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ON A NON-HOMOGENEOUS DIFFERENCE EQUATION FROM PROBABILITY THEORY

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ABSTRACT. The so-called gambler's ruin problem in probability theory is considered for a Markov chain having transition probabilities depending on the current state. This problem leads to a non-homogeneous difference equation with non-constant coefficients for the expected duration of the game. This mathematical expectation is computed explicitly.

1. Introduction

We consider a Markov chain $\{X_n, n = 0, 1, \ldots\}$ having state space $S := \{0, \ldots, N\}$ and for which the (time-homogeneous) transition probabilities

$$p_{i,j} := P\left[X_{n+1} = j \mid X_n = i\right]$$

are given by

$$p_{i,i+1} = \frac{1}{4}(1 - ci), \quad p_{i,i-1} = \frac{1}{4}(1 + ci) \quad \text{and} \quad p_{i,i} = \frac{1}{2} \tag{1}$$

for $i = 1, \ldots, N-1$. The states 0 and N are assumed to be absorbing, so that $p_{0,j} = \delta_{0,j}$ and $p_{N,j} = \delta_{N,j}$. Moreover, because $p_{i,j} \in [0,1]$ for all $i, j \in S$ and the state N must be accessible from $N-1$, the positive constant c must satisfy the condition

$$c < \frac{1}{N-1}. \tag{2}$$

Using the results in Cox and Miller (1965, p. 213), we can state that the Markov chain defined above may be regarded as a discrete version of the Ornstein-Uhlenbeck process. Various discrete versions of the Ornstein-Uhlenbeck process have been considered by, in particular, Renshaw (1997), Anishchenko *et al.* (2002, p. 53), Sprott (2003, p. 234), Larralde (2004 a,b), Kontoyiannis and Meyn (2005), and Milstein *et al.* (2007).

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Assume that $X_0 = i \in \{1, \ldots, N-1\}$, and let $T := \inf\{n > 0 : X_n = 0 \text{ or } N\}$. We want to determine the value of

$$D_i := E[T \mid X_0 = i]. \tag{3}$$

The quantity D_i is thus the expected value of the time (that is, the number of transitions) needed for the Markov chain to hit either 0 or N, starting from i. An application of this type of problem is the following: we can imagine that a *gambler*, having initially i units of money, bets 1 unit at each play of a game for which he has a probability $p_{i,i+1}$ (respectively $p_{i,i-1}$, $p_{i,i}$) of winning (resp. losing, drawing). If $X_T = 0$, we say that the gambler is *ruined*, hence the classical problem called the *gambler's ruin problem*. Here, D_i is the *expected duration of the game*.

Conditioning on the outcome of the first play, we can write that D_i satisfies the non-homogeneous difference equation with non-constant coefficients

$$D_i = \frac{1-ci}{4} D_{i+1} + \frac{1+ci}{4} D_{i-1} + \frac{1}{2} D_i + 1, \tag{4}$$

that is,

$$\frac{\big[1-c(i+1)\big]}{2} D_{i+2} - D_{i+1} + \frac{\big[1-c(i+1)\big]}{2} D_i = -2. \tag{5}$$

The boundary conditions are

$$D_0 = D_N = 0. \tag{6}$$

When $\{X_n, n = 0, 1, \ldots\}$ is a *random walk*, that is, when $p_{i,i+1} = p = 1 - p_{i,i-1}$ for $i = 1, \ldots, N-1$, where $p \in (0,1)$, equation (4) becomes

$$D_i = pD_{i+1} + (1-p)D_{i-1} + 1.$$

It is not difficult to find the solution to this difference equation that satisfies $D_0 = D_N = 0$ (see for instance Feller, 1968, p. 348). However, it turns out that solving (5), (6) is much more difficult. In Section 2, we will compute D_i explicitly, and we will end this work with a few remarks in Section 3.

2. Expected duration of the game

In Lefebvre and Guilbault (2008), the authors calculated the first hitting place probability

$$p_i := P[X_T = N \mid X_0 = i]. \tag{7}$$

This probability is the solution of

$$p_i = \frac{(1-ci)}{2} p_{i+1} + \frac{(1+ci)}{2} p_{i-1}, \tag{8}$$

for $i = 1, \dots, N-1$, that satisfies the boundary conditions $p_0 = 0$ and $p_N = 1$. To obtain an explicit expression for p_i, the authors first set $x = i - 1$ and let $y(x) = p_{i-1}$, so that equation (8) could be rewritten as

$$\left[\frac{1 - c(x+1)}{2}\right] y(x+2) - y(x+1) + \left[\frac{1 + c(x+1)}{2}\right] y(x) = 0 \tag{9}$$

for $x = 0, \dots, N-2$. The boundary conditions were $y(0) = 0$ and $y(N) = 1$. Notice that (9) is a *hypergeometric difference equation* (see Batchelder, 1967, p. 68).

Next, they transformed (9) into its *normal form*:

$$(x - a)y(x+2) + ay(x+1) - xy(x) = 0, \tag{10}$$

where $a := \frac{2}{c}$ and $x \in \left\{1 + \frac{1}{c}, \dots, N - 1 + \frac{1}{c}\right\}$. They found (see Batchelder, 1967, Chapter III) that a fundamental system of solutions of equation (10) is $y_1(x) \equiv 1 \in \mathbb{R}$ and

$$y_2(x) = (-1)^x \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x - a, \frac{1}{2}\right), \tag{11}$$

in which $F(\cdot, \cdot, \cdot, \cdot)$ is a hypergeometric function (see Abramowitz and Stegun, 1965, p. 556).

Finally, they showed that the general solution of (10) can be expressed as

$$y(x) = c_1 + c_2(-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x - a, \frac{1}{2}\right), \tag{12}$$

where [ ] denotes the integer part, and c_1 and c_2 are arbitrary constants.

Remarks.

i) The integer part of x was used because the function $y(x)$ must be real-valued in the application considered. Moreover, the constants c_1 and c_2 are also assumed to be real.

ii) When the constant c is of the form $c = 1/(N+j)$ or $c = 2/[2(N+j) - 1]$, where $j \in \{0, 1, \dots\}$, equation (10) is *completely reducible* (see Batchelder, 1967, p. 123–124). The case when $c = 2/[2(N+j) - 1]$ does not cause any problem, but when $c = 1/(N+j)$ the function $y_2(x)$ can be expressed as

$$y_2(x) = (-1)^{a+1} \frac{a!}{2^a} \qquad \text{for} \quad x = 1, 2, \dots, a+1.$$

Therefore, $y_1(x) \equiv 1$ and $y_2(x)$ are not linearly independent. However, $y_2(x)$ being a continuous function of the parameter c, the authors could take the limit as c tends to $1/(N+j)$ to obtain the solution for this value of c.

Now, with $x = i - 1$ as above, and $z(x) := D_{i-1}$, the normal form of equation (5) is

$$(x-a)z(x+2) + az(x+1) - xz(x) = 2a. \tag{13}$$

We already know the general solution of the corresponding homogeneous equation. Hence, we must find a particular solution of (13) to obtain its general solution. To do so, we will make use of a method that is similar to the method of variation of parameters for ordinary differential equations. We assume that the constant c is different from $1/(N+n)$. As mentioned above, the solution for this value of c is obtained as a limit.

If $z_1(x)$ and $z_2(x)$ form a fundamental system of solutions for the associated homogeneous equation, and if $D(x)$ is the Casorati determinant, then a particular solution of (13) is (see Batchelder, 1967, p. 13)

$$z_p(x) = c_1(x)z_1(x) + c_2(x)z_2(x),$$

where

$$\Delta c_n(x) = \frac{M_{2n}(x)}{D(x+1)} \frac{2a}{(x-a)} \qquad \text{for} \quad n = 1, 2$$

and $M_{2n}(x)$ is the cofactor of the second element in the nth column of

$$D(x+1) = \begin{vmatrix} 1 & z_2(x+1) \\ 1 & z_2(x+2) \end{vmatrix}.$$

Hence,

$$\Delta c_1(x) = -\frac{z_2(x+1)}{D(x+1)} \frac{2a}{(x-a)} \tag{14}$$

and

$$\Delta c_2(x) = \frac{1}{D(x+1)} \frac{2a}{(x-a)}. \tag{15}$$

Remark. In the sequel, we assume that $\frac{\Gamma(z)}{\Gamma(w)} = \lim_{\epsilon \to 0} \frac{\Gamma(\epsilon + z)}{\Gamma(\epsilon + w)}$.

Lemma 2.1. *When $c - a \neq 0$ and $z \neq 0$,*

$$F(a, 1, c+1, z) = \frac{\Gamma(c+1)}{\Gamma(c)} \frac{\Gamma(c-a)}{\Gamma(c-a+1)} \frac{1}{z} \{1 + (z-1)F(a, 1, c, z)\}.$$

Proof. The result follows directly from the following relation (see Abramowitz and Stegun, 1965, p. 558):

$$c(1-z)F(a, b, c, z) - cF(a, b-1, c, z) + (c-a)zF(a, b, c+1, z) = 0,$$

and from the fact that $F(a, 0, c, z) \equiv 1$. □

Next, for equation (13) the Casorati determinant is

$$\begin{aligned} D(x) &= (-1)^{[x]+1}\frac{\Gamma(x+1)}{\Gamma(x+1-a)}F\left(-a,1,x+1-a,\frac{1}{2}\right) \\ &\quad -(-1)^{[x]}\frac{\Gamma(x)}{\Gamma(x-a)}F\left(-a,1,x-a,\frac{1}{2}\right) \end{aligned} \tag{16}$$

and, making use of Lemma 2.1, we may write that

$$D(x) = 2(-1)^{[x]+1}\frac{\Gamma(x)}{\Gamma(x-a)}.$$

Hence, the determinant is never equal to zero if $c \neq 1/(N+n)$, with $n \in \mathbb{N}$. Furthermore, from equation (15),

$$\Delta c_2(x) = (-1)^{[x]}a\frac{\Gamma(x-a)}{\Gamma(x+1)}.$$

Remark. When $\frac{1}{c} = N+n-\frac{1}{2}$, x is not an integer and the Casorati determinant is never equal to zero for all admissible values of x, as should be. In this case,

$$\Delta c_2(x) = (-1)^{[x]}a\prod_{j=0}^{a}(x-j)^{-1}.$$

However, if $\frac{1}{c} = N+n$, $D(x)$ is a polynomial of degree $a = 2(N+n)$, which is zero over the set $S_1 = \{1,2,\ldots,a\}$, but non-zero over $\mathbb{N} \setminus S_1$, that is, to the right of the singularity at $x = a$.

Now, an inverse difference $\Delta^{-1}f(x)$ of the function $f(x)$ is known every time either of the following formal series: $-\sum_{n=0}^{\infty} f(x+n)$, $\sum_{n=1}^{\infty} f(x-n)$ converges (see Batchelder, 1967, p. 6). However, these series usually diverge. Nevertheless, they enable us to demonstrate the following proposition.

Proposition 2.1. *Let*

$$f(x) = \eta^x\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)}.$$

Then, everywhere the limit $\lim_{\epsilon\to 0}\frac{\Gamma(x+\epsilon+\alpha)}{\Gamma(x+\epsilon+\beta)}$ *exists, we have*

$$\Delta^{-1}f(x) = -\eta^x\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)}F(x+\alpha,1,x+\beta,\eta)$$

or, equivalently,

$$\Delta^{-1}f(x) = -\frac{\eta^x}{(1-\eta)}\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)}F\left[\beta-\alpha,1,x+\beta,\frac{\eta}{(\eta-1)}\right], \tag{17}$$

provided that $\eta < 1$ *or that* $\beta-\alpha > 1$ *if* $\eta = 1$. *In the case of the second equation,* η *must be strictly smaller than* 1.

P r o o f. Making use of the formal series, we may write that

$$\Delta^{-1} f(x) = -\eta^x \frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} F(x+\alpha, 1, x+\beta, \eta).$$

We can show (see E r d é l y i *et al.*, 1953, p. 68) that this function is entire when $|\eta| < 1$. The series remains convergent when $\eta = 1$ if $\beta - \alpha > 1$. Moreover, by analytic continuation, this formula is valid for any $\eta < 1$ and can then be written as in (17) (see A b r a m o w i t z and S t e g u n, 1965, p. 558), which is an entire function for any $\eta < 1$. □

Remark. When $\eta < 0$, it is better to take the integer part of x and write $\eta^{[x]}$ instead of η^x, in order to obtain only real values.

Next, from the previous proposition, if $c \neq 1/(N+n)$, with $n \in \mathbb{N}$, we have

$$c_2(x) = \frac{a}{2}(-1)^{[x]+1} \frac{\Gamma(x-a)}{\Gamma(x+1)} F\left(a+1, 1, x+1, \frac{1}{2}\right).$$

Finally [see equation (11)],

$$c_2(x) z_2(x) = -\frac{a}{2x} F\left(a+1, 1, x+1, \frac{1}{2}\right) F\left(-a, 1, x-a, \frac{1}{2}\right). \tag{18}$$

Remarks.

i) In the preceding equation, the second hypergeometric function is absolutely convergent when the state space of the Markov chain is $\{0, 1, \ldots, N\}$, because $x - 1 > N - 1 > 0$.

ii) In the particular case when $a = 2(N+n) - 1$, since $x - a$ is not a relative integer, we may write (see A b r a m o w i t z and S t e g u n, 1965, p. 561)

$$c_2(x) z_2(x) = -\frac{a}{2x} F\left(a+1, 1, x+1, \frac{1}{2}\right) \sum_{n=0}^{a} \frac{(-a)_n}{2^n (x-a)_n}.$$

The term $\Delta c_1(x)$ can be written as follows [see equation (14)]:

$$\Delta c_1(x) = aF\left(-a, 1, x+1-a, \frac{1}{2}\right) \frac{\Gamma(x-a)}{\Gamma(x-a+1)}. \tag{19}$$

In order to obtain an explicit expression for $c_1(x)$, we will use the *Fundamental Theorem* on summations.

Proposition 2.2. *If $\Delta F(x) = f(x)$, and if a, b $(\geq a)$ are not integers, then*

$$\sum_{x=a}^{b} f(x) = \Delta^{-1} f(x)\Big|_a^{b+1} = F(b+1) - F(a).$$

Now, since

$$\Delta \sum_{t=a}^{x-1} f(t) = \Delta\big[F(x) - F(a)\big] = f(x),$$

we have

$$\Delta^{-1} f(x) = \sum_{t=a}^{x-1} f(t)$$

(with the convention that $\sum_{t=a}^{a-1} f(t) = 0$). Using this principle, equation (14) and the linearity of the inverse difference operator Δ^{-1}, we deduce that

$$\begin{aligned} c_1(x) &= a\Delta^{-1}\left\{F\left(-a, 1, x+1-a, \frac{1}{2}\right)\frac{\Gamma(x-a)}{\Gamma(x-a+1)}\right\} \\ &= a\sum_{t=1+\frac{1}{c}}^{x-1} F\left(-a, 1, t+1-a, \frac{1}{2}\right)\frac{\Gamma(t-a)}{\Gamma(t-a+1)}, \end{aligned}$$

where the summation $\sum_{t=1+\frac{1}{c}}^{x-1} f(t)$ is over $x-1-\frac{1}{c}$ elements. Furthermore, because $z_1(x) \equiv 1$, we have $c_1(x)z_1(x) = c_1(x)$, which is indeed a difference.

From the preceding results, we may state that a particular solution of equation (13) is given by

$$\begin{aligned} z_p(x) &= -\frac{a}{2x}F\left(a+1, 1, x+1, \frac{1}{2}\right)F\left(-a, 1, x-a, \frac{1}{2}\right) \\ &\quad + a\sum_{t=1+\frac{1}{c}}^{x-1} F\left(-a, 1, t+1-a, \frac{1}{2}\right)\frac{\Gamma(t-a)}{\Gamma(t-a+1)}. \end{aligned} \tag{20}$$

This solution, evaluated at $x = 1 + \frac{a}{2}$, becomes (see Abramowitz and Stegun, 1965, p. 557)

$$z_p\left[1 + \left(\frac{a}{2}\right)\right] = \frac{a\pi}{2}\left\{\frac{\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)} - \frac{1}{\sqrt{\pi}}\right\}\frac{\Gamma\left(\frac{-a}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)}.$$

It follows that $z_p\big[1 + \left(\frac{a}{2}\right)\big] = 0$ when $c = 2/\big[2(N+n) - 1\big]$, with $n \in \mathbb{N}$.

The general solution of equation (13), subject to the boundary conditions

$$z\left[1 + \left(\frac{a}{2}\right)\right] = z\left[N + 1 + \left(\frac{a}{2}\right)\right] = 0,$$

is now given in the following proposition.

Proposition 2.3. *The average number of transitions D_i defined in* (3) *is*

$$D_i = z\left[i + 1 + \left(\frac{1}{c}\right)\right],$$

where

$$z(x) = c_1 + c_2(-1)^{[x]} \frac{\Gamma(x)}{\Gamma(x-a)} F\left(-a, 1, x-a, \frac{1}{2}\right) + z_p(x),$$

$z_p(x)$ *is given in* (20),

$$c_2 = \frac{z_p\Big[1+\left(\frac{a}{2}\right)\Big] - z_p\Big[N+1+\left(\frac{a}{2}\right)\Big]}{z_2\Big[N+1+\left(\frac{a}{2}\right)\Big] - z_2\Big[1+\left(\frac{a}{2}\right)\Big]}$$

and

$$c_1 = -\,c_2 z_2\left[1+\left(\frac{a}{2}\right)\right] - z_p\left[1+\left(\frac{a}{2}\right)\right],$$

in which the function $z_2(x)$ *is the same as the function* $y_2(x)$ *in* (11), *but with* $(-1)^{[x]}$ *instead of* $(-1)^x$.

In the particular case when

$$c = \frac{2}{[2(N+n)-1]}, \qquad \textit{with} \quad n \in \mathbb{N},$$

we have

$$c_1 = 0 \quad \textit{and} \quad c_2 = \frac{-\,z_p\Big[N+1+\left(\frac{a}{2}\right)\Big]}{z_2\Big[N+1+\left(\frac{a}{2}\right)\Big]}.$$

3. Concluding remarks

We have obtained an explicit expression for the average number D_i of transitions needed for the Markov chain having the transition probabilities defined in (1) and starting at $X_0 = i$ to hit either 0 or N. To do so, we solved the appropriate difference equation, subject to the boundary conditions $D_0 = D_N = 0$. We saw that the problem of finding a particular solution of (4) is not trivial.

Instead of proceeding as above, we could have solved a system of linear equations. This is straightforward when N is small enough. However, in order to obtain an exact solution for any N, this technique is not convenient.

As in Lefebvre and Guilbault (2008), we could extend the problem solved in the current work by considering a Markov chain $\{X_n, n = 0, 1, \ldots\}$ with state space $\{-M, \ldots, -1, 0, 1 \ldots, N\}$, where $M \in \{1, 2, \ldots\}$. Moreover, the probabilities $p_{i,j}$ could be different when i is a negative state. That is, we could set

$$p_{i,i+1} = \frac{1-di}{4}, \; p_{i,i-1} = \frac{1+di}{4} \quad \text{and} \quad p_{i,i} = \frac{1}{2} \qquad \text{for} \quad i \in \{-M+1, \ldots, -1\},$$

in which the constant d (> 0) must be such that $d < 1/(M-1)$ for the condition $0 \le p_{i,j} \le 1$ for all i, j to be respected and the state $-M$ to be accessible from $-M+1$. When the process hits the origin, we could set

$$p_{0,1} = p_0,\ p_{0,-1} = \frac{1}{2} - p_0 \quad \text{and} \quad p_{0,0} = \frac{1}{2}, \qquad \text{with} \quad p_0 \in (0,1).$$

This type of Markov chain can have some interesting applications [see Lefebvre (2006)].

Finally, the fact that the Markov chain always has a $\frac{1}{2}$ probability of remaining in its current state does not influence the probability that it will hit 0 before N, for instance. However, it does of course increase the average number of transitions needed for the process to hit either 0 or N. We could also calculate the quantity D_i when

$$p_{i,i+1} = \frac{1-ci}{2} \quad \text{and} \quad p_{i,i-1} = \frac{1+ci}{2} \qquad \text{for} \quad i \in \{1, \dots, N-1\}.$$

In this case, taking the limit as c decreases to zero, we should retrieve the formula (see Feller, 1968, p. 349) $D_i = i(N-i)$.

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