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A TELESCOPING PRINCIPLE FOR OSCILLATION OF THE SECOND ORDER HALF-LINEAR DYNAMIC EQUATIONS ON TIME SCALES

Jiří Vítovec

ABSTRACT. We establish the so-called "*telescoping principle*" for oscillation of the second order half-linear dynamic equation

$$\Big[r(t)\Phi\big(x^{\Delta}\big)\Big]^{\Delta} + c(t)\Phi(x^{\sigma}) = 0$$

on a time scale. This principle provides a method enabling us to construct many new oscillatory equations. Unlike previous works concerning the telescoping principle, we formulate some oscillation results under the weaker assumption $r(t) \neq 0$ (instead $r(t) > 0$).

1. Introduction

In the paper, we study oscillatory properties of solutions of the second-order half-linear dynamic equation

$$\Big[r(t)\Phi\big(x^{\Delta}\big)\Big]^{\Delta} + c(t)\Phi(x^{\sigma}) = 0 \qquad (\mathrm{HL}^{\Delta}\mathrm{E})$$

on a time scale $\mathbb{T}$ (i.e., an arbitrary closed subset of $\mathbb{R}$), where r and c are real rd-continuous functions on $\mathbb{T}$ with $r(t) \neq 0$, and $\Phi(x) = |x|^{p-1}\operatorname{sgn} x$ with $p > 1$. We will suppose that $\sup \mathbb{T} = \infty$. For an infinite time scale interval, we will use the symbol $\mathcal{I}_a := \{t \in \mathbb{T} : a \leq t < \infty\}$, where $a \in \mathbb{T}$.

The terminology *half-linear* is motivated by the fact that the space of all solutions of ($\mathrm{HL}^{\Delta}\mathrm{E}$) is homogeneous, but not generally additive. Thus, it has just "half of the properties" of a linear space.

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Equation (HL$^\Delta$E) covers the half-linear differential equation (if $\mathbb{T} = \mathbb{R}$)

$$\big[r(t)\Phi(x')\big]' + c(t)\Phi(x) = 0 \tag{HLDE}$$

as well as the half-linear difference equation (if $\mathbb{T} = \mathbb{Z}$)

$$\Delta\big[r_k\Phi(\Delta x_k)\big] + c_k\Phi(x_{k+1}) = 0. \tag{HLΔE}$$

Furthermore, the linear differential equation (frequently called as a Sturm-Liouville differential equation)

$$\big(r(t)x'\big)' + c(t)(x) = 0 \tag{LDE}$$

is a special case of (HLDE) (when $p = 2$). If $\Phi = \mathrm{id}$ (i.e., $p = 2$), then (HLΔE) reduces to the linear (Sturm-Liouville) difference equation

$$\Delta(r_k\Delta x_k) + c_k x_{k+1} = 0. \tag{LΔE}$$

Finally, the linear dynamic equation

$$\Big(r(t)x^\Delta\Big)^\Delta + c(t)x^\sigma = 0, \tag{L$^\Delta$E}$$

which covers (LDE) and (LΔE) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, is a special case of (HL$^\Delta$E) (when $p = 2$). The last equation is important for us, because our results extend and modify the results done just for this equation, see [10].

Oscillation and nonoscillation criteria have been established at first for equations (LDE) and (LΔE), see, for example [1, 3, 11, 17, 18], and later successively extended on (HLDE), (HLΔE), (L$^\Delta$E) and (HL$^\Delta$E), see, for example [2, 6–9, 14, 15, 16]. Many of these criteria require to know the properties of the integral (resp. sum or Δ-integral, when $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T}$ is arbitrary) of $c(t)$ on the whole interval $\mathcal{I}_a$. According to the behaviour of this integral (or sum), we can sometime decide whether our equation is oscillatory or nonoscillatory. On the other hand, from the Sturm separation theorem, it is clear that oscillation can be taken as an interval property. Consider equation (HL$^\Delta$E). If there exists a sequence of subsets $[a_i, b_i]\cap\mathbb{T}$ of $\mathcal{I}_a$, $a_i \to \infty$ as $i \to \infty$, such that for each i there is a nontrivial solution of equation (HL$^\Delta$E) which has at least two zeros (resp. generalized zeros) in $[a_i, b_i]$, then every solution of (HL$^\Delta$E) is oscillatory with at least one zero (resp. generalized zero) in each $[a_i, b_i]$ and it doesn't matter, what behavior of the functions $c(t)$ and $r(t)$ is on the remaining parts of $[a_i, b_i]$.

M. K. Kwong and A. Zettl [13] applied this idea to oscillation of equation (LDE) and constructed so-called "*telescoping principle*" which allows to trim off "problem" parts of $\int_0^t c(s)\Delta s$ and use any known oscillation criterion to the remaining "good" parts. Q. Kong and A. Zettl [12] came up with an analogic telescoping principle for equation (LΔE) and used it to obtain some new oscillation results for difference equations. P. Řehák [16] extended this telescoping principle to equation (HLΔE). Finally, L. H. Erbe, L. Kong and Q. Kong [10] unified and generalized the telescoping principle for equations

(LDE) and (LΔE) into the only one telescoping principle on time scales for equation (L$^\Delta$E) and found many new oscillation results for this equation.

The aim of this paper is to extend, modify and generalize the results of previous articles to equation (HL$^\Delta$E) and make an analogical telescoping principle for half-linear time scale case. Unlike previous works we formulate the telescoping principle under the weaker assumption $r(t) \neq 0$ (instead $r(t) > 0$), which is new even in the linear case.

2. Essentials on time scales and equation (HL$^\Delta$E)

In this chapter we recall basic information concerning the calculus on time scales and equation (HL$^\Delta$E) that are needed for further considerations.

Definition 2.1. Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$ with the inherited standard (Euclidean) topology on the real numbers $\mathbb{R}$. Define the *forward jump operator* σ for all $t \in \mathbb{T}$ such that $t < \sup \mathbb{T}$, by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the *backward jump operator* for all $t \in \mathbb{T}$ such that $t > \inf \mathbb{T}$, by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. If $\sigma(t) = t$, we say that t is *right-dense*, while if $\rho(t) = t$, we say that t is *left-dense*. Finally, we also define $\mu(t) := \sigma(t) - t$ which is called the *graininess function*.

We will use the notation $f^\sigma(t) = f\big(\sigma(t)\big)$, i.e., $f^\sigma = f \circ \sigma$. From the definition of the set $\mathbb{T}$ is evident that $\sigma(t)$, $\rho(t) \in \mathbb{T}$. If $\sup \mathbb{T} < \infty$, we define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$. Similarly, if $\inf \mathbb{T} > -\infty$, we define $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$. (In our case always $\sup \mathbb{T} = \infty$ and $\inf \mathbb{T} = a > -\infty$.)

Definition 2.2. The function $f\colon \mathbb{T} \to \mathbb{R}$ is called Δ-*differentiable* at $t \in \mathbb{T}$ with Δ-*derivative* $f^\Delta(t) \in \mathbb{R}$, if for any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\big|\big[f^\sigma(t) - f(s)\big] - f^\Delta(t)\big[\sigma(t) - s\big]\big| \leq \varepsilon|\sigma(t) - s| \qquad \text{for all} \quad s \in U.$$

We say that f is Δ-*differentiable* on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}$.

It can be shown that if $f\colon \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$ then

$$f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t).$$

If $t \in \mathbb{T}$ is right-dense and $f\colon \mathbb{T} \to \mathbb{R}$ is differentiable at t, then

$$f^{\Delta}(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}\,.$$

Next useful statement is that if $f^{\Delta}(t)$ exists, then

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.1}$$

Definition 2.3. Let $f\colon \mathbb{T} \to \mathbb{R}$ be a function. We say that f is *rd-continuous* on $\mathbb{T}$ if it is continuous at each right-dense point in $\mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists as a finite number for all left-dense points $t \in \mathbb{T}$. We write $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. If f is Δ-differentiable on a set $\mathbb{T}$ with $f^{\Delta}(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, we write $f \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.4. Let $f, F\colon \mathbb{T} \to \mathbb{R}$ be the functions and $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. If $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}$, then we say that F is *antiderivative* of function f and define the Δ-*integral* of f on $[a, b] \cap \mathbb{T}$ with $a, b \in \mathbb{T}$ by

$$\int_a^b f(t) \Delta t = F(b) - F(a)$$

and the Δ-*integral* of f on $[a, \infty] \cap \mathbb{T}$ by

$$\int_a^\infty f(s) \Delta s = \lim_{t \to \infty} \int_a^t f(s) \Delta s.$$

Note that the function F from Definition 2.4 always exists and is determined unambiguously. For further results on the calculus on time scales, see, for example [4, 5] and the references therein.

Now we recall some concepts and facts concerning equation (HL$^{\Delta}$E). The symbol Φ^{-1} denotes the inverse function of Φ. It can be shown that $\Phi^{-1}(x) = |x|^{q-1} \operatorname{sgn} x$, where q is the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. We assume throughout that $r, c \in C_{\mathrm{rd}}(\mathcal{I}_a, \mathbb{R})$ with $r(t) \neq 0$, if it is not said otherwise. We say that $x \in C^1_{\mathrm{rd}}(\mathcal{I}_a, \mathbb{R})$, is a solution of (HL$^{\Delta}$E) provided

$$\left\{ \left[r\, \Phi\big(x^{\Delta}\big) \right]^{\Delta} + c\, \Phi\big(x^{\sigma}\big) \right\}(t) = 0 \qquad \text{holds for all} \quad t \in \mathcal{I}_a.$$

Let us consider the initial value problem (IVP)

$$\left[r(t) \Phi\big(x^{\Delta}\big) \right]^{\Delta} + c(t) \Phi(x^{\sigma}) = 0, \qquad x(t_0) = A, \quad x^{\Delta}(t_0) = B \tag{2.2}$$

on $\mathcal{I}_a$, where $A, B \in \mathbb{R}$, $t_0 \in \mathcal{I}_a$.

Theorem 2.5 (Existence and Uniqueness [15, p. 373]). *Let r, c be rd-continuous functions on I_a. Then IVP* (2.2) *has exactly one solution.*

Definition 2.6. We say that a nontrivial solution x of (HL$^\Delta$E) has a *generalized zero* at t, if $r(t)x(t)x\big(\sigma(t)\big) \leq 0$. If $x(t) = 0$, we say that solution x has a *common zero* at t (common zero is a special case of generalized zero).

Definition 2.7. We say that a solution x of equation (HL$^\Delta$E) is *nonoscillatory* on $\mathcal{I}_a$, if there exists $\tau \in \mathcal{I}_a$ such that x has no generalized zero at t for $t > \tau$. Otherwise, it is *oscillatory*. Oscillation may be equivalently defined as follows. A nontrivial solution x of (HL$^\Delta$E) is called *oscillatory* on $\mathcal{I}_a$, if for every $\tau \in \mathcal{I}_a$ x has a generalized zero on $\mathcal{I}_\tau$.

Theorem 2.8 (Sturm separation theorem [15, p. 388]). *Let x, y be two linearly independent solutions of* (HL$^\Delta$E). *If there are $c_1, c_2 \in \mathcal{I}_a$, with $c_1 < c_2$, such that $(rxx^\sigma)(c_1) \leq 0$ and $(rxx^\sigma)(c_2) \leq 0$ (we exclude the case where $\sigma(c_1) = c_2$ and $y(c_2) = 0$), then there is $d \in [c_1, c_2]$ such that $(ryy^\sigma)(d) \leq 0$. Two nontrivial solutions of* (HL$^\Delta$E)*, which are not proportional, can not have a common zero.*

From Theorem 2.8 it is clear that if one solution of (HL$^\Delta$E) is oscillatory (resp. nonoscillatory), then every solution of (HL$^\Delta$E) is oscillatory (resp. nonoscillatory). Hence we can speak about *oscillation* or *nonoscillation* of equation (HL$^\Delta$E).

Remark 2.9 (Important). In most literature one supposes only $r(t) > 0$, hence a generalized zero of a solution x is defined as a point $t \in \mathcal{I}_a$ such that only $x(t)x^\sigma(t) \leq 0$. This situation is common, in particular, in the continuous case (equation (HLDE) and a classical Sturm-Liouville differential equation), where the assumption of the continuity of $r(t)$ implies a preservation of $\operatorname{sgn} r(t)$, hence is natural to suppose that $r(t) > 0$ for all $t \in [a, \infty)$.

The reason why we use, with the assumption $r(t) \neq 0$, Definition 2.6 is that we can classify every equation as osillatory or nonoscillatory, which is not assured if we use a "more simple" definition $x(t)x^\sigma(t) \leq 0$ of a generalized zero. For example, consider the linear difference equation

$$\Delta\big[(-1)^k \Delta x_k\big] + (-1)^k x_{k+1} = 0, \tag{2.3}$$

which is the special case of (HL$^\Delta$E). Equation (2.3) has two independently solutions

$$[x_k]^1 = \left(\frac{1+\sqrt{5}}{2}\right)^k \qquad \text{and} \qquad [x_k]^2 = \left(\frac{1-\sqrt{5}}{2}\right)^k,$$

where the first solution is nonoscillatory and the second is oscilatory (if we use definition with $x(t)x^\sigma(t) \leq 0$), but by Definition 2.6 both solutions are oscillatory, so we can say, that (2.3) is oscillatory and Sturmian theory works well.

On the other hand, one problem may arise, when we allow $r(t) \neq 0$ and use Definition 2.6. Concretely, consider a pair of linear dynamic equations on time scales

$$(x^{\Delta})^{\Delta} - x^{\sigma} = 0, \tag{2.4}$$

$$(-x^{\Delta})^{\Delta} + x^{\sigma} = 0, \tag{2.5}$$

which are the special cases of ($\mathrm{HL}^{\Delta}\mathrm{E}$). It is easy to see, that every solution of (2.4) is nonoscilatory, so that (2.4) is nonoscillatory. The equation (2.5) is evidently equivalent to (2.4) having the same space of solutions. When we use Definition 2.6 on equation (2.5) and its arbitrary solution, we find out, that every solution of (2.5) has a generalized zero at each $t \in \mathbb{T}$, so that (2.5) is oscillatory. If we want to evade this problem with "different classification of equivalent equations" and to obtain a reasonable oscillation theory, it is convenient to consider equation ($\mathrm{HL}^{\Delta}\mathrm{E}$) in a form, which satisfies the following conditions:

There exists $\tau \in \mathcal{I}_a$ such that $r(t) > 0$ for all right-dense points $t \in \mathcal{I}_a$ satisfying $t > \tau$. (C_1)

If for every $\tau \in \mathcal{I}_a$, $\mathcal{I}_\tau$ has infinitely many right-scattered points, then $r(t) > 0$ at infinitely many of them. (C_2)

Observe that it is always possible to reach a validity of a condition (C_2). If the condition (C_2) does not hold, it is enough to multiply the whole equation by -1 and (C_2) holds. Similarly, sometimes (but not always) it is enough to multiply the whole equation by -1 and (C_1) holds (like in a case (2.5)). Note that if (C_1) or (C_2) does not hold, then every solution of ($\mathrm{HL}^{\Delta}\mathrm{E}$) is (by Definition 2.6) oscillatory, thus ($\mathrm{HL}^{\Delta}\mathrm{E}$) is classified as oscillatory.

Theorem 2.10 (Sturm comparison theorem [15, p. 388])**.** *Consider the equation*

$$\Big[R(t)\Phi\big(x^{\Delta}\big)\Big]^{\Delta} + C(t)\Phi\big(x^{\sigma}\big) = 0, \tag{2.6}$$

and equation ($\mathrm{HL}^{\Delta}\mathrm{E}$)*,* $R, r, C, c \in C_{rd}(\mathcal{I}_a, \mathbb{R})$ *with* $r(t), R(t) \neq 0$*. Suppose that we have* $R(t) \geq r(t)$ *and* $c(t) \geq C(t)$ *for every* $t \in \mathcal{I}_a$*. If* ($\mathrm{HL}^{\Delta}\mathrm{E}$) *is nonoscillatory on* $\mathcal{I}_a$*, then* (2.6) *is also nonoscillatory on* $\mathcal{I}_a$*.*

Our approach to the oscillation problems of ($\mathrm{HL}^{\Delta}\mathrm{E}$) is based mainly on the application of a Riccati type transformation involving the generalized Riccati dynamic equation

$$w^{\Delta}(t) + c(t) + \mathcal{S}[w, r, \mu](t) = 0, \tag{$\mathrm{R}^{\Delta}\mathrm{E}$}$$

where

$$\mathcal{S}[w, r, \mu] = \lim_{\lambda \to \mu} \frac{w}{\lambda}\left(1 - \frac{r}{\Phi\big(\Phi^{-1}(r) + \lambda\Phi^{-1}(w)\big)}\right). \tag{2.7}$$

The relation between (HL$^\Delta$E) and (R$^\Delta$E) is following. If $x(t)$ is a solution of (HL$^\Delta$E) with $x(t)x^\sigma(t) \neq 0$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$ we let

$$w(t) = \frac{r(t)\Phi\big(x^\Delta(t)\big)}{\Phi\big(x(t)\big)}\,. \tag{2.8}$$

Then for $t \in [t_1, t_2] \cap \mathcal{I}_a$, $w = w(t)$ satisfies the Riccati equation (R$^\Delta$E). Note that

$$\mathcal{S}(w, r, \mu)(t) = \begin{cases} \left\{\frac{p-1}{\Phi^{-1}(r)}|w|^q\right\}(t) & \text{at right-dense } t, \\ \left\{\frac{w}{\mu}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r)+\mu\Phi^{-1}(w))}\right)\right\}(t) & \text{at right-scattered } t. \end{cases}$$

If $t \in [t_1, t_2] \cap \mathcal{I}_a$ is right-scattered, then from (2.1) and (R$^\Delta$E), we have

$$w\big(\sigma(t)\big) = \frac{r(t)w(t)}{\Phi\Big[\Phi^{-1}\big(r(t)\big) + \mu(t)\Phi^{-1}\big(w(t)\big)\Big]} - \mu(t)c(t). \tag{2.9}$$

Note, that (2.9) is trivially satisfied at a right-dense t.

3. Telescoping principle

In this section we establish the telescoping principle for oscillation of equation (HL$^\Delta$E). Consider the set

$$J = \left(\bigcup_{i=1}^{\infty} J_i\right) \cap \mathcal{I}_a, \qquad J_i = \big(a_i, \sigma(b_i)\big), \quad i \in \mathbb{N}, \tag{3.1}$$

where $a_i, b_i \in \mathcal{I}_a$ with $a < a_i < b_i < a_{i+1}$ and if $\mu(a_i) = 0$ then $\mu\big(\sigma(b_i)\big) = 0$ for all $i \in \mathbb{N}$. We call J *an interval shrinking set* in $\mathcal{I}_a$. With the help of the set J we define following "shrinking" transformation on the time scale $\mathcal{I}_a$.

At first, we define a new time scale $\widehat{\mathcal{I}_a}$ by:

$$\begin{aligned}\widehat{\mathcal{I}_a} &:= \{s \in \mathcal{I}_a : s \leq a_1\} \\ &\cup \left\{\bigcup_{j=1}^{\infty}\left\{s = t - \sum_{i=1}^{j}\big(\sigma(b_i) - \sigma(a_i)\big) : t \in \big[\sigma(b_j), a_{j+1}\big] \cap \mathcal{I}_a\right\}\right\},\end{aligned} \tag{3.2}$$

which is, anyway, the set $\mathcal{I}_a$ without the set J. More precisely, it is the set $\mathcal{I}_a$, where we trim off the time scales intervals $\big(\sigma(a_i), \sigma(b_i)\big)$ (or if one wants, it is the set $\mathcal{I}_a$, where each (time scale) subinterval $\big(a_i, \sigma(b_i)\big)$ is collapsed to its left point).

Now we define an interval shrinking transformation $\mathcal{T}\colon \mathcal{I}_a \to \widehat{\mathcal{I}_a}$ by:

$$s = \mathcal{T}t = \begin{cases} t, & t \in [a, a_1] \cap \mathcal{I}_a, \\ a_1, & t \in \big(a_1, \sigma(b_1)\big) \cap \mathcal{I}_a, \\ a_{j+1} - \sum_{i=1}^{j} \big(\sigma(b_i) - \sigma(a_i)\big), & t \in \big(a_{j+1}, \sigma(b_{j+1})\big) \cap \mathcal{I}_a, \\ t - \sum_{i=1}^{j} \big(\sigma(b_i) - \sigma(a_i)\big), & t \in \big[\sigma(b_j), a_{j+1}\big] \cap \mathcal{I}_a, \end{cases} \tag{3.3}$$

where $j \in \mathbb{N}$. For $s \in \widehat{\mathcal{I}_a}$ we define an inverse transformation $\mathcal{T}^{-1}\colon \widehat{\mathcal{I}_a} \to \mathcal{I}_a$ by:

$$\mathcal{T}^{-1}s = \inf\{t \in \mathcal{I}_a : \mathcal{T}t = s\}. \tag{3.4}$$

Note, that the condition if $\mu(a_i) = 0$ then $\mu\big(\sigma(b_i)\big) = 0$ implies $\widehat{\mu}(s) = \mu(t)$ for all $t = \mathcal{T}^{-1}s$, where $\widehat{\mu}$ denotes the graininess in $\widehat{\mathcal{I}_a}$.

Lemma 3.1. *A solution x of equation* (HL$^\Delta$E) *satisfies $r(t)x(t)x\big(\sigma(t)\big) > 0$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$ if and only if the corresponding solution $w(t)$ of the Riccati equation* (R$^\Delta$E) *satisfies $-\mu^{p-1}(t)w(t) < r(t)$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$.*

P r o o f. From Roundabout theorem [15, p. 383] follows that equation (HL$^\Delta$E) has a solution x satisfying $r(t)x(t)x\big(\sigma(t)\big) > 0$ for $t \in [t_1, t_2] \cap \mathcal{I}_a$ if and only if the corresponding solution $w(t)$ of the Riccati equation (R$^\Delta$E) satisfies

$$\Phi^{-1}\big(r(t)\big) + \mu(t)\Phi^{-1}\big(w(t)\big) > 0, \tag{3.5}$$

for $t \in [t_1, t_2] \cap \mathcal{I}_a$. However, (3.5) is equivalent to $-\mu^{p-1}(t)w(t) < r(t)$, so Lemma 3.1 holds. □

Let $\widehat{\mathcal{I}_a}$ be defined by (3.2) and consider the telescoped equation of (HL$^\Delta$E)

$$\Big[\widehat{r}(s)\Phi\big(y^\Delta\big)\Big]^\Delta + \widehat{c}(s)\Phi\big(y^{\widehat{\sigma}}\big) = 0, \qquad s \in \widehat{\mathcal{I}_a}, \tag{$\widehat{\mathrm{HL^\Delta E}}$}$$

where $\widehat{r}(s) = r(t)$, $\widehat{c}(s) = c(t)$, for $t = \mathcal{T}^{-1}s$, and where $\widehat{\sigma}$ denotes the forward jump operator in $\widehat{\mathcal{I}_a}$.

The following theorem is similar to the comparison type result. In simple terms, it says that if a certain solution $y(s)$ of the telescoped equation ($\widehat{\mathrm{HL^\Delta E}}$) has a generalized zero in $[a, b] \cap \widehat{\mathcal{I}_a}$, then a corresponding solution of the original equation (HL$^\Delta$E) has a generalized zero in $[\mathcal{T}^{-1}a, \mathcal{T}^{-1}b] \cap \mathcal{I}_a$.

Theorem 3.2. *Assume*

$$\int_{\sigma(a_i)}^{\sigma(b_i)} c(t)\Delta t \geq 0, \qquad i \in \mathbb{N}, \tag{3.6}$$

and let $d \in \widehat{\mathcal{I}_a}$ be such that $d > a$. Suppose that y is a solution of ($\widehat{\mathrm{HL^\Delta E}}$) *with $\widehat{r}(s)y(s)y\big(\widehat{\sigma}(s)\big) > 0$ for $s \in [a, d) \cap \widehat{\mathcal{I}_a}$ and $\widehat{r}(d)y(d)y\big(\widehat{\sigma}(d)\big) \leq 0$. Let x be* *a solution of* (HL$^\Delta$E) *with* $x(a) \neq 0$,

$$\frac{r(a)\Phi\big(x^\Delta(a)\big)}{\Phi\big(x(a)\big)} \leq \frac{\widehat{r}(a)\Phi\big(y^\Delta(a)\big)}{\Phi\big(y(a)\big)}.$$

Then there exists $e \leq \mathcal{T}^{-1}d$ *such that* $r(e)x(e)x\big(\sigma(e)\big) \leq 0$. *More precisely, if* $d \leq \mathcal{T}a_i$, *then there exists* $e \leq a_i$ *such that* $r(e)x(e)x\big(\sigma(e)\big) \leq 0$.

Proof. In this proof, by $w \not< v$ we mean either $w \geq v$ or w does not exist. The proof is by induction (with respect to the location of the point $d \in \widehat{\mathcal{I}_a}$). Assume the contrary. Then $w(t)$ defined by (2.8) satisfies the Riccati equation (R$^\Delta$E) and (2.9), and by Lemma 3.1 $-\mu^{p-1}(t)w(t) < r(t)$ holds for $t \in [a, \mathcal{T}^{-1}d] \cap \mathcal{I}_a$. For $s \in \widehat{\mathcal{I}_a}$, let

$$v(s) = \frac{\widehat{r}(s)\Phi\big(y^\Delta(s)\big)}{\Phi\big(y(s)\big)}.$$

Then it follows that v satisfies the telescoped Riccati equation

$$v^\Delta(s) + \widehat{c}\,(s) + \mathcal{S}[v, \widehat{r}, \widehat{\mu}\,](s) = 0, \qquad s \in \widehat{\mathcal{I}_a}, \tag{$\widehat{\mathrm{R}^\Delta\mathrm{E}}$}$$

and

$$v\big(\widehat{\sigma}(s)\big) = \frac{\widehat{r}(s)v(s)}{\Phi\big[\Phi^{-1}\big(\widehat{r}(s)\big) + \widehat{\mu}(s)\Phi^{-1}\big(v(s)\big)\big]} - \widehat{\mu}(s)\widehat{c}(s), \qquad s \in \widehat{\mathcal{I}_a}. \tag{3.7}$$

By Lemma 3.1,

$$-\widehat{\mu}^{\,p-1}(s)v(s) < \widehat{r}(s)$$

for $s \in [a, d) \cap \widehat{\mathcal{I}_a}$ and moreover

$$-\widehat{\mu}^{\,p-1}(d)v(d) \not< \widehat{r}(d).$$

(i) Assume that $d \leq \mathcal{T}a_1 = a_1$, then $t = \mathcal{T}^{-1}s = s$ for $s \in [a, d] \cap \widehat{\mathcal{I}_a}$, so we have $\widehat{r}(t) = r(t)$, $\widehat{c}(t) = c(t)$, and Riccati equations (R$^\Delta$E) and ($\widehat{\mathrm{R}^\Delta\mathrm{E}}$) are the same on $[a, d] \cap \mathcal{I}_a$. We wish to show that

$$w(t) \leq v(t), \qquad t \in [a, d] \cap \mathcal{I}_a. \tag{3.8}$$

From Theorem 2.5 (Existence and Uniqueness), it follows that the initial value problem

$$w_n^\Delta(t) + c(t) + \mathcal{S}[w_n, r, \mu](t) + \frac{1}{n} = 0, \qquad w_n(a) = w(a), \tag{3.9}$$

has a unique solution $w_n(t)$ on $t \in [a, d] \cap \mathcal{I}_a$. It is clear that $w_n(t) \to w(t)$ as $n \to \infty$ for $t \in [a, d] \cap \mathcal{I}_a$. We want to show that for all large $n \in \mathbb{N}$,

$$w_n(t) \leq v(t), \qquad t \in [a, d] \cap \mathcal{I}_a. \tag{3.10}$$

Assume the contrary. Since $w_n(a) \le v(a)$, suppose that there exist points $t_*,\ t^* \in \mathcal{I}_a$ with $a \le t_* < t^* \le d$ such that

$$w_n(t) \le v(t), \quad t \in (a, t_*] \cap \mathcal{I}_a \quad \text{and} \quad w_n(t) > v(t), \quad t \in (t_*,\ t^*] \cap \mathcal{I}_a. \tag{3.11}$$

If t_* is right-scattered in $\mathcal{I}_a$ (thus in $\widehat{\mathcal{I}_a}$), then from (3.9) and (2.9)

$$w_n\big(\sigma(t_*)\big) = \frac{r(t_*)w_n(t_*)}{\Phi\big[\Phi^{-1}\big(r(t_*)\big) + \mu(t_*)\Phi^{-1}\big(w_n(t_*)\big)\big]} - \mu(t_*)c(t_*) - \frac{\mu(t_*)}{n}. \tag{3.12}$$

Let the function $\tilde{\mathcal{S}}(w, r, \mu)$ represent the first term of the right-hand side of equations (3.12) and (3.7), so

$$\tilde{\mathcal{S}}(w, r, \mu) = \frac{rw}{\Phi\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]}\,.$$

Then from the continuity of $\tilde{\mathcal{S}}$ with respect to the first variable and from the positivity of term $\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]$ (see, (3.5)), we obtain

$$\begin{aligned}\frac{\partial\tilde{\mathcal{S}}}{\partial w} &= r\left(\frac{\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^{p-1} - \big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^{p-2}\mu\Phi^{-1}(w)}{\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^{2p-2}}\right)\\ &= r\left(\frac{\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big] - \mu\Phi^{-1}(w)}{\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^{p}}\right)\\ &= \frac{|r|^q}{\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^{p}}\,.\end{aligned}$$

Hence

$$\frac{\partial\tilde{\mathcal{S}}(w, r, \mu)}{\partial w} > 0, \tag{3.13}$$

which means that the function $\tilde{\mathcal{S}}$ is increasing with respect to w. If we compare (3.7) and (3.12), we obtain a contradiction to (3.11).

If t_* is right-dense in $\mathcal{I}_a$ (thus in $\widehat{\mathcal{I}_a}$), then $w_n(t_*) = v(t_*)$ and, moreover, $w_n(t) > v(t)$ for $t \in (t_*,\ t^*] \cap \mathcal{I}_a$. From $\widehat{(\mathrm{R}^{\Delta}\mathrm{E})}$ and (3.9), $w_n^{\Delta}(t_*) < v^{\Delta}(t_*)$, so there exists $\bar{t} \in (t_*,\ t^*] \cap \mathcal{I}_a$ such that $w_n(\bar{t}) < v(\bar{t})$, which is a contradiction to (3.11). Hence (3.10) holds.

Therefore, from $w_n(t) \to w(t)$ as $n \to \infty$ for $t \in [a, d] \cap \mathcal{I}_a$, we get (3.8), and so letting $t = d$ in (3.8), we have (with the use of the validity of $\widehat{\mu}(s) = \mu(t)$ for all $t = \mathcal{T}^{-1}s$)

$$-\mu^{p-1}(d)w(d) \ge -\widehat{\mu}^{\,p-1}(d)v(d) \not< \widehat{r}(d) = r(d). \tag{3.14}$$

Hence (with use the fact $d = \mathcal{T}^{-1}d$),

$$-\mu^{p-1}(\mathcal{T}^{-1}d)w(\mathcal{T}^{-1}d) \not< r(\mathcal{T}^{-1}d),$$

which is the contradiction to assumption.

(ii) Assume that $\mathcal{T}a_1 < d \le \mathcal{T}a_2$, then arguing as in the first part above, we see that $w(\sigma(a_1)) \le v\big(\widehat{\sigma}(a_1)\big) = v\big(\widehat{\sigma}(\mathcal{T}a_1)\big)$. Now we integrate ($\mathrm{R}^{\Delta}\mathrm{E}$) from $\sigma(a_1)$ to $\sigma(b_1)$ and obtain

$$w\big(\sigma(b_1)\big) - w\big(\sigma(a_1)\big) = -\int_{\sigma(a_1)}^{\sigma(b_1)} \mathcal{S}[w,r,\mu](t)\Delta t - \int_{\sigma(a_1)}^{\sigma(b_1)} c(t)\Delta t. \tag{3.15}$$

We wish to show, that the right-hand side of (3.15) is nonpositive and so the relation

$$w\big(\sigma(b_1)\big) \le w\big(\sigma(a_1)\big) \tag{3.16}$$

holds. Due to (3.6) it is enough to show that the function $\mathcal{S}$ is nonnegative. Under the assumption that $-\mu^{p-1}(t)w(t) < r(t)$ for $t \in [a, \mathcal{T}^{-1}d] \cap \mathcal{I}_a$, which is equivalent to (3.5), it is enough to show that function $\mathcal{S}$ satisfies

$$\mathcal{S}(w,r,\mu) \ge 0 \qquad \text{for} \quad \Phi^{-1}(r) + \mu\,\Phi^{-1}(w) > 0. \tag{3.17}$$

The statement (3.17) is obvious if $\mu = 0$. Hence, suppose $\mu > 0$. It is not difficult to compute (similar like in case of the function $\tilde{\mathcal{S}}$) that

$$\frac{\partial \mathcal{S}(w,r,\mu)}{\partial w} = \frac{\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^p - |\Phi^{-1}(r)|^p}{\mu\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big]^p}. \tag{3.18}$$

For the case $r > 0$, function $\frac{\partial \mathcal{S}(w,r,\mu)}{\partial w}$ is negative for $w < 0$ and positive for $w > 0$. Hence for $w = 0$ has function $\mathcal{S}(w,r,\mu)$ minimum, which is 0 for every $r > 0$, thus (3.17) holds.

One can observe that function $\mathcal{S}(w,r,\mu)$ with arbitrary fixed $r < 0$ and $\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big] > 0$ is increasing with the respect to the variable w for $w > (2/\mu)^{p-1}|r|$, decreasing for $|r| < w < (2/\mu)^{p-1}|r|$ and $\mathcal{S}\big((2/\mu)^{p-1}|r|, r, \mu\big)$ is positive. The statement (3.17) now follows from the continuity of $\mathcal{S}$.

The case $r, w < 0$ is excluded due to $\big[\Phi^{-1}(r) + \mu\Phi^{-1}(w)\big] > 0$. Hence (3.17) holds and thus (3.16) holds, too.

Because of (3.16),

$$w\big(\sigma(b_1)\big) \le w\big(\sigma(a_1)\big) \le v\big(\widehat{\sigma}(\mathcal{T}a_1)\big) = v\big(\mathcal{T}\sigma(b_1)\big). \tag{3.19}$$

Now since $\mathcal{T}\sigma(b_1) = \widehat{\sigma}(\mathcal{T}a_1)$, it follows that $w(t)$ and $v(s)$ satisfy the same Riccati equation for $\sigma(b_1) \le t \le \mathcal{T}^{-1}d$ and $\mathcal{T}\sigma(b_1) \le s \le d$, respectively, and also from (3.19), $w\big(\sigma(b_1)\big) \le v\big(\mathcal{T}\sigma(b_1)\big)$ holds. As before, we see that

$$-\mu^{p-1}(\mathcal{T}^{-1}d)w(\mathcal{T}^{-1}d) \ge -\widehat{\mu}^{\,p-1}(d)v(d) \not< \widehat{r}(d) = r(\mathcal{T}^{-1}d). \tag{3.20}$$

This implies that $-\mu^{p-1}(\mathcal{T}^{-1}d)w(\mathcal{T}^{-1}d) \not< r(\mathcal{T}^{-1}d)$, which is the contradiction to assumption.

The proof of the induction step from i to $i+1$ is similar and hence is omitted. $\square$

Theorem 3.3 (Telescoping principle). *Under the same conditions and with the same notation as in Theorem 3.2, if the telescoped equation* $(\widehat{\mathrm{HL}^\Delta \mathrm{E}})$ *is oscillatory, then* $(\mathrm{HL}^\Delta \mathrm{E})$ *is oscillatory, too.*

Proof. Let $x(t)$ be a solution of $(\mathrm{HL}^\Delta \mathrm{E})$ with $x(a) \neq 0$ and let $y_1(s)$ be a solution of $(\widehat{\mathrm{HL}^\Delta \mathrm{E}})$ with $y_1(a) \neq 0$ satisfying

$$\frac{r(a)\Phi\big(x^\Delta(a)\big)}{\Phi\big(x(a)\big)} \leq \frac{\widehat{r}(a)\Phi\big(y_1^\Delta(a)\big)}{\Phi\big(y_1(a)\big)}.$$

Since $y_1(s)$ is oscillatory, there exists a smallest $d_1 > a$ in $\widehat{\mathcal{I}_a}$, which satisfies $\widehat{r}(s)y_1(s)y_1\big(\widehat{\sigma}(s)\big) > 0$ for $s \in [a, d_1) \cap \widehat{\mathcal{I}_a}$ and $\widehat{r}(d_1)y_1(d_1)\big(y_1(\widehat{\sigma}(d_1))\big) \leq 0$. By Theorem 3.2, there exists $e_1 \leq \mathcal{T}^{-1}d_1$ in $\mathcal{I}_a$ with $r(e_1)x(e_1)\big(x(\sigma(e_1))\big) \leq 0$. Now, we will be interested in behaviour of the solution $x(t)$ for $t > e_1$. Let $f_1 \in \mathcal{I}_a$ with $f_1 \geq e_1$ satisfy $x(f_1) \neq 0$. Let $y_2(s)$ be a solution of $(\widehat{\mathrm{HL}^\Delta \mathrm{E}})$ with $y_2(f_1) \neq 0$ satisfying

$$\frac{r(f_1)\Phi\big(x^\Delta(f_1)\big)}{\Phi\big(x(f_1)\big)} \leq \frac{\widehat{r}(\mathcal{T}f_1)\Phi\big(y_1^\Delta(\mathcal{T}f_1)\big)}{\Phi\big(y_1(\mathcal{T}f_1)\big)}.$$

Proceeding as before, we show that there exists $e_2 \in \mathcal{I}_a$ with $e_2 > e_1$ such that $r(e_2)x(e_2)x\big(\sigma(e_2)\big) \leq 0$. Continuing this process leads to the conclusion that x is oscillatory and therefore the equation $(\mathrm{HL}^\Delta \mathrm{E})$ is oscillatory, too. $\square$

This principle can be applied to get many new examples of oscillatory equations. We use a process which is the reverse of the construction in Theorem 3.3. We begin with any known oscillatory equation $(\widehat{\mathrm{HL}^\Delta \mathrm{E}})$. We choose a sequence of numbers $s_i \in \widehat{\mathcal{I}_a}$ such that $s_i \to \infty$. Now we cut $\widehat{\mathcal{I}_a}$ at each s_i and pull the two halves of $\widehat{\mathcal{I}_a}$ apart to form a gap of arbitrary new (bounded) time scale interval. Now we define a function r on the new time scale interval and create an arbitrary function c, whose integral over the new time scale interval is nonnegative. When we do it at each s_i and relabel the so-constructed new coefficient functions by $r(t)$ and $c(t)$, then we obtain equation $(\mathrm{HL}^\Delta \mathrm{E})$, which is oscillatory.

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Department of Mathematical Analysis
Faculty of Science
Masaryk University Brno
Kotlářská 2
CZ–611-37 Brno
CZECH REPUBLIC
E-mail: vitovec@math.muni.cz