \), see [11].) First, choose an arbitrary point \((x_1, y_1)\) of \( B_1 \). Suppose that for some natural number \( n \) we have already chosen \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) such that \((x_i, y_i) \in B_i \) and if \( i \neq j \), then \( x_i \neq x_j \) and \( y_i \neq y_j \). Since every set in \( \mathcal{B} \) is of cardinality \( \mathfrak{c} \), by the Pigeonhole Principle we can pick \((x_{n+1}, y_{n+1}) \in B_{n+1} \) such that \( x_{n+1} \neq x_i \) and \( y_{n+1} \neq y_i \) for \( i = 1, 2, \ldots, n \). Let \( D = \{(x_n, y_n) : n \in \mathbb{N}\} \). By construction, the set \( D \) is countable and dense. Now, let \( G = (I \times I) \setminus D \). It is easy to see that \( G \) is separately open. The above construction can be generalized to fairly general topological spaces, e.g., both spaces in the product being Baire spaces having countable \( \pi \)-weight. (For results on thin and very thin dense sets, see [10], [13], and [5].)

Finally, Hart and Kunen [6, Remark 2.2] give the following example. Let \( f: \mathbb{R} \to \mathbb{R} \) be a 1–1 function whose graph \( \Gamma \) is dense in the plane. Then \( \mathbb{R}^2 \setminus \Gamma \) is separately open. However, since \( \Gamma \) is dense in the plane, \( \mathbb{R}^2 \setminus \Gamma \) has an empty interior, so it cannot be derived by adding points to a nonempty open set.

**Example 3.** The Maltese Cross \( A \) is a \( G_\delta \) set in the Euclidean topology. If we let \( A_n = A \cup B_{1/n}(0,0) \), then each \( A_n \) is Euclidean open and \( \cap A_n = A \).

**Theorem 1.** If \( C \) is a separately open subset of \( \mathbb{R}^2 \) and is Euclidean open at all points except those in a set \( E \subset C \) that is a \( G_\delta \) set in the Euclidean topology, then \( C \) is a \( G_\delta \) set in the Euclidean topology.
Proof. Since $E$ is a $G_\delta$ set, there is a countable collection of Euclidean open sets $U_n$ such that $E = \cap U_n$. The set $C_n = C \cup U_n$ is Euclidean open for each $n$ and $C = \cap C_n$. \qed

Question. The set $E$ can be finite or in some cases countably infinite, but how far can we extend this exceptional set? Will any countable set do? What about a nowhere dense set?

It is not the case, though, that every separately open set is a Euclidean $G_\delta$ one.

Example 4. Let $S = \{(x, x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ and let $G = \mathbb{R}^2 \setminus S$. Then each $x$-section and each $y$-section is open in $\mathbb{R}$ so $G$ is separately open. However, $G$ is not a Euclidean $G_\delta$ set because if it were, then $G \cap \{(x, x) : x \in \mathbb{R}\}$ would be a $G_\delta$ subset of the line $y = x$. This is impossible since this set is homeomorphic to $\mathbb{Q}$.

This example shows that it is not sufficient for the set $E$ in Theorem 1 to be countable. We note that in this example the set $\mathbb{R} \setminus \mathbb{Q}$ could be replaced by any other subset of $\mathbb{R}$ ($G_\delta$ or not, nor even Borel) and the resulting set $G$ would be separately open. This shows that the cardinality of the collection of separately open sets in $\mathbb{R}^2$ is $2^\mathfrak{c}$, and since the cardinality of the collection of Borel subsets of $\mathbb{R}^2$ is $\mathfrak{c}$, there must exist separately open sets that are not Borel sets.

An interesting fact about the usual topology on $\mathbb{R}^2$ is that each open set can be expressed as the inverse image of an open set in $\mathbb{R}$ under some continuous function. In particular, if $G \subset \mathbb{R}^2$ is open and $f(x)$ is the distance from $x$ to $\mathbb{R}^2 \setminus G$, then $f^{-1}((0, \infty)) = G$.

Question. Is every separately open set in $\mathbb{R}^2$ the inverse image of an open set in $\mathbb{R}$ under a separately continuous function?

The answer to this question is no, as can be seen by the following cardinality argument. The cardinality of $\{G \subset \mathbb{R} : G$ is open$\}$ is $\mathfrak{c}$, and since a separately continuous function on $\mathbb{R}^2$ is uniquely determined by its values on a dense subset (such as $\mathbb{Q}^2$) of $\mathbb{R}^2$ (see [14]), the cardinality of the set of separately continuous functions is $\mathfrak{c}$. Hence the cardinality of

$$\{f^{-1}(G) : G$ is open in $\mathbb{R} \text{ and } f : \mathbb{R}^2 \to \mathbb{R}$ is separately continuous$\}$$

is $\mathfrak{c}$. However, the cardinality of the collection of all separately open sets in $\mathbb{R}^2$ is $2^\mathfrak{c}$. It follows that most separately open sets in $\mathbb{R}^2$ cannot be expressed as the inverse image of an open set in $\mathbb{R}$ under a separately continuous function.
3. Generalized separate oscillation

In this section we will assume that all spaces are Hausdorff.

Let $Z$ be a topological space. A sequence $\{G_n : n \in \mathbb{N}\}$ of open covers of $Z$ is called a development of $Z$ if for each $z \in Z$ the set $\{st(z, G_n) : n \in \mathbb{N}\}$ is a base at $z$. A regular developable space is called a Moore space.

Further, a completely regular space $Z$ is a $p$-space if and only if there exists a sequence $\{G_n : n \in \mathbb{N}\}$ of families of open subsets of $\beta Z$ such that

1. each $G_n$ covers $Z$;
2. for each $z \in Z$, $\cap \{st(z, G_n) : n \in \mathbb{N}\} \subset Z$.

The following term was introduced in [8]:

DEFINITION 3. We say that a topological space $Z$ has the property (*) if there is a sequence $\{G_n : n \in \mathbb{N}\}$ of open covers of $Z$ such that if $z \in G_n \in G_n$ for each $n$, and if $W$ is an open set in $Z$ that contains $z$, then $\cap \{G_j : 1 \leq j \leq n\} \subset W$ for some $n$.

In the class of completely regular spaces, $p$-spaces with a $G_\delta$-diagonal coincide with spaces that have property (*). Also, every developable space has the property (*). (See [8] for additional information.)

Refining the generalized oscillation $\omega_f$ introduced in [8], we will now define a generalized separate oscillation $\omega_f^{\text{sep}}$ of $f : X \times Y \to Z$. Define the generalized separate oscillation $\omega_f^{\text{sep}}$ of $f$ on the $(U,V)$-plus $P = P_{U,V}(p,q)$ by

$$\omega_f^{\text{sep}}(P) = \inf \left\{ \frac{1}{n} : n \in \mathbb{N}, \exists G \in G_n \text{ such that } f(P) \subset G \right\}.$$  

The generalized oscillation $\omega_f^{\text{sep}}$ of $f$ is defined by

$$\omega_f^{\text{sep}}(p,q) = \inf \left\{ \omega_f^{\text{sep}}(P) : P \in \mathcal{P}(p,q) \right\},$$

where $\mathcal{P}(p,q)$ stands for the collection of all $(U,V)$-pluses at $(p,q)$.

4. An extension theorem

It is well-known [10, p. 422] that if $f$ is a continuous function defined on a subset $A$ of a metric space $X$ with values in a complete metric space $Y$, then there exists a continuous extension $f^*$ of $f$ to a $G_\delta$ subset $A^*$ of $X$. This motivates us to look for an analogous result for separately continuous functions defined on subsets of the Cartesian plane $\mathbb{R}^2$. To begin, let $A$ be a subset of $\mathbb{R}^2$, and let $f$ be a real-valued separately continuous function defined on $A$; that is, the restrictions of $f$ to each horizontal and vertical section of $A$ are continuous.
Call a point \( p \) a weak plus-accumulation point of \( A \) if \( p \) is an accumulation point of \( P_1(p) \cap A \) in the usual topology on \( \mathbb{R}^2 \). Call a point \( p = (x, y) \) a plus-accumulation point of \( A \) if \( p \) is an accumulation point of both \( (\{x\} \times \mathbb{R}) \cap A \) and \( (\mathbb{R} \times \{y\}) \cap A \) in the usual topology on \( \mathbb{R}^2 \). Let \( A^+ \) denote the set \( A \) together with all its plus-accumulation points. For each point \( p \) in \( A^+ \) define \( \omega^+(f, p) \), the separate oscillation of \( f \) at \( p \), to be the oscillation considered only over pluses at \( p \); that is,

\[
\omega^+(f, p) = \lim_{r \to 0} \sup \{ |f(q_1) - f(q_2)| : q_1, q_2 \in A \cap P_r(p) \}.
\]

(Notice that if \( p \) is an isolated point of \( A \), in the sense that \( P_r(p) \cap A = \{p\} \) for some \( r \), then \( \omega^+(f, p) = 0 \).) Let \( A^* \) be the set of points \( p \) in \( A^+ \) for which \( \omega^+(f, p) = 0 \). To each \( p = (x_0, y_0) \) in \( A^* \) assign the sequence \( \{p_n\} \) in \( A \) with \( p_n \to p \) and \( p_n = (x_n, y_n) \) or \( p_n = (x_n, y_0) \). Since \( \omega^+(f, p) = 0 \), we have

\[
\lim_{n \to \infty} \text{diam}\left( f\{p_n, p_{n+1}, \ldots \} \right) = 0.
\]

So \( \{f(p_n)\} \) is a Cauchy sequence whose limit we will denote as \( f^*(p) \). Then \( f^* \) is the extension of \( f \) to \( A^* \). Separate continuity follows directly from the fact that \( \omega^+(f^*, p) = 0 \). Hence we have the following:

**Theorem 2.** Let \( f : A \to \mathbb{R} \) be a separately continuous function where \( A \subset \mathbb{R}^2 \) and let \( A^* \) be as defined above. If \( A \) is a proper subset of \( A^* \), then \( f \) has a separately continuous extension to \( A^* \).

**Remark 1.** The statement of this theorem is far weaker than we would have liked, which would have been to say that \( A^* \) is a separately \( G_\delta \) set (that is, the intersection of a countable collection of separately open sets). While it is true that the set of points in \( A^+ \) where \( \omega^+(f^*, p) = 0 \) is the intersection of the sets

\[
A_n = \left\{ p \in A^+ : \omega^+(f^*, p) < \frac{1}{n} \right\},
\]

we cannot say that the sets \( A_n \) are separately open. For suppose \( p \in A_n \). To show that \( A_n \) is open, we would need to show that there is \( r > 0 \) such that \( P_r(p) \cap A^+ \) is contained in \( A_n \). However, for any \( r > 0 \) there may exist points \( q \) in \( P_r(p) \cap A^+ \) such that \( \omega^+(f^*, q) \geq 1/n \), simply because there are points from \( A \) that lie on a plus centered at \( q \) that do not lie on a plus centered at \( p \).

Even if all of the sets \( A_n \) were separately open, we still would not be able to say that \( f \) could be extended to a separately \( G_\delta \) set, because it is not clear that \( A^+ \) is a separately \( G_\delta \) set. While it is true that every horizontal and vertical section of a separately \( G_\delta \) subset of \( \mathbb{R}^2 \) is a \( G_\delta \) subset of \( \mathbb{R} \), the following question

1Separate oscillation and generalized separate oscillation (mentioned in the previous section) are related. However, the former is an extended real-valued function, while the latter is bounded above by 1.
remains: If every horizontal and vertical section of a subset $A$ of $\mathbb{R}^2$ is a $G_\delta$ subset of $\mathbb{R}$, is $A$ a separately $G_\delta$ set?

Upon examining the proof of the preceding theorem, one might think that instead of using plus accumulation points in the definition of $A^+$, we could have used weak plus accumulation points instead. The following example will show that this is not always possible.

**Example 5.** Let

$$f(p) = \begin{cases} 1 & \text{if } p \in \mathbb{Q} \times (\mathbb{Q} \setminus \{0\}); \\
0 & \text{if } p \in (\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}), \end{cases}$$

and let

$$A = \left[ \mathbb{Q} \times (\mathbb{Q} \setminus \{0\}) \right] \cup \left[ (\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \right].$$

Then $f$ is separately continuous on $A$, because it is constant on each horizontal and each vertical section. Each point $p = (x,0)$ is a weak plus-accumulation point of $A$, and $\omega^+(p) = 0$. However, $f^*(p)$ will be either 0 or 1 depending on whether $x$ is irrational or rational. Hence, $f^*$ is not separately continuous on $A^*$.

The next example demonstrates a limitation on the above method used to obtain an extension.

**Example 6.** Let $A$ be a countable dense subset of $(0,1)^2$ having at most one point in common with each horizontal and each vertical line. (The construction of such a set $D$ is demonstrated in the text following Example 2.) Also, let $B$ and $C$ be disjoint subsets of $A$ such that both $B$ and $C$ are dense in $A$ and $A = B \cup C$. Now, consider the following two functions:

1. $f_1: A \rightarrow \mathbb{R}$, defined by $f_1(p) = 1$ for each $p \in A$, and
2. $f_2: A \rightarrow \mathbb{R}$, defined by $f_2(p) = 1$ for each $p \in B$ and $f_2(p) = -1$ for each $p \in C$.

Note that the extension function $f_1^*$ is given by $f_1^*(p) = 1$ for each $p \in (0,1)^2$, but $f_1^*$ cannot be obtained by the “sequence techniques” used above, because $A$ has no plus-accumulation points. For the same reason, our technique does not extend $f_2$ continuously either.

The authors are grateful to the referee for supplying the previous example.

For abstract topological spaces, a corresponding result is Theorem 1.1 of [2].
5. Separation axioms

In this section we will discuss which separation axioms the plus topology satisfies. To distinguish between the space $X \times Y$ with the product topology and the space $X \times Y$ with the plus topology, we will denote the latter by $X \otimes Y$.

Henriksen and Woods [7] have shown that if each of $X$ and $Y$ has a countable $\pi$-weight and $Y$ is a Baire space, then $X \otimes Y$ is not regular. A more explicit construction showing that $\mathbb{R} \otimes \mathbb{R}$ is not regular is provided by Hart and Kunen [6], where it is shown that if $D \subset \mathbb{R} \times \mathbb{R}$ is dense in the Tychonoff topology and can be viewed as the graph of a 1–1 function that is closed and discrete in the plus topology, then the non-regularity of $\mathbb{R} \otimes \mathbb{R}$ follows from Sierpinski’s theorem (see [6]), which asserts that every such separately open set is dense in the plus topology. Yet another construction showing the non-regularity of $\mathbb{R} \otimes \mathbb{R}$, based on the Baire Category theorem, was provided by Popvassilev [12].

The space $\mathbb{R} \otimes \mathbb{R}$ is clearly Hausdorff because its topology is stronger than the usual topology, which is Hausdorff. More generally, it is shown in [6] that $X \otimes Y$ is Hausdorff if and only if both $X$ and $Y$ are Hausdorff.

Similar arguments can be made for the properties Urysohn, completely Hausdorff, and strongly Hausdorff. A space $X$ is Urysohn (see [15]) if for each pair of distinct points $x$ and $y$ in $X$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. The space $\mathbb{R} \otimes \mathbb{R}$ is Urysohn because $\mathbb{R}^2$ is Urysohn, and a continuous function on $\mathbb{R}^2$ is also continuous on $\mathbb{R} \otimes \mathbb{R}$.

A space $X$ is completely Hausdorff (see [15]) if for each pair of distinct points $x$ and $y$ there exist disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. If $X$ is a Urysohn space, then it is completely Hausdorff. Hence, $\mathbb{R} \otimes \mathbb{R}$ is completely Hausdorff.

A Hausdorff space $X$ is strongly Hausdorff (see [9]) if every infinite subset of $X$ contains a sequence $\{x_n\}$ such that the terms $x_n$ have pairwise disjoint neighborhoods in $X$. Again, since $\mathbb{R}^2$ is strongly Hausdorff and the plus topology is stronger than the usual topology, $\mathbb{R} \otimes \mathbb{R}$ is strongly Hausdorff as well; that is, a collection of pairwise disjoint neighborhoods in $\mathbb{R}^2$ is also a collection of pairwise disjoint neighborhoods in $\mathbb{R} \otimes \mathbb{R}$.

6. Other topological properties

**Theorem 3.** A neighborhood base for a point in $\mathbb{Q} \otimes \mathbb{Q}$ must be uncountable.

**Proof.** Suppose that $\{B_n\}$ is a countable neighborhood base of the point $(x, y)$ in $\mathbb{Q} \otimes \mathbb{Q}$. We will construct inductively an open set $G$ containing $(x, y)$ such
that $B_n \not\subset G$ for each $n$. Let $(x_1, y_1)$ be a point in $B_1 \setminus \{(x, y)\}$. Suppose that points $(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1})$ different from $(x, y)$ have been selected respectively from $B_1, B_2, \ldots, B_{n-1}$ so that no two of these points lie on the same horizontal or vertical line. Since $B_n$ contains a plus centered at $(x, y)$, there is a point $(x_n, y)$ in $B_n$ with $x_n \neq x_i$ for all $i = 1, 2, \ldots, n - 1$. Now $B_n$ contains a plus centered at $(x_n, y)$, so there is a point $(x_n, y_n)$ in $B_n$ with $y_n \neq y$ and $y_n \neq y_i$ for all $i = 1, 2, \ldots, n - 1$. Hence $(x_n, y_n) \in B_n \setminus \{(x, y)\}$ and $(x_n, y_n)$ does not lie on any horizontal or vertical line containing $(x_i, y_i)$ for any $i < n$. Now, $G = \mathbb{Q}^2 \setminus \{(x_n, y_n) : n \in \mathbb{N}\}$ is an open set and $B_n \not\subset G$ for each $n \in \mathbb{N}$. Hence, a neighborhood base of $(x, y)$ cannot be countable.

\section*{Remark 2.} Since there are at most $\mathfrak{c}$ subsets of $\mathbb{Q}^2$ and a neighborhood base of $\mathbb{Q} \otimes \mathbb{Q}$ must be uncountable, under the Continuum Hypothesis there must be exactly $\mathfrak{c}$ open neighborhoods of a point.

In view of the above construction, the cardinality of the neighborhood base of $\mathbb{R} \otimes \mathbb{R}$ must be uncountable. In fact, a neighborhood base for a point in $\mathbb{R} \otimes \mathbb{R}$ must have $2^{\mathfrak{c}}$ elements. This is an immediate corollary of the following theorem (see also [18, p. 739]).

\section*{Theorem 4} ([6, Lemma 2.1, p. 105]). Suppose that $X$ and $Y$ are Hausdorff spaces, that $w(X) \leq \mathfrak{c}$, and that each non-empty open subset of $X$ has size at least $\mathfrak{c}$. Suppose that there are disjoint countable sets $D_\alpha \subset X$ for $\alpha < \mathfrak{c}$ such that each $D_\alpha$ is dense in $Y$. Then,

$$\chi((p, q), X \otimes Y) \geq 2^\mathfrak{c} \quad \text{for all } (p, q) \in X \times Y.$$ 

(For a discussion of the weight $w(X)$ of a topological space $X$ and the character $\chi(p, X)$ of a point in $X$, see [3, pp. 27–28].) Note that our Theorem 3 does not imply nor is implied by this result.

\section*{Remark 3.} A. V. A r h a n g e l s k i i [4] introduced a class of spaces, called $p$-spaces, in the following way: $X$ is called a $p$-space (cf. [4, p. 444]) if there exists a sequence $\{G_n\}$ of open covers of $X$ satisfying the following condition: For each $x \in X$ and each $n$, if $G_n$ satisfies $x \in G_n \subset G_n$, then

1. $\cap_n G_n$ is compact, and
2. $\{\cap_{i \leq n} G_i : n \in \omega\}$ is an outer network for the set $\cap_n G_n$; that is, every open set containing $\cap_n G_n$ contains some $\cap_{i \leq n} G_i$.

The class of $p$-spaces is rather large; it contains all metric spaces and all Čech-complete spaces. In the same article [4], A r h a n g e l s k i i showed that if $X$ is a $p$-space, $w(X) \leq \text{card}(X)$ (see [9], Remark, p. 10).

Obviously, $\text{card}(\mathbb{Q}^2) = \omega$, but we have just shown that $w(\mathbb{Q} \otimes \mathbb{Q})$ is uncountable. This proves that $\mathbb{Q} \otimes \mathbb{Q}$ is not a $p$-space.
Remark 4. It is natural to ask whether $\mathbb{Q} \otimes \mathbb{Q}$ is a regular space. In fact, it is. Recall (see [6]) that a $\sigma$-set is a separable metric space in which every $F_\sigma$ set is also a $G_\delta$ set. Since every countable metric space (in particular, $\mathbb{Q}$ is a $\sigma$-set and $\mathbb{Q}$ is a countable non-discrete metric space, it follows from [6, Theorem 5.5, p. 118] that $\mathbb{Q} \otimes \mathbb{Q}$ is regular.

One of the cardinals used in set theory is the cardinal $p$ (see [17, p. 115]). It is known [17, Theorem 3.1(a), p. 116] that $p \geq \omega_1$. It follows from [6, Corollary 5.8, p. 119] that $\mathbb{Q} \otimes \mathbb{Q}$ is normal and strongly 0-dimensional. (For a definition of strongly 0-dimensional, see [3, p. 443].) Of course, it would be nice to see an elementary proof of the normality of $\mathbb{Q} \otimes \mathbb{Q}$.

REFERENCES

ON THE SEPARATELY OPEN TOPOLOGY


Received September 22, 2007

Zbigniew Piotrowski
Eric Wingler
Department of Mathematics
Youngstown State University
Youngstown, OH 44555
U.S.A.
E-mail: zpiotr@math.ysu.edu
wingler@math.ysu.edu

Robert W. Vallin
Department of Mathematics
Slippery Rock University of PA
Slippery Rock, PA 16057
U.S.A.
E-mail: robert.vallin@sru.edu