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COMPARISON OF RAABE'S AND SCHLÖMILCH'S TESTS

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ABSTRACT. The exact relationship between the Raabe's test and Schlömilch's test is described and a useful improved version of the divergence part of these tests is given. Furhter, a more general "continuous" scale of tests is introduced; it includes not only both commonly used versions of Raabe's test, but also Schlömilch's test.

The aim of this note is to make a precise comparison between two classical tests of convergence (and divergence) of real series with positive terms.

Raabe's Test. *Let $\sum a_n$ be a series of positive terms and let*

$$\mathcal{R}_n := n\left(\frac{a_n}{a_{n+1}} - 1\right).$$

If $\liminf \mathcal{R}_n > 1$, then the series $\sum a_n$ converges.
If $\mathcal{R}_n \le 1$ for all sufficiently large n, then the series $\sum a_n$ diverges.

The strength of the well-known Raabe's test lies in a comparison between the series under investigation and the family of harmonic series $\sum \frac{1}{n^p}$ of order $p > 0$. The same idea is used in the proof of the forgotten Schlömilch's test.

Schlömilch's Test. *Let $\sum a_n$ be a series of positive terms and let*

$$\mathcal{S}_n := n \ln \frac{a_n}{a_{n+1}}.$$

If $\liminf \mathcal{S}_n > 1$, then the series $\sum a_n$ converges.
If $\mathcal{S}_n \le 1$ for all sufficiently large n, then the series $\sum a_n$ diverges.

I have decided to formulate the test for ratios $\frac{a_n}{a_{n+1}}$ (as Bromwich [1], Fichtenholz [2] and Vorobev [5] do) instead of using ratios $\frac{a_{n+1}}{a_n}$ (as Knopp [3] does), despite the fact that my investigation of the relationship between the two tests was inspired by a remark made by K. Knopp who wrote that the Schlömilch's test does not differ from Raabe's test essentially

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[3, p. 287]. The difference between these two formulations of Raabe's test has been explained in my note [4].

It is worth mentioning that in the Fichtenholz's book [2] the condition $\mathcal{R}_n < 1$ (for all sufficiently large n) is given as a sufficient condition for divergence of an investigated series while the proof of it supplied in the book works for the weaker condition $\mathcal{R}_n \leq 1$ (for all sufficiently large n) as well.

Of course, the Raabe's associated sequence $(\mathcal{R}_n)$ is easier to analyse than the Schlömilch's one $(\mathcal{S}_n)$, and this seems to be the reason behind Raabe's test shining in numerous textbooks and Schlömilch's test becoming obscure. Knopp's monography [3] is the only source I know that mentions the latter test. The only reasonable example of a series $\sum a_n$ for which the sequence $(\mathcal{S}_n)$ works better than the sequence $(\mathcal{R}_n)$ is

$$a_n = \frac{1}{x\sqrt{x}\sqrt[3]{x}\ldots\sqrt[n-1]{x}} \qquad \text{for} \quad n \geq 2,$$

where the parameter x is positive.

1. The comparison of the tests

That these tests are not equivalent it can be seen from the following example.

Example 1. For the series

$$\sum_{n=1}^{\infty} \frac{4^{n-1}\big[(n-1)!\big]^2}{[(2n-1)!!]^2}$$

we have $\mathcal{R}_n = 1 + \frac{1}{4n} \to 1^+$, and hence the Raabe's test is inconclusive. On the other hand, the Maclaurin's formula

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

yields

$$\mathcal{S}_n = n\,\ln\left(1 + \frac{1}{n} + \frac{1}{4n^2}\right) = 1 - \frac{1-o(1)}{4n} < 1$$

for all sufficiently large values of n, that is, the series diverges by Schlömilch's test.

It is perhaps surprising that the above series with its first term removed

$$\sum_{n=1}^{\infty} b_n := \sum_{n=2}^{\infty} \frac{4^{n-1}\big[(n-1)!\big]^2}{[(2n-1)!!]^2} = \sum_{n=1}^{\infty} \frac{4^n(n!)^2}{\big((2n+1)!!\big)^2}$$

discloses divergence by both tests. Indeed, we get

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} - \frac{\frac{3}{4}}{n^2} + o\left(\frac{1}{n^2}\right),$$

and hence

$$\mathcal{R}_n = 1 + \frac{-\frac{3}{4} + o(1)}{n} \xrightarrow[n\to\infty]{} 1^-$$

which proves divergence by Raabe's test, and also

$$\mathcal{S}_n = 1 \ + \frac{-\frac{5}{4}}{n} + o\left(\frac{1}{n}\right) \xrightarrow[n\to\infty]{} 1^-$$

which proves divergence by Schlömilch's test.

Our first task is to compare the convergence parts of the two tests. We start with a lemma that is a little bit more than we need, but its own beauty deserves a full statement.

Lemma 1. *For any sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers the sequences*

$$(n \ln a_n)_{n\in\mathbb{N}} \quad \text{and} \quad \big(n(a_n-1)\big)_{n\in\mathbb{N}}$$

have exactly the same accumulation points.

Proof. We will actually show even more. Namely, for any sequence (a_n) of positive numbers and any value $g \in \overline{\mathbb{R}}$ the sequence $\big(n(a_n-1)\big)$ converges to g if and only if the sequence $(n \ln a_n)$ converges to g.

If $n(a_n-1) \to g \in \mathbb{R}$ then $a_n \to 1$, and hence the Maclaurin's formula

$$n \ln a_n = n \ln\big(1 + (a_n-1)\big) = n(a_n-1)\left[1 + \frac{o(a_n-1)}{a_n-1}\right] \tag{1}$$

yields $n \ln a_n \to g[1+0] = g$.

If $n(a_n-1) \to +\infty$, then by the Bernoulli's inequality we get

$$a_n^n = \big[1 + (a_n-1)\big]^n \geq 1 + n(a_n-1) \to +\infty$$

and hence $n \ln a_n = \ln a_n^n \to +\infty$.

If $n(a_n-1) \to -\infty$, then for every $c > 0$ we have $n(a_n-1) \leq -c$ for all sufficiently large values of n. Hence

$$a_n^n \leq \left(1 - \frac{c}{n}\right)^n$$

for sufficiently large n, and therefore $\limsup a_n^n \leq e^{-c}$ for every $c > 0$. Thus $\limsup a_n^n \leq 0$. Since all terms a_n are positive, it follows that $\lim a_n^n = 0$, and hence $n \ln a_n = \ln a_n^n \to -\infty$, which completes the proof of one implication.

If $n \ln a_n \to g \in \mathbb{R}$, then $a_n \to 1$. Hence by (1) we get $n(a_n-1) \to g$.

If $n \ln a_n \to +\infty$, then the inequality $\ln x \leq x - 1$ valid for $x > 0$ implies that

$$n(a_n - 1) \geq n \ln a_n \to +\infty.$$

If $n \ln a_n \to -\infty$, then the inequality $e^x \leq \frac{1}{1-x}$ valid for $x < 1$ implies that

$$a_n \leq e^{-\frac{c}{n}} \leq \frac{1}{1 + \frac{c}{n}}$$

for any $c > 0$ and for sufficiently large n. Thus one has

$$n(a_n - 1) \leq \frac{-nc}{n + c}$$

for those indices. Therefore, $\limsup n(a_n - 1) \leq -c$ for all $c > 0$ and hence $\lim n(a_n - 1) = -\infty$. □

It follows from the lemma that $\liminf \mathcal{R}_n = \liminf \mathcal{S}_n$ for every series Σa_n of positive terms, and thus the convergence parts of Raabe's test and of Schlömilch's test are equivalent.

The same cannot be said about the divergence parts of the tests as we know by our Example 1. In order to explain the exact relationship between divergence parts of the Raabe's test and the Schlömilch's test, let us note first that a slight modification of the proof of the divergence part of Raabe's test yields the following observation.

Proposition 1. *Let $\sum a_n$ be a series of positive terms. If*

$$\mathcal{R}_n^{(k)} := (n - k)\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1 \qquad (\mathcal{R}^+)$$

for a nonnegative integer k and for all sufficiently large indices n, then the series $\sum a_n$ diverges.

The full discussion of this extension of Raabe's test and also of relationship of Knoppp's version and Bromwich's version of Raabe's test can be found in [4]. A divergence test provided by condition $(\mathcal{R}^+)$ is in fact slightly stronger than the divergence part of Raabe's test for the latter one and requires that $\mathcal{R}_n^{(k)} \leq 1$ holds in the particular case $k = 0$. Actually, as it can be seen from its proof, the new divergence test rests on ratio comparison of the series under investigation not with the single harmonic series but with with all series of the form $\sum \frac{1}{n-k}$, where k runs over nonnegative integers (and where the first k terms are tacitly assumed to be equal to 1).

There is an analogous extension of the divergence part of Schlömilch's test which we formulate as our next proposition.

Proposition 2. *Let $\sum a_n$ be a series of positive terms. If*

$$\mathcal{S}_n^{(k)} := (n - k) \ln \frac{a_n}{a_{n+1}} \leq 1 \qquad (\mathcal{S}^+)$$

for a nonnegative integer k and for all sufficiently large indices n, then the series $\sum a_n$ diverges.

P r o o f. If $\mathcal{S}_n^{(k)} \le 1$ for some nonnegative integer k and all sufficiently large indices n, then $\ln \frac{a_n}{a_{n+1}} \le \frac{1}{n-k}$, which together with the inequality $e^x \le \frac{1}{1-x}$ valid for $x < 1$ implies that

$$\frac{a_n}{a_{n+1}} \le \frac{1}{1 - \frac{1}{n-k}} = \frac{\frac{1}{n-(k+1)}}{\frac{1}{(n+1)-(k+1)}}$$

for all n large enough. Since the series $\sum \frac{1}{n-(k+1)}$ diverges, so does the series $\sum a_n$ by the ratio comparison test [3, p. 114]. $\square$

It is easy to see that for any sequence (c_n) of positive terms the sequences $\big((n-k)\ln c_n\big)_{n=1}^{\infty}$ and $(n \ln c_n)_{n=1}^{\infty}$ have the same accumulation points, and thus $\liminf_n \mathcal{S}_n^{(k)} = \liminf_n \mathcal{S}_n$ for any positive integer k, so that the new associated sequences $\big(\mathcal{S}_n^{(k)}\big)$ do not improve the Schlömilch's convergence test. But they do strengthen the divergence part as the following example shows.

Example 2. The Schlömilch's test is inconclusive when being applied to the series

$$\sum_{n=1}^{\infty} \frac{(n-1)!(n+2)!}{\big[(n+1)!\big]^2},$$

because

$$\frac{a_n}{a_{n+1}} = 1 + \frac{n+4}{n^2+3n},$$

and thus

$$\mathcal{S}_n = 1 + \frac{\frac{1}{2}}{n} + o\Big(\frac{1}{n}\Big) \to 1^+.$$

On the other hand

$$\begin{aligned} \mathcal{S}_n^{(1)} &= (n-1)\ln\frac{a_n}{a_{n+1}} \\ &= (n-1)\left(\tfrac{n+4}{n^2+3n} - \tfrac{1}{2}\left[\tfrac{n+4}{n^2+3n}\right]^2 + o\left(\tfrac{1}{n^2}\right)\right) \\ &= 1 - \frac{\frac{1}{2}}{n} + o\Big(\frac{1}{n}\Big), \end{aligned}$$

and hence $\mathcal{S}_n^{(1)} < 1$ for sufficiently large n, which proves divergence of the investigated series.

Our next proposition describes the relationship between the strengthened versions of divergence parts of Raabe's and Schlömilch's tests.

Proposition 3. *A series* $\sum a_n$ *of positive terms satisfies the condition* $(\mathcal{R}^+)$*if and only if satisfies the condition* $(\mathcal{S}^+)$

Proof. Let $\sum a_n$ be a series with positive terms such that $\mathcal{R}_n^{(k)} \leq 1$ for some $k \in \mathbb{N}_0$ and for all n large enough. The inequality

$$(n-k)\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1$$

implies that

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n-k}\,.$$

Because of the inequality $1 + x < e^x$ valid for $x \neq 0$, we get

$$\frac{a_n}{a_{n+1}} < e^{\frac{1}{n-k}}$$

and thus

$$(n-k)\,\ln\frac{a_n}{a_{n+1}} < 1$$

for all sufficiently large indices n.

Now, if $\sum a_n$ is a series of positive terms such that

$$(n-k)\,\ln\frac{a_n}{a_{n+1}} \leq 1$$

for some $k \in \mathbb{N}_0$ and for sufficiently large n, then the inequality $e^x < \frac{1}{1-x}$ valid for $x < 1$, $x \neq 0$, implies that

$$\frac{a_n}{a_{n+1}} \leq e^{\frac{1}{n-k}} < \frac{1}{1 - \frac{1}{n-k}} = 1 + \frac{1}{n-k-1}\,.$$

Thus, if additionally n is greater than $k+1$, we obtain

$$\big(n - (k+1)\big)\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$$

for all sufficiently large n. □

Roughly speaking, the first part of the preceding proof validates the implication

$$\mathcal{R}_n^{(k)} \leq 1 \Rightarrow \mathcal{S}_n^{(k)} \leq 1 \tag{2}$$

and the second part shows that

$$\mathcal{S}_n^{(k)} \leq 1 \Rightarrow \mathcal{R}_n^{(k+1)} \leq 1. \tag{3}$$

The shift of the parameter k by 1 in the second implication is the cause of the classical Raabe's divergence test being weaker than the Schlömilch's one. This completes the comparison of the two tests.

However, the Knopp's version of basic Raabe's test [3, p. 285] with ratios $\frac{a_{n+1}}{a_n}$ is better than the basic Schlömilch's test, since Knopp's version is equivalent to $\mathcal{R}_n^{(1)}$-test [4].

Example 3. Divergence of the series $\sum \frac{1}{n \ln n}$ cannot be discovered either by the strengthened Raabe's test or by the strengthened Schlömilch's test. Indeed, given a nonnegative integer k, one has

$$\mathcal{R}_n^{(k)} = (n-k)\left(\frac{n+1}{n}\frac{\ln(n+1)}{\ln n} - 1\right) = \frac{n-k}{n \ln n} \cdot \frac{(n+1)\ln(n+1) - n\ln n}{(n+1)-n}.$$

Hence by the Mean Value Theorem there is a number $\xi_n \in (n, n+1)$ such that

$$\mathcal{R}_n^{(k)} = \frac{n-k}{n\ln n}\big(1+\ln \xi_n\big) = \frac{\ln \xi_n}{\ln n} + \frac{n-k(1+\ln\xi_n)}{n \ln n}.$$

Since $\lim \frac{n}{1+\ln(n+1)} = +\infty$, we get $\frac{n}{1+\ln \xi_n} > k$ for all sufficiently large values of n and thus

$$\mathcal{R}_n^{(k)} > \frac{\ln \xi_n}{\ln n} > 1$$

for those n. Since k was an arbitrary nonnegative integer, it follows that the series $\sum \frac{1}{n\ln n}$ does not satisfy the condition $(\mathcal{R}^+)$

The above example suggests that the strengthened Raabe's test is weaker than Bertrand's test. In fact the strengthened Raabe's test is about as strong as the original Gauss's test [3, p. 288], that is, if we only consider series that satisfy the condition (4) below, then it is equivalent to the original Gauss's test.

Original Gauss's Test. *Let $\sum a_n$ be a series of positive terms such that the quotient $\frac{a_n}{a_{n+1}}$ can be written in the form*

$$\frac{a_n}{a_{n+1}} = \frac{n^k + \alpha_1 n^{k-1} + \ldots + \alpha_k}{n^k + \beta_1 n^{k-1} + \ldots + \beta_k} \tag{4}$$

for some $k \in \mathbb{N}, \alpha_i, \beta_i \in \mathbb{R}$ and for all sufficiently large values of n.

If $\alpha_1 - \beta_1 > 1$, then the series $\sum a_n$ converges.

If $\alpha_1 - \beta_1 \le 1$, then the series $\sum a_n$ diverges.

Indeed, if (4) holds, then $\lim \mathcal{R}_n = \alpha_1 - \beta_1$, and by the virtue of the original Raabe's test, the series $\sum a_n$ converges if $\alpha_1 - \beta_1 > 1$, and it diverges if $\alpha_1 - \beta_1 < 1$.

In the remaining case $\lim \mathcal{R}_n = 1$, we have $\alpha_1 - \beta_1 = 1$. Then

$$\begin{aligned}\mathcal{R}_n^{(p)} &= (n-p)\left(\frac{a_n}{a_{n+1}} - 1\right)\\ &= \frac{n^k + (\alpha_2 - \beta_2 - p)n^{k-1} + (\alpha_3 - \beta_3 - p)n^{k-2} + \ldots - p(\alpha_k - \beta_k)}{n^k + \beta_1 n^{k-1} + \ldots + \beta_k}\end{aligned}$$

and hence $\mathcal{R}_n^{(p)} < 1$ for p so big that $\alpha_2 - \beta_2 - p < \beta_1$ and for sufficiently large n. Thus the series $\sum a_n$ satisfies the condition $(\mathcal{R}^+)$and therefore it diverges.

We conclude the first part of the paper with one more example.

EXAMPLE 4. Neither the basic Raabe's test nor the basic Schlömilch's test proves divergence of

$$\sum_{n=3}^{\infty} \left(2 - \sqrt[2]{e}\right)\left(2 - \sqrt[3]{e}\right) \dots \left(2 - \sqrt[n-1]{e}\right) = \sum_{n=3}^{\infty} a_n \,,$$

but the strengthened Raabe's test discloses its divergence.

$$\mathcal{S}_n = n \ln \frac{1}{2 - e^{\frac{1}{n}}} = -n \ln \left(1 - \frac{1}{n} - \frac{1}{2n^2} - o\left(\tfrac{1}{n^2}\right)\right) = 1 + \frac{1}{n} + o\left(\frac{1}{n}\right)$$

and hence $\mathcal{S}_n > 1$ for all sufficiently large values of n. Therefore the basic Raabe's test fails to disclose divergence of the series as well, because of (2). Even the level one strengthened Raabe's test is inconclusive here, since

$$\begin{aligned}\mathcal{R}_n^{(1)} = (n-1)\,\frac{e^{\frac{1}{n}} - 1}{2 - e^{\frac{1}{n}}} &= (n-1)\,\frac{\frac{1}{n} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{1 - \frac{1}{n} - \frac{1}{2n^2} - o\left(\frac{1}{n^2}\right)} \\ &= \frac{1 + \frac{-\frac{1}{2}}{n} + o(\frac{1}{n})}{1 + \frac{-1}{n} + o(\frac{1}{n})} \xrightarrow[n\to\infty]{} 1^+,\end{aligned}$$

but the next level associated sequence works well:

$$\mathcal{R}_n^{(2)} = (n-2)\,\frac{e^{\frac{1}{n}} - 1}{2 - e^{\frac{1}{n}}} = \frac{1 + \frac{-\frac{3}{2} + o(1)}{n}}{1 + \frac{-1 + o(1)}{n}} < 1$$

for all sufficiently large n, and therefore the series under investigation diverges for it satisfies the condition $(\mathcal{R}^+)$.

2. A unified Raabe-Schlömilch's test

The following idea of unification of Raabe's test and Schlömilch's test has been suggested to author by Zsolt Páles during the 20th Summer Conference on Real Functions theory in Liptovský Ján, Slovakia. Consider the function

$$f(r,\, x) = \begin{cases} \frac{x^r - 1}{r} & \text{if} \quad r \neq 0, \\ \ln x & \text{if} \quad r = 0, \end{cases}$$

defined for $r \in \mathbb{R}$ and for $x > 0$. Then the following proposition is true.

Proposition 4. *Let r be a real number and let $\sum a_n$ be a series of positive terms. Define*

$$\mathcal{P}_n^r := n\, f\left(r,\, \frac{a_n}{a_{n+1}}\right).$$

If $\liminf \mathcal{P}_n^r > 1$, *then the series* $\sum a_n$ *converges.*

If $\mathcal{P}_n^r \leq 1$ *for all sufficiently large* n, *then the series* $\sum a_n$ *diverges.*

Since the function $f(r, x)$ is continuous in the first variable, the above Proposition provides a "continuous" family of convergence/divergence tests depending on a parameter r and embracing the classical tests. Indeed, taking $r = 1$, Proposition 4 turns into Raabe's test. The value $r = 0$ yields Schlömilch's test and the value $r = -1$ leads us to the Knopp's version of Raabe's test (see the second paragraph of [4]).

Of course, $f(r, 1) = 0$ for every r and $f(r, x) < 0$ for $x \in (0, 1)$, $r \in \mathbb{R}$, and $f(r, x) > 0$ for $x \in (1, +\infty)$, $r \in \mathbb{R}$. It is not difficult to see that for a fixed $x \neq 1$ the function $f(r, x)$ as a function of the first variable is increasing on each of the open intervals $(-\infty, 0)$ and $(0, +\infty)$. Since $f(r, x) \to \ln x$ as $r \to 0$, it follows that $f(r, x)$ is a continuous and increasing function of r for any fixed $x \neq 1$. In particular, given $r < s$ and a sequence (c_n) of positive numbers, if $\liminf_{n\to\infty} nf(r, c_n) > 1$, then $\liminf_{n\to\infty} nf(s, c_n) > 1$. This implication can be reversed as it follows from the next lemma.

Lemma 2. *For any sequence (c_n) of positive numbers and for any $r \neq 0$ the sequences $(n \ln c_n)_{n\in\mathbb{N}}$ and $\bigl(n\frac{c_n^r - 1}{r}\bigr)_{n\in\mathbb{N}}$ have exactly the same accumulation points.*

P r o o f. It is an extension of our Lemma 1 and an easy proof can be based on the fact that for any sequence (a_n) of positive numbers and any value $g \in \overline{\mathbb{R}}$ the sequence $\bigl(n(a_n - 1)\bigr)_{n\in\mathbb{N}}$ converges to g if and only if the sequence $(n \ln a_n)_{n\in\mathbb{N}}$ converges to g (the equivalence has been proved in the course of the proof of Lemma 1). Thus, we will consider only two particular cases because the other cases can be treated analogously.

First, if $n\frac{c_n^r - 1}{r} \to g \in \mathbb{R}$ for a given $r \neq 0$, then $n(c_n^r - 1) \to rg$. Hence by the above mentioned equivalence, $n \ln c_n^r \to rg$ which implies $n \ln c_n \to g$.

Second, if $n \ln c_n \to -\infty$, then given $r > 0$, one has $n \ln c_n^r \to -\infty$, and hence $n(c_n^r - 1) \to -\infty$ which implies $n\frac{c_n^r - 1}{r} \to -\infty$. On the other hand, given $r < 0$, one has $n \ln c_n^r \to +\infty$ provided that $n \ln c_n \to -\infty$. Thus $n(c_n^r - 1) \to +\infty$ which implies that $n\frac{c_n^r - 1}{r} \to -\infty$. □

It follows from Lemma 2 that for any sequence (c_n) of positive numbers and any parameters $r, s \in \mathbb{R}$ the sequences

$$\bigl(nf(r, c_n)\bigr)_{n\in\mathbb{N}} \quad \text{and} \quad \bigl(nf(s, c_n)\bigr)_{n\in\mathbb{N}}$$

have exactly the same accumulation points. In particular

$$\liminf_{n\to\infty} nf(r,\, c_n) > 1 \quad \text{if and only if} \quad \liminf_{n\to\infty} nf(s,\, c_n) > 1$$

and this equivalence together with Raabe's test ($r = 1$) proves the convergence part of Proposition 4. Thus the strength of the convergence tests given by the associated sequences $(\mathcal{P}_n^r)_{n\in\mathbb{N}}$ does not depend on the value of the parameter, r and it is equal to the sensitiveness of the classical Raabe's test.

We are now going to prove the divergence test given by Proposition 4. Since the function $f(r,\, x)$ for $x \in (0,\, 1]$ takes values not exceeding 1 and since it is increasing in the first variable for any $x > 1$, it suffices to show that given any positive integer k the condition $\mathcal{P}_n^{-k} \le 1$ implies $n\big(\frac{a_n}{a_{n+1}} - 1\big) \le 1 + \mathrm{O}\left(\frac{1}{n}\right)$, because the latter condition (if satisfied for all sufficiently large n) implies divergence of the series $\sum a_n$ ([4, Corollary]).

If

$$n\,\frac{\left(\frac{a_n}{a_{n+1}}\right)^{-k} - 1}{-k} \le 1$$

for all large indices n, then

$$\frac{a_n}{a_{n+1}} \le \sqrt[k]{1 + \frac{k}{n-k}}$$

for those n. Now, the expansion $(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \mathrm{o}(x^2)$ yields

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \;\le\; n\left[\left(1 + \tfrac{k}{n-k}\right)^{\frac{1}{k}} - 1\right] \;=\; 1 + \tfrac{1}{n-k} + \mathrm{o}\left(\tfrac{1}{n}\right) \;=\; 1 + \mathrm{O}\left(\tfrac{1}{n}\right)$$

which completes the proof of the divergence part of Proposition 4.

Observe that since the Corollary from [4] is a consequence of the divergence test given by the condition $(\mathcal{R}^+)$, the divergence test provided by Proposition 4 might be less sensitive than the one given by $(\mathcal{R}^+)$. Actually, it is not the case. These two divergence tests are in fact equivalent.

Clearly, $\mathcal{R}_n^{(0)} = \mathcal{P}_n^1$. Further, if k is a positive integer such that

$$\mathcal{R}_n^{(k)} = (n-k)\left(\frac{a_n}{a_{n+1}} - 1\right) \le 1,$$

that is

$$\frac{a_n}{a_{n+1}} \le 1 + \frac{1}{n-k},$$

for all sufficiently large n, then

$$\left(\frac{a_n}{a_{n+1}}\right)^{-2k} \ge \left(1 + \frac{1}{n-k}\right)^{-2k},$$

and hence for $r := -2k$ one has

$$\begin{aligned}\mathcal{P}_n^r &= n\frac{\left(\frac{a_n}{a_{n+1}}\right)^r - 1}{r}\\ &= n\frac{\left(\frac{a_n}{a_{n+1}}\right)^{-2k} - 1}{-2k} \le n\frac{\left(1+\frac{1}{n-k}\right)^{-2k} - 1}{-2k}\\ &= \tfrac{n}{n-k} + \tfrac{-2k-1}{2}\cdot\tfrac{n}{(n-k)^2} + n\cdot \mathrm{o}\left(\tfrac{1}{(n-k)^2}\right)\\ &= 1 - \tfrac{n+2k^2}{2(n-k)^2} + \mathrm{o}\left(\tfrac{1}{n}\right) \le 1\end{aligned}$$

for all sufficiently large indices n.

Thus, the test given in Proposition 4 is equivalent to our extension of Raabe's test.

On the other hand, the divergence tests given by the associated sequences $(\mathcal{P}_n^r)_{n\in\mathbb{N}}$ for various values of the parameter r, are not equivalent.

Example 5. Given a positive integer k, consider a series $\sum_{n=k+2}^{\infty} a_n$ with the general term

$$a_n := \sqrt[k]{\left(1-\frac{k}{k+1}\right)\left(1-\frac{k}{k+2}\right)\cdots\left(1-\frac{k}{n-1}\right)}.$$

Then

$$\frac{a_{n+1}}{a_n} = \sqrt[k]{1-\frac{k}{n}}$$

and hence

$$\mathcal{P}_n^{-k} = n\frac{\left(\frac{a_n}{a_{n+1}}\right)^{-k} - 1}{-k} = 1$$

for all $n \ge k+2$. Thus, the series diverges by Proposition 4. However, given an $\epsilon \in (0, 1)$, one has

$$\begin{aligned}\mathcal{P}_n^{-k+\epsilon} &= n\frac{\left(\frac{a_n}{a_{n+1}}\right)^{-k+\epsilon} - 1}{-k+\epsilon}\\ &= \frac{n}{-k+\epsilon}\left[\frac{k-\epsilon}{k}\left(-\frac{k}{n}\right) + \frac{\left(1-\frac{\epsilon}{k}\right)\left(-\frac{\epsilon}{k}\right)}{2}\left(\frac{k}{n}\right)^2 + \mathrm{o}\left(\left(\frac{k}{n}\right)^2\right)\right]\\ &= 1 + \frac{\epsilon}{2n} + \mathrm{o}\left(\frac{1}{n}\right) > 1\end{aligned}$$

for all sufficiently large n. Thus, the test $(\mathcal{P}_n^r)_{n\in\mathbb{N}}$ does not disclose divergence of the series for any $r > -k$, while the test with $r = -k$ shows the divergence of the series.

REFERENCES

[1] BROMWICH, T. J. I'A.: *An Introduction to the Theory of Infinite Series* (3rd ed.), Chelsea, New York, NY, 1991.
[2] FICHTENHOLZ, G. M.: *Differential- und Integralrechnung*, 14. Aufgabe, H. Deutsch, Frankfurt am Main, 1999.
[3] KNOPP, K.: *Theory and Application of Infinite Series* (2nd. ed.), Blackie & Son, Ltd., London, 1964.
[4] PRUS-WIŚNIOWSKI, F.: *A refinement of Raabe's test*, Amer. Math. Monthly **115** (2008), 249–252.
[5] VOROBEV, N. N.: *Teoria Riadov*, Nauka, Moskva, 1975. (In Russian)

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