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PRODUCT OF *M*-MEASURES

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ABSTRACT. M-measures have been introduced as a useful tool for probability theory on IF-events. We deal with the product of M-measures, a notion which is important for the construction of a joint observable — an analogue of a random vector in the Kolmogorov theory.

It is well-known that, in probability theory, the σ-additivity plays a key role. It can be expressed as a conjunction of continuity and additivity:

$$P(A) + P(B) = P(A \cap B) + P(A \cup B), \qquad A, B \subset \Omega.$$

Of course, if instead of (crisp) sets, we consider fuzzy sets (i.e., functions $f\colon \Omega \longrightarrow [0,1]$), then some operations with fuzzy sets, instead of sets, will be considered. In [1], IF-sets, i.e., pairs

$$A = (\mu_A, \nu_A), \quad \mu_A, \nu_A\colon \Omega \longrightarrow < 0, 1 >, \quad \mu_A + \nu_A \le 1,$$

are considered. If $\mathcal{F}$ is a set of IF-sets, then the probability can be regarded as a mapping

$$m\colon \mathcal{F} \longrightarrow [0,1].$$

The second important concept of probability theory is the notion of a random variable. It is a measurable mapping

$$\xi\colon \Omega \longrightarrow R_1$$

(if A is a Borel set, then $\xi^{-1}(A) \in \mathcal{S}$, where $\mathcal{S}$ is a given σ-algebra of subsets of Ω). In the IF-probability theory instead of random variables, the so-called observables

$$x\colon \mathcal{B}(R) \longrightarrow \mathcal{F}$$

are considered. For $C \in \mathcal{B}(R)$ we denote

$$x(C) = \big(x^\flat(C), 1 - x^\sharp(C)\big) \in \mathcal{F}.$$

Then the mappings

$$x^\flat, x^\sharp\colon \mathcal{B}(R) \longrightarrow [0,1]$$

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have nice properties leading to the notion of an M-measure (see Definition 1).

In the present paper we deal with the existence of products of M-measures. It is motivated by the construction of a joint observable — an analogue of a random vector in the Kolmogorov theory. In Theorem 4 we construct a product of two M-measures defined on σ-algebras, our main result.

Definition 1. Let $\mathcal{R}$ be an algebra of subsets of a set Ω. A mapping $\mu\colon\mathcal{R}\to[0,1]$ is called an M-measure if the following properties are satisfied:

(i) $\mu(\Omega)=1, \mu(\emptyset)=0$;

(ii) $\mu(A\cup B)=\mu(A)\vee\mu(B)$, $\mu(A\cap B)=\mu(A)\wedge\mu(B)$ for any $A,B\in\mathcal{R}$;

(iii) $A_n\nearrow A, B_n\searrow B, A_n,B_n,A,B\in\mathcal{R}\Longrightarrow\mu(A_n)\nearrow\mu(A),\mu(B_n)\searrow\mu(B)$.

Theorem 2. *For every M-measure μ defined on an algebra $\mathcal{R}$ there exists exactly one M-measure $\overline{\mu}$ on $\sigma(\mathcal{R})$ extending μ.*

Proof. See [3]. □

Definition 3. Let $\mathcal{C}$ be a family of subsets of Ω. We say that $\mathcal{C}$ is a compact family if for every sequence $(C_n)_n\subset\mathcal{C}$ the following implication holds:

$$\left(\forall n\colon \bigcap_{i=1}^{n} C_i\neq\emptyset\right) \Longrightarrow \bigcap_{i=1}^{\infty} C_i\neq\emptyset.$$

A mapping $\lambda\colon\mathcal{R}\longrightarrow[0,1]$ is called compact if there exists a compact family $\mathcal{C}$ such that to every $A\in\mathcal{R}$ and every $\varepsilon>0$

$$B\in\mathcal{R}, C\in\mathcal{C}, B\subset C\subset A \quad\text{with}\quad \lambda(A\setminus B)<\varepsilon.$$

Theorem 4. *Let $(\Omega,\mathcal{S},\mu)$, $(Y,\mathcal{T},\nu)$ be two spaces with compact M-measures, where $\mu\colon\mathcal{S}\longrightarrow[0,1]$ and $\nu\colon\mathcal{T}\longrightarrow[0,1]$. Then there exists exactly one M-measure $\overline{\kappa}\colon\sigma(\mathcal{R})=\mathcal{S}\otimes\mathcal{T}\longrightarrow[0,1]$ such that*

$$\overline{\kappa}(A\times B)=\mu(A)\wedge\nu(B)\quad\textit{for all}\quad A\in\mathcal{S},\ B\in\mathcal{T}.$$

Proof. Let $\mathcal{R}$ be an algebra of all sets of the form

$$\bigcup_{i=1}^{n}(A_i\times B_i),$$

where

$$n\in N,\ A_i\in\mathcal{S},\ B_i\in\mathcal{T},\quad\text{and}\quad (A_i\times B_i)\cap(A_j\times B_j)=\emptyset\quad\text{for}\quad i\neq j.$$

Since the operations $a\vee b=\max(a,b)$ and $a\wedge b=\min(a,b)$ satisfy the distributive law, the expression

$$\bigvee_{i=1}^{n}\big(\mu(A_i)\wedge\nu(B_i)\big)$$

does not depend on the choice of A_i and B_i. Therefore, we can define

$$\kappa\left(\bigcup_{i=1}^{n}(A_i \times B_i)\right) = \bigvee_{i=1}^{n}\big(\mu(A_i) \wedge \nu(B_i)\big).$$

This way we obtain a mapping $\kappa\colon \mathcal{R} \to [0,1]$. Moreover,

$$\kappa(A \times B) = \mu(A) \wedge \nu(B). \tag{1}$$

We shall prove that κ is an M-measure. Evidently,

$$\begin{aligned} \kappa(X \times Y) &= \mu(X) \wedge \nu(Y) = 1 \wedge 1 = 1, \\ \kappa(\emptyset) &= \mu(\emptyset) \wedge \nu(\emptyset) = 0. \end{aligned}$$

Further,

$$\begin{aligned} &\kappa(A \cup B) \\ &= \kappa\left(\bigcup_{i=1}^{n}(A_i \times B_i) \cup \bigcup_{j=1}^{m}(C_j \times D_j)\right) = \kappa\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(A_i \times B_i) \cup (C_j \times D_j)\right) \\ &= \kappa\left(\bigcup_{i=1}^{n}\bigcup_{j=1}^{m}\big((A_i \setminus C_j) \times B_i\big) \cup \big((A_i \cap C_j) \times (B_i \cup D_j)\big) \cup \big((C_j \setminus A_i) \times D_j\big)\right) \\ &= \bigvee_{i=1}^{n}\bigvee_{j=1}^{m}\big(\mu(A_i \setminus C_j) \wedge \nu(B_i)\big) \vee \big(\mu(A_i \cap C_j) \wedge \nu(B_i \cup D_j)\big) \vee \big(\mu(C_j \setminus A_i) \wedge \nu(D_j)\big) \\ &= \bigvee_{i=1}^{n}\bigvee_{j=1}^{m}\big(\mu(A_i) \wedge \nu(B_i)\big) \vee \big(\mu(C_j) \wedge \nu(D)j)\big) \\ &= \kappa(A) \vee \kappa(B). \end{aligned}$$

Similarly,

$$\kappa(A \cap B) = \kappa(A) \wedge \kappa(B) \qquad \text{for all} \quad A, B \in \mathcal{R}.$$

Now, we shall prove

$$A_n \in \mathcal{R}\ (n = 1, 2, \dots), \quad A_n \searrow \emptyset \Longrightarrow \kappa(A_n) \searrow 0. \tag{2}$$

Since μ, ν are compact M-measures, there exist compact families $\mathcal{K}_1 \subset 2^X$, $\mathcal{K}_2 \subset 2^Y$ such that

$$\forall\, A \in \mathcal{S} \quad \forall\, \varepsilon > 0 \quad \exists\, E \in \mathcal{S}, \quad C \in \mathcal{K}_1, \quad E \subset C \subset A, \quad \mu(A \setminus E) < \varepsilon$$

and

$$\forall\, B \in \mathcal{T} \quad \forall\, \varepsilon > 0 \quad \exists\, F \in \mathcal{T}, \quad D \in \mathcal{K}_2, \quad F \subset D \subset B, \quad \nu(B \setminus F) < \varepsilon\,.$$

Denote

$$\mathcal{K} = \left\{ C; \quad C = \bigcup_{i=1}^{n} (C_i \times D_i), \quad n \in N, \quad C_i \in \mathcal{K}_1, \quad D_i \in \mathcal{K}_2 \right\}.$$

First, we show that

$$\forall \varepsilon > 0 \quad \forall A \in \mathcal{R} \quad \exists B \in \mathcal{R}, \quad C \in \mathcal{K}, \quad B \subset C \subset A, \quad \kappa(A \setminus B) < \varepsilon\,. \tag{3}$$

Let $A = \bigcup_{i=1}^{n} (A_i \times B_i)$, $A_i \in \mathcal{S}$, $B_i \in \mathcal{T}$ $(i = 1, 2, \ldots, n)$.

Since μ, ν are compact M-measures, then $\exists H_i \in \mathcal{K}_1$, $E_i \in \mathcal{S}$ such that

$$E_i \subset H_i \subset A_i, \quad \mu(A_i \setminus E_i) < \varepsilon \quad \text{and} \quad \exists G_i \in \mathcal{K}_2,\ F_i \in \mathcal{S}$$

such that

$$F_i \subset G_i \subset B_i, \qquad \nu(B_i \setminus F_i) < \varepsilon.$$

Put

$$C = \bigcup_{i=1}^{n} (H_i \times G_i), \quad B = \bigcup_{j=1}^{n} (E_j \times F_j),$$

and

$$C \subset B \subset A\,.$$

Then,

$$\begin{aligned}
\kappa(A \setminus B) &= \kappa\left(\bigcup_{i=1}^{n} (A_i \times B_i) \setminus \bigcup_{j=1}^{n} (E_j \times F_j) \right) \\
&= \kappa\left(\bigcup_{i=1}^{n} \big((A_i \setminus E_i) \times B_i\big) \cup \bigcup_{j=1}^{n} \big(A_j \times (B_j \setminus F_j)\big) \right) \\
&= \bigvee_{i=1}^{n} \big(\mu(A_i \setminus E_i) \wedge \nu(B_i)\big) \vee \bigvee_{j=1}^{n} \big(\mu(A_j) \wedge \nu(B_j \setminus F_j)\big) \\
&\le \bigvee_{i=1}^{n} \mu(A_i \setminus E_i) \vee \bigvee_{j=1}^{n} \nu(B_j \setminus F_j) < \varepsilon.
\end{aligned}$$

Now return to the sequence (A_n), $A_n \in \mathcal{R}$, $A_n \searrow \emptyset$.

Using (3) we construct

$$B_n \in \mathcal{R}, \quad C_n \in \mathcal{K}, \quad B_n \subset C_n \subset A_n, \quad \kappa(A_n \setminus B_n) < \varepsilon.$$

Putting

$$D_n = \bigcap_{i=1}^{n} C_i, \quad C_i \in \mathcal{K}.$$

We get

$$\bigcap_{n=1}^{\infty} D_n \subset \bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Since $\mathcal{K}$ is a compact family, there exists m such that

$$\bigcap_{i=1}^{m} B_i \subset D_m = \bigcap_{i=1}^{m} C_i = \emptyset.$$

We have

$$\begin{aligned}\kappa(A_m) &= \kappa\left(A_m \setminus \bigcap_{i=1}^{m} B_i\right)\\ &= \kappa\left(\bigcup_{i=1}^{m}(A_m \setminus B_i)\right)\\ &\le \kappa\left(\bigcup_{i=1}^{m}(A_i \setminus B_i)\right)\\ &= \bigvee_{i=1}^{m} \kappa(A_i \setminus B_i) < \varepsilon.\end{aligned}$$

Further

$$\kappa(A_n) \le \kappa(A_m) < \varepsilon \qquad \text{for any} \quad n \ge m.$$

Therefore,

$$\lim_{n\to\infty} \kappa(A_n) = 0.$$

Now if

$$B_n \in \mathcal{R},\ B_n \nearrow B,\ B \in \mathcal{R}, \quad \text{then} \quad B \setminus B_n \searrow \emptyset$$

and hence

$$\begin{aligned}\kappa(B) &= \kappa\big((B \setminus B_n) \cup B_n\big)\\ &= \kappa(B \setminus B_n) \vee \kappa(B_n)\\ &\le \kappa(B \setminus B_n) \vee \bigvee_{i=1}^{\infty} \kappa(B_i).\end{aligned}$$

Therefore,

$$\kappa(B) \le \lim_{n\to\infty} \kappa(B \setminus B_n) \vee \left(\bigvee_{i=1}^{\infty} \kappa(B_i)\right) = \bigvee_{i=1}^{\infty} \kappa(B_i) \le \kappa(B)$$

and hence

$$\kappa(B) = \lim_{i\to\infty} \kappa(B_i).$$

On the other hand, $C_n \searrow C$ implies $C_n \setminus C \searrow \emptyset$,

$$\kappa(C_n) = \kappa\big((C_n \setminus C) \vee C\big) = \kappa(C_n \setminus C) \vee \kappa(C),$$

$$\bigwedge_{n=1}^{\infty} \kappa(C_n) = \left(\bigwedge_{n=1}^{\infty} \kappa(C_n \setminus C)\right) \vee \kappa(C) = 0 \vee \kappa(C) = \kappa(C).$$

We have proved that $\kappa\colon \mathcal{R} \to [0,1]$ is an M-measure. By Theorem 1.2 there exists exactly one M-measure $\overline{\kappa}\colon \sigma(\mathcal{R}) \to [0,1]$ such that $\overline{\kappa}|\mathcal{R} = \kappa$.

By (1) we have

$$\overline{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \qquad \text{for} \quad A \times B \in \mathcal{R}.$$

Fix B and put

$$\mathcal{L} = \big\{A \in \mathcal{S};\ \overline{\kappa}(A \times B) = \mu(A) \wedge \nu(B)\big\}.$$

Since $\mathcal{L}$ is monotone and $\mathcal{L} \supset X$, it follows that

$$\mathcal{L} \supset \sigma(X) = \mathcal{S}.$$

Therefore

$$\overline{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \qquad \text{for each} \qquad A \in \mathcal{S}.$$

Further, for fixed $A \in \mathcal{S}$, consider the family

$$\mathcal{G} = \big\{B \in \mathcal{T};\ \overline{\kappa}(A \times B) = \mu(A) \wedge \nu(B)\big\}.$$

Clearly, $\mathcal{G} \supset Y$. Since $\mathcal{G}$ is monotone,

$$\mathcal{G} \supset \sigma(Y) = \mathcal{T}$$

and hence,

$$\overline{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \qquad \text{whenever} \qquad A \in \mathcal{S},\ B \in \mathcal{T}.$$

This completes the proof. □

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