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ψ-CONTINUOUS FUNCTIONS AND FUNCTIONS PRESERVING ψ-DENSITY POINTS

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ABSTRACT. Let $\mathcal{T}_\psi$ be the ψ-density topology for a fixed function ψ. We will examine some new properties of the family of ψ-continuous functions (that means continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with ψ-density topology $\mathcal{T}_\psi$ in its domain and range). In the second part of the article we will discuss functions preserving ψ-density points.

We will use the following notations: $\mathbb{R}$ will denote the set of real numbers, $\mathcal{L}$ – the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, m (m^*) – the Lebesgue measure (outer measure) on $\mathbb{R}$, A' – the complement of the set $A \subset \mathbb{R}$. Let $\mathcal{C}$ be a family of nondecreasing continuous functions $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lim_{t\to 0^+} \psi(t) = 0$.

We say that $x \in \mathbb{R}$ is *a ψ-density point* of a measurable set $A \subset \mathbb{R}$ if and only if

$$\lim_{h\to 0^+} \frac{m\big(A' \cap [x-h, x+h]\big)}{2h\psi(2h)} = 0\,.$$

A point $x \in \mathbb{R}$ is said to be a ψ-dispersion point of a measurable set A if it is a ψ-density point of the complement of A.

In the definition of a ψ-density point we use a symmetric interval of the length $2h$ and with the center in the point x. As shown in [TW-B, Theorem 0.1], we cannot replace such an interval with an arbitrary interval I including x. Despite this fact, we can prove the following property.

Property 1. The point x_0 is a ψ-density point of a measurable set S if and only if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, I \ni x_0 \ \left(m(I) < \delta \Longrightarrow \frac{m(I \setminus S)}{m(I) \quad \psi\big(2m(I)\big)} < \varepsilon \right). \qquad (1)$$

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P r o o f. If the condition (1) holds then it is sufficient to put $I = [x_0 - h, x_0 + h]$ and we get that x_0 is a ψ-density point of the set S. Suppose that x_0 is a ψ-density point of a measurable set S. Hence for any $\varepsilon_1 > 0$ there exists a positive number δ_1 such that for all $h \in (0, \delta_1)$ we have

$$\frac{m\big([x_0 - h, x_0 + h] \setminus S\big)}{2h\psi(2h)} < \varepsilon_1.$$

For a fixed $\varepsilon > 0$ we match δ_1 to $\varepsilon_1 = \frac{1}{2}\varepsilon$. Let I be an interval containing x_0 and having the length less than $\frac{1}{2}\delta_1$. By h we denote the least positive number such that $I \subset [x_0 - h, x_0 + h]$. Then $h \le m(I) \le 2h < \delta_1$ and

$$\frac{m(I \setminus S)}{m(I)\, \psi\big(2m(I)\big)} \le \frac{m\big([x_0 - h, x_0 + h] \setminus S\big)}{h\psi(2h)} < 2\varepsilon_1 = \varepsilon.$$

□

For any $A \in \mathcal{L}$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}.$$

From [TW-B, Theorem 1.4], we obtain that the family $\mathcal{T}_\psi = \big\{A \in \mathcal{L} \colon A \subset \Phi_\psi(A)\big\}$ is a topology which is stronger than the natural topology $\mathcal{T}_o$ and weaker than the density topology $\mathcal{T}_d$. Topology $\mathcal{T}_\psi$ is invariant under multiplication if and only if for any $\alpha > 0$.

$$\limsup_{x \to 0^+} \frac{\psi(\alpha x)}{\psi(x)} < \infty. \tag{2}$$

If the condition (2) is not fulfilled for a certain $\alpha > 1$, then there is a set $A \in \mathcal{T}_\psi$ such that $\frac{1}{\alpha}A$ is not a set from $\mathcal{T}_\psi$ (compare with [TW-B, Theorem 2.8]).

Fix a function $\psi \in \mathcal{C}$. We will examine some properties of continuous functions $f \colon (\mathbb{R}, \mathcal{T}_\psi) \to (\mathbb{R}, \mathcal{T}_\psi)$. We will call such functions ψ-continuous and the family of them will be denoted by $\mathcal{C}_{\psi\psi}$. Each ψ-continuous function is ψ-approximately continuous, hence it is $\mathcal{D}B_1$. In [FT1] we showed that

$$\mathcal{C}_{oo} \setminus \mathcal{C}_{\psi\psi} \ne \emptyset \quad \text{and} \quad \mathcal{C}_{\psi\psi} \setminus \mathcal{C}_{oo} \ne \emptyset,$$

where $\mathcal{C}_{oo}$ is the family of continuous functions.

The idea of ψ-continuity of a function is strictly connected with the notion of a function preserving ψ-density points.

Definition 2. We will say that a homeomorphism h preserves ψ-density points if for any measurable set $S \subset \mathbb{R}$ and any $x_0 \in \Phi_\psi(S)$

$$\lim_{t \to 0^+} \frac{m^*\Big(\big(h(S)\big)' \cap \big[h(x_0) - t, h(x_0) + t\big]\Big)}{2t\psi(2t)} = 0.$$

Proposition 3. *If h is a homeomorphism preserving ψ-density points, then h satisfies Lusin's condition (N).*

Proof. Let Z be a set of Lebesgue measure zero. There exists a G_δ-set $A \supset Z$ of measure zero. Then $h(A)$ is also a G_δ-set, so it is measurable. Suppose that $m\big(h(A)\big) > 0$. Hence $h(A)$ has density 1 at a certain point $y_0 \in h(A)$, so

$$\lim_{t \to 0^+} \frac{m\big(h(A) \cap [y_0 - t, y_0 + t]\big)}{2t} = 1.$$

The point $h^{-1}(y_0)$ is from A and it is a ψ-density point of the complement of A. Let $S = A'$. For any $t > 0$ such that $\psi(2t) \le 1$, we have

$$\begin{aligned} \frac{m\Big(\big(h(S)\big)' \cap [y_0 - t, y_0 + t]\Big)}{2t\psi(2t)} &= \frac{m\big(h(S') \cap [y_0 - t, y_0 + t]\big)}{2t\psi(2t)} \\ &\ge \frac{m\big(h(A) \cap [y_0 - t, y_0 + t]\big)}{2t}. \end{aligned}$$

Therefore, h does not preserve ψ-density points. $\square$

Corollary 4. *If a homeomorphism $h\colon [0,1] \to [0,1]$ preserves ψ-density points then it is an absolutely continuous function.*

From Proposition 3, it follows that if homeomorphism h preserves ψ-density points then, for any measurable set $S \subset \mathbb{R}$, $h(S)$ is a measurable set and we need not use the outer measure from Definition 2.

Theorem 5. *A homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ preserves ψ-density points if and only if h^{-1} is a ψ-continuous function.*

Proof. First, we assume that h preserves ψ-density points. We will show that h^{-1} is a ψ-continuous function at any point. Fix a point y_0 and a set $V \in \mathcal{T}_\psi$ such that $x_0 = h^{-1}(y_0) \in V$. We will show that there exists a set $U \in \mathcal{T}_\psi$ such that $y_0 \in U$ and $h^{-1}(U) \subset V$. Since V is open in $\mathcal{T}_\psi$, for any $x \in V$ we have $x \in \Phi_\psi(V)$. The homeomorphism preserves ψ-density points, so $h(x) \in \Phi_\psi\big(h(V)\big)$. Hence $h(V)$ is open in $\mathcal{T}_\psi$ and putting $U = h(V)$, we complete the proof of this implication.

Suppose now that h does not preserve ψ-density points. Fix a point $x_0 \in \mathbb{R}$ and a measurable set S such that x_0 is a ψ-dispersion point of S and

$$\limsup_{t \to 0^+} \frac{m^*\Big(h(S) \cap \big[h(x_0) - t, h(x_0) + t\big]\Big)}{2t\psi(2t)} > 0.$$

Let $A \supset S$ be a G_δ-set such that $m(A \setminus S) = 0$. Then x_0 is a ψ-dispersion point of A, $h(A)$ is measurable and $h(S) \subset h(A)$. Hence, there exists a sequence

$(a_n)_{n\in\mathbb{N}}$ decreasing to 0 and a number $\alpha > 0$ such that

$$\frac{m\Big(h(A)\cap\big[h(x_0)-a_n, h(x_0)+a_n\big]\Big)}{2a_n\psi(2a_n)} > \alpha$$

for all $n \in \mathbb{N}$. Therefore, there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of the sequence $(a_n)_{n\in\mathbb{N}}$ such that for any natural k

$$\frac{m\Big(h(A)\cap\big[h(x_0), h(x_0)+a_{n_k}\big]\Big)}{2a_{n_k}\psi(2a_{n_k})} > \frac{\alpha}{2}$$

or

$$\frac{m\Big(h(A)\cap\big[h(x_0)-a_{n_k}, h(x_0)\big]\Big)}{2a_{n_k}\psi(2a_{n_k})} > \frac{\alpha}{2}, \tag{3}$$

For simplicity, we can assume that inequality (3) holds for all elements of the sequence $(a_n)_{n\in\mathbb{N}}$ and for any natural n we have

$$m\Big(h(A)\cap\big[h(x_0)+a_{n+1}, h(x_0)+a_n\big]\Big) > 0.$$

From the properties of measurable sets it follows that for any $n \in \mathbb{N}$ there exists a closed set $B_n \subset h(A)\cap\big[h(x_0)+a_{n+1}, h(x_0)+a_n\big]$ such that

$$m(B_n) > m\Big(h(A)\cap\big[h(x_0)+a_{n+1}, h(x_0)+a_n\big]\Big) - \frac{\alpha}{4}\cdot\frac{1}{2^n}\cdot 2a_n\psi(2a_n).$$

We put $B = \bigcup_{n=1}^{\infty} B_n \cup \big\{h(x_0)\big\}$. Observe that B is a closed set in natural topology. Moreover, for any natural i

$$\begin{aligned}
\frac{m\Big(B\cap\big[h(x_0), h(x_0)+a_i\big]\Big)}{2a_i\psi(2a_i)} &= \frac{\sum\limits_{n=i}^{\infty} m(B_n)}{2a_i\psi(2a_i)} \geq \frac{1}{2a_i\psi(2a_i)} \\
&\times\left(\sum_{n=i}^{\infty} m\Big(h(A)\cap\big[h(x_0)+a_{n+1}, h(x_0)+a_n\big]\Big) - \frac{\alpha}{4}\sum_{n=i}^{\infty} 2a_n\psi(2a_n)\frac{1}{2^n}\right) \\
&\geq \frac{m\Big(h(A)\cap\big[h(x_0), h(x_0)+a_i\big]\Big)}{2a_i\psi(2a_i)} - \frac{\frac{\alpha}{4}\cdot 2a_i\psi(2a_i)\sum\limits_{n=i}^{\infty}\frac{1}{2^n}}{2a_i\psi(2a_i)} \geq \frac{\alpha}{4} > 0\,.
\end{aligned}$$

Hence, $h(x_0)$ is not a ψ-dispersion point of the set B.

Since $h^{-1}(B) \subset A$, x_0 is a ψ-dispersion point of $h^{-1}(B)$. Moreover, $h^{-1}(B)$ is closed in natural topology, hence the set $C = \mathbb{R} \setminus h^{-1}(B) \cup \{x_0\} \in \mathcal{T}_\psi$. Simultaneously, $h(C) = \mathbb{R} \setminus B \cup \big\{h(x_0)\big\} \notin \mathcal{T}_\psi$, so the function h^{-1} is not ψ-continuous. □

Property 6. If a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, A \in \mathcal{L} \quad \forall\, I \supset A$$
$$\Big(m(A) < \delta\, m(I)\, \psi\big(2m(I)\big) \Longrightarrow m\big(h(A)\big) < \varepsilon\, m\big(h(I)\big)\, \psi\Big(2m\big(h(I)\big)\Big)\Big), \tag{4}$$

then h preserves ψ-density points.

P r o o f. Suppose that x_0 is a ψ-density point of a set $S \in \mathcal{L}$. Using Property 1, we will show that for any $\varepsilon > 0$ there exists a positive number $\gamma > 0$ such that for each interval $J \ni h(x_0)$ of the length less than γ

$$\frac{m\big(J \setminus h(S)\big)}{m(J)\, \psi\big(2m(J)\big)} < \varepsilon\,.$$

Let $\varepsilon > 0$ and choose δ from condition (4). Since $x_0 \in \Phi_\psi(S)$, from Property 1 it follows that there exists $\lambda > 0$ such that for any $I \ni x_0$ of length less than λ

$$\frac{m(I \setminus S)}{m(I)\, \psi\big(2m(I)\big)} < \delta\,. \tag{5}$$

Since h^{-1} is continuous at $h(x_0)$, there is $\gamma > 0$ such that for any interval $J \ni h(x_0)$ of length less than γ, the length of the interval $h^{-1}(J)$ is less than λ.

Let J be an interval containing $h(x_0)$, of the length less than γ. Using (5), we have

$$m\left(h^{-1}(J) \setminus S\right) \;<\; \delta\, m\left(h^{-1}(J)\right)\; \psi\left(2m(h^{-1}(J)\right).$$

From condition (4), taking $A = h^{-1}(J) \setminus S$ and $I = h^{-1}(J)$, we obtain

$$m\left(J \setminus h(S)\right) \;=\; m\left(h(h^{-1}(J) \setminus S)\right) < \varepsilon\, m(J)\, \psi\big(2m(J)\big).$$

$\square$

Theorem 7. *Let a function $\psi \in \mathcal{C}$ satisfy the condition* (2). *Suppose that $h\colon \mathbb{R} \to \mathbb{R}$ is an absolutely continuous homeomorphism such that for almost all $x \in \mathbb{R}$*

$$0 < \alpha \le h'(x) \le \beta < \infty. \tag{6}$$

Then h preserves ψ-density points.

P r o o f. Since h is an increasing function, it is sufficient to show that if x_0 is a right-hand ψ-density point of a measurable set S, then $h(x_0)$ is a right-hand ψ-density point of a set $h(S)$. The left-hand side case can be shown in the similar way. Let S be a measurable set and $\varepsilon > 0$. We assume that x_0 is a right-hand ψ-density point of S. The function ψ satisfies the condition (2), so there exist numbers $M > 0$ and $\delta_1 > 0$ such that

$$\frac{\psi(t)}{\psi(\alpha t)} < M \tag{7}$$

for all $t < \delta_1$. The point x_0 is a right-hand ψ-density point of the set S, hence there exists a positive number $\delta_2 < \frac{1}{2}\delta_1$ such that for all $t \in (0, \delta_2)$

$$\frac{m\big([x_0, x_0 + t] \setminus S\big)}{2t\psi(2t)} < \frac{\varepsilon\alpha}{\beta M}. \tag{8}$$

The fact that h^{-1} is continuous at the point $h(x_0)$ results in the existence of $\delta_3 > 0$ such that $h^{-1}(y) < x_0 + \delta_2$ for any $y \in \big(h(x_0), h(x_0) + \delta_3\big)$. To complete the proof it is sufficient to show that for any interval of the form $J = \big[h(x_0), h(x_0) + y\big]$, where $y < \delta_3$, we have

$$\frac{m\big(h(S) \setminus J\big)}{2m(J)\psi\big(2m(J)\big)} < \varepsilon.$$

The function h is a homeomorphism, so there exists an interval I which has the following properties: its length is less than δ_2, so x_0 is the endpoint of the interval $I = h^{-1}(J)$. We have $m(I) < \delta_2$. From the inversibility of h we obtain

$$m\big(h(S) \setminus J\big) = m\big(h(S) \setminus h(I)\big) = m\big(h(S \setminus I)\big).$$

From the absolute continuity of h and (6) we have

$$\alpha\, m(I) \le m\big(h(I)\big) \le \beta \cdot m(I)$$

and

$$m\big(h(S \setminus I)\big) \le \beta m(S \setminus I).$$

From this, (7) and (8) it follows that

$$\frac{m\big(h(S) \setminus J\big)}{2m(J)\psi\big(2m(J)\big)} \le \frac{\beta m(S \setminus I)}{2\alpha m(I)\psi\big(2\alpha m(I)\big)} = \frac{m(S \setminus I)}{2m(I)\psi\big(2m(I)\big)}\,\frac{\beta}{\alpha}\,\frac{\psi\big(2m(I)\big)}{\psi\big(2\alpha m(I)\big)} < \varepsilon.$$

□

Corollary 8. *Suppose that $h\colon \mathbb{R} \to \mathbb{R}$ is an absolutely continuous homeomorphism such that for almost all $x \in \mathbb{R}$*

$$1 \le \alpha \le h'(x) \le \beta < \infty. \tag{9}$$

Then h preserves ψ-density points for arbitrary function $\psi \in \mathcal{C}$.

We follow the proof of Theorem 7. Notice that if $\alpha \ge 1$ and $\psi \in \mathcal{C}$, then for each $t > 0$ we have

$$\frac{\psi(t)}{\psi(\alpha t)} \le 1.$$

□

Remark 9. Observe that if a function $\psi \in \mathcal{C}$ does not satisfy the condition (2) then for any $\alpha > 1$

$$\limsup_{x \to 0^+} \frac{\psi(\alpha x)}{\psi(x)} = \infty.$$

It means that for any $\alpha > 1$ there exists a set $A \in \mathcal{T}_\psi$ such that $\frac{1}{\alpha}A \notin \mathcal{T}_\psi$. Therefore any linear function $f(x) = ax$, with $a > 1$, is not ψ-continuous. By Corollary 8, such the functions preserve ψ-density points.

Corollary 10. *Let $f\colon I \to \mathbb{R}$. If there are numbers α, β such that*

$$0 < \alpha < \frac{f(x) - f(y)}{x - y} < \beta < \infty \tag{10}$$

for any $x, y \in I$, $x \neq y$, then f is ψ-continuous for each function $\psi \in \mathcal{C}$ satisfying (2)*.*

Moreover, if $\beta \leq 1$, then f is ψ-continuous for any function $\psi \in \mathcal{C}$.

P r o o f. From (10) the function f is strictly monotonic and continuous, so $f\colon I \to f(I)$ has the inverse function $g = f^{-1}$. This function satisfies the condition

$$0 < \frac{1}{\beta} < \frac{g(u) - g(v)}{u - v} < \frac{1}{\alpha} < \infty$$

for all $u, v \in f(I)$, $u \neq v$. As it fulfils the local Lipschitz condititon on $f(I)$, so g is absolutely continuous. Its derivative g' is bounded from above and below by positive numbers, hence g preserves ψ-density points (Theorem 7). From Theorem 5 the function $g^{-1} = f$ is ψ-continuous.

If $\beta \leq 1$, then for each function $\psi \in \mathcal{C}$, the function g preserves ψ-density points (we use Corollary 8), whence f is ψ-continuous. □

Corollary 11. *Let $\psi \in \mathcal{C}$ fulfil the condition* (2)*. If a function $h\colon \mathbb{R} \to \mathbb{R}$ is a homeomorphism and h, h^{-1} satisfy the local Lipschitz condition, then h, h^{-1} are ψ-continuous.*

It is well-known fact that every function $f(x) = x^\tau$ on the interval $[0, \infty)$, where $\tau > 0$, is density continuous. We will examine, whether such functions are ψ-continuous.

Theorem 12. *Consider the function $f\colon [0, \infty) \ni x \mapsto x^\tau$, where $\tau > 0$. If*

$$\liminf_{x \to 0^+} \frac{\psi(2x)}{\psi\bigl(2f(x)\bigr)} > 0, \tag{11}$$

then f is ψ-continuous at $x = 0$ for each $\psi \in \mathcal{C}$.

P r o o f. We will show that if 0 is not a right-hand ψ-dispersion point of a measurable set A, then $f(0) = 0$ is not a right-hand ψ-dispersion point of $f(A)$. Suppose that there exists a number $\alpha > 0$ and a sequence $(y_n)_{n \in \mathbb{N}}$ decreasing to zero such that for all natural n

$$\frac{m\bigl(A \cap [0, y_n]\bigr)}{2y_n \psi(2y_n)} > \alpha. \tag{12}$$

Consider the interval $[0, y_1]$. Let $x_k^{(1)} = \frac{y_1}{2^k}$ for $k = 0, 1, 2, \ldots$ Notice that there exists a number k_0 such that

$$m\left(A \cap \left[\frac{1}{2} x_{k_0}^{(1)},\, x_{k_0}^{(1)}\right]\right) > x_{k_0}^{(1)}\ \alpha\ \psi(2y_1)\,. \tag{13}$$

Indeed, if for all k

$$m\left(A \cap \left[\frac{1}{2} x_{k}^{(1)},\, x_{k}^{(1)}\right]\right) \le x_{k}^{(1)}\ \alpha\ \psi(2y_1)\,,$$

then

$$\frac{m\big(A \cap [0, y_1]\big)}{2y_1\psi(2y_1)} = \frac{\sum\limits_{k=0}^{\infty} m\Big(A \cap \big[\frac{1}{2} x_k^{(1)},\, x_k^{(1)}\big]\Big)}{2y_1\psi(2y_1)} \le \frac{\alpha\psi(2y_1) \sum\limits_{k=0}^{\infty} x_k^{(1)}}{2y_1\psi(2y_1)} \le \alpha\,,$$

contrary to (12). We put $a_1 = x_{k_0}^{(1)}$ and obtain

$$\frac{m\Big(A \cap \big[\frac{1}{2}a_1, a_1\big]\Big)}{2a_1\psi(2a_1)} \ge \frac{m\Big(A \cap \big[\frac{1}{2}a_1, a_1\big]\Big)}{2a_1\psi(2y_1)} > \frac{\alpha}{2}\,.$$

There exists $n_2 \in \mathbb{N}$ such that $y_{n_2} < a_1$. Analogously, we can find $a_2 \in (0, y_{n_2}]$ such that

$$\frac{m\Big(A \cap \big[\frac{1}{2}a_2, a_2\big]\Big)}{2a_2\psi(2a_2)} > \frac{\alpha}{2}\,.$$

By induction we define the sequence $(a_n)_{n\in\mathbb{N}}$ decreasing to zero such that for all $n \in \mathbb{N}$ we have

$$\frac{m\Big(A \cap \big[\frac{1}{2}a_n, a_n\big]\Big)}{2a_n\psi(2a_n)} > \frac{\alpha}{2}\,.$$

Notice that for any $n \in \mathbb{N}$ the derivative of the function f on the interval $\big[\frac{1}{2}a_n, a_n\big]$ fulfils the condition

$$f'(x) \ge \frac{\tau a_n^{\tau-1}}{2^{\tau-1}},$$

so

$$m\left(f\left(A \cap \left[\frac{1}{2}a_n, a_n\right]\right)\right) \ge \frac{\tau a_n^{\tau-1}}{2^{\tau-1}} \cdot m\left(A \cap \left[\frac{1}{2}a_n, a_n\right]\right).$$

Let us estimate the following expression

$$\begin{aligned}
\frac{m\Big(f(A)\cap\big[0,a_n^\tau\big]\Big)}{2a_n^\tau\psi\big(2a_n^\tau\big)} &= \frac{m\Big(f(A)\cap f\big([0,a_n]\big)\Big)}{2a_n^\tau\psi\big(2a_n^\tau\big)} \\
&= \frac{m\Big(f\big(A\cap[0,a_n]\big)\Big)}{2a_n^\tau\psi\big(2a_n^\tau\big)} \\
&\ge \frac{m\Big(f\big(A\cap\big[\frac{1}{2}a_n,a_n\big]\big)\Big)}{2a_n^\tau\psi\big(2f(a_n)\big)} \\
&\ge \frac{\tau a_n^{\tau-1}}{2^{\tau-1}}\cdot\frac{m\Big(A\cap\big[\frac{1}{2}a_n,a_n\big]\Big)}{2a_n^\tau\psi\big(2f(a_n)\big)} \\
&\ge \frac{\tau}{2^{\tau-1}}\cdot\frac{m\Big(A\cap\big[\frac{1}{2}a_n,a_n\big]\Big)}{2a_n\psi(2a_n)}\cdot\frac{\psi(2a_n)}{\psi\big(2f(a_n)\big)} \\
&> \frac{\tau}{2^{\tau-1}}\cdot\frac{\alpha}{2}\cdot\frac{\psi(2a_n)}{\psi\big(2f(a_n)\big)}\,.
\end{aligned}$$

Since

$$\liminf_{n\to\infty}\frac{\psi(2a_n)}{\psi\big(2f(a_n)\big)} > 0,$$

hence

$$\limsup_{x\to 0^+}\frac{m\big(f(A)\cap[0,x]\big)}{2x\psi(2x)} > 0\,.$$

We obtain that 0 is not a ψ-dispersion point of the set $f(A)$. We have shown that if 0 is a ψ-dispersion point of the set $f(A)$, then 0 is a ψ-dispersion point of the set A. Hence f is ψ-continuous at 0. □

Corollary 13. *Let $\psi\in\mathcal{C}$. Consider the function $f\colon[0,\infty)\ni x\mapsto x^\tau$, where $\tau\ge 1$. Then f is ψ-continuous on $\big[0,\tau^{-1/(\tau-1)}\big]$.*

In fact, for any $x\in\big(0,\tau^{-1/(\tau-1)}\big]$ for the derivative $f'(x)$ we have $0<f'(x)\le 1$, so, from Corollary 8, f is ψ-continuous at x. Continuity at 0 follows from the last theorem.

Remark 14. If ψ satisfies (2), then $f(x)=x^\tau$, where $\tau\ge 1$, is ψ-continuous on $[0,\infty)$.

Theorem 15. *Let $\tau\in(0,1)$ and $f(x)=x^\tau$ for $x\in[0,\infty)$. If*

$$\lim_{x\to 0^+}\frac{\psi(2x)}{\psi\big(2f(x)\big)} = 0, \tag{14}$$

then f is not ψ-continuous at $x=0$.

P r o o f. We will construct an interval set

$$A = \bigcup_{n=1}^{\infty} [a_n, b_n], \qquad 0 < a_n < b_n < a_{n-1}, \ \lim_{n\to\infty} b_n = 0$$

such that 0 is a ψ-dispersion point of the set A and 0 is not a ψ-dispersion point of the set $f^{-1}(A)$.

Let c_1 be a point from $(0,1)$ such that $\psi(2c_1) < \frac{1}{2}$. From the assumption of a function ψ, there exists $d_1 \in (0, c_1)$ such that for any $x \in (0, d_1]$

$$\frac{\psi(2x)}{\psi\big(2f^{-1}(x)\big)} > \frac{1}{c_1}.$$

Since

$$\frac{d_1}{2d_1\psi(2d_1)} \geq \frac{1}{2\psi(2c_1)} > 1 > c_1,$$

there exists a point a_1 such that

$$\frac{m\big([a_1, d_1]\big)}{2d_1\psi(2d_1)} = c_1.$$

Let us consider the function

$$f_1 \colon [a_1, c_1] \ni x \mapsto \frac{m\big([a_1, x]\big)}{2x\,\psi(2x)}.$$

It is a continuous function attaining the value c_1. The set $f_1^{-1}\big(\{c_1\}\big)$ is closed in natural topology and not empty. Let $b_1 = \min f_1^{-1}\big(\{c_1\}\big)$. Notice that for any $x \in (a_1, b_1)$,

$$\frac{m\big([a_1, x]\big)}{2x\,\psi(2x)} < c_1 \quad \text{and} \quad \frac{m\big([a_1, b_1]\big)}{2b_1\,\psi(2b_1)} = c_1.$$

Let c_2 be a point from $(0, a_1)$ such that

$$\frac{c_2}{2a_1\,\psi(2a_1)} < c_1.$$

In an analogous way we can find $0 < a_2 < b_2 < d_2 < a_1$ such that

$$\frac{\psi(2x)}{\psi\big(2f^{-1}(x)\big)} > \frac{1}{c_2}\,; \qquad \text{for each} \quad x \in (0, d_2],$$

$$\frac{m\big([a_2, b_2]\big)}{2b_2\,\psi(2b_2)} = c_2 \quad \text{and} \quad \frac{m\big([a_2, x]\big)}{2x\,\psi(2x)} < c_2 \qquad \text{for} \quad x \in (a_2, b_2).$$

By induction we construct the sequence of intervals $\big([a_n, b_n]\big)_{n\in\mathbb{N}}$ and the decreasing sequence $(c_n)_{n\in\mathbb{N}}$ which have the following properties:

(1) $0 < a_n < b_n \leq c_n$,

(2) $\frac{m([a_n, b_n])}{2b_n\,\psi(2b_n)} = c_n$,

(3) $\frac{\psi(2b_n)}{\psi(2f^{-1}(b_n))} > \frac{1}{c_n}$,

(4) $\frac{m([a_n,x])}{2x\ \psi(2x)} < c_n$ for $x \in (a_n, b_n)$,

(5) $\frac{c_{n+1}}{2a_n\ \psi(2a_n)} < c_n$.

The point 0 is a ψ-dispersion point of the set $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$. Indeed, let us take $x \in (0, b_1)$. There exists a natural number n such that $x \in (b_{n+1}, b_n]$. If $x \in (b_{n+1}, a_n]$, then

$$\begin{aligned}\frac{m\big(A \cap [0,x]\big)}{2x\ \psi(2x)} &= \frac{m\big(A \cap [0,b_{n+1}]\big)}{2x\ \psi(2x)} \\ &\leq \frac{m\big([a_{n+1}, b_{n+1}]\big)}{2b_{n+1}\ \psi(2b_{n+1})} + \frac{b_{n+2}}{2b_{n+1}\ \psi(2b_{n+1})} \leq 2c_{n+1}.\end{aligned}$$

If $x \in (a_n, b_n]$, then

$$\begin{aligned}\frac{m\big(A \cap [0,x]\big)}{2x\ \psi(2x)} &\leq \frac{b_{n+1} + m\big([a_n, x]\big)}{2x\ \psi(2x)} \\ &\leq \frac{b_{n+1}}{2a_n\ \psi(2a_n)} + \frac{m\big([a_n, x]\big)}{2x\ \psi(2x)} < 2c_n.\end{aligned}$$

For $n \to \infty$ the sequence $(c_n)_{n\in\mathbb{N}}$ tends to zero, so $0 \in \Phi_\psi(A')$. For completeness of the proof it remains to show that 0 is not a ψ-dispersion point of the set $f^{-1}(A)$. Denote, for the simplicity, by α the number $\frac{1}{\tau}$ and observe that

$$f^{-1}(A) = \bigcup_{n=1}^{\infty}\big[a_n^\alpha, b_n^\alpha\big].$$

For any $b > a > 0$

$$b^\alpha - a^\alpha > b^\alpha - ab^{\alpha-1} = (b-a)b^{\alpha-1}.$$

Therefore, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\frac{m\big(f^{-1}(A) \cap [0, b_n^\alpha]\big)}{2b_n^\alpha \psi(2b_n^\alpha)} &\geq \frac{b_n^\alpha - a_n^\alpha}{2b_n^\alpha \psi(2b_n^\alpha)} \\ &> \frac{b_n - a_n}{2b_n \psi(2b_n)} \cdot \frac{\psi(2b_n)}{\psi(2b_n^\alpha)} \\ &= c_n \cdot \frac{\psi(2b_n)}{\psi\big(2f^{-1}(b_n)\big)} > 1.\end{aligned}$$

$\square$

Corollary 16. *Functions $f(x) = x^\tau$ for $\tau \in (0,1)$ are not ψ-continuous at zero for $\psi = \mathrm{id}$. In particular, the function $f(x) = x^2$ is ψ-continuous and its inverse function is not ψ-continuous for $\psi = \mathrm{id}$.*

Remark 17. From Theorem 12 it follows that the function $f(x) = x^\tau$ for $\tau \in (0,1)$ is ψ-continuous at zero if ψ satisfies (11). It is not ψ-continuous at zero if ψ satisfies (14). A natural question arises whether such a function is ψ-continuous at zero if ψ satisfies $\liminf_{x\to 0^+} \frac{\psi(2x)}{\psi(2f(x))} = 0$. Another natural question is if $f(x) = x^\tau$, where $\tau \geq 1$, is not ψ-continuous on $\left[\tau^{-1/(\tau-1)}, \infty\right)$ for ψ which does not satisfy the condition (2).

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