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DISCRETE-TIME COHEN-GROSSBERG NEURAL NETWORKS WITH TRANSMISSION DELAYS AND IMPULSES

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ABSTRACT. A discrete-time analogue is formulated for an impulsive Cohen-
-Grossberg neural network with transmission delay in a manner in which the
global exponential stability characterisitics of a unique equilibrium point of the
network are preserved. The formulation is based on extending the existing semi-
discretization method that has been implemented for computer simulations of neu-
ral networks with linear stabilizing feedback terms. The exponential convergence
in the p-norm of the analogue towards the unique equilibrium point is analysed
by exploiting an appropriate Lyapunov sequence and properties of an M-matrix.
The main result yields a Lyapunov exponent that involves the magnitude and
frequency of the impulses. One can use the result for deriving the exponential sta-
bility of non-impulsive discrete-time neural networks, and also for simulating the
exponential stability of impulsive and non-impulsive continuous-time networks.

1. Introduction

Numerical (or computer) simulations of continuous-time neural networks gov-
erned by differential equations involving transmission delays (*i.e.*, discrete or
fixed delays, time-varying delays and distributed delays) have been developed
steadily over the years [14]–[21], [23]–[34]. These simulations are produced by
discrete-time analogues formulated from semi-discretization method, in a man-
ner in which the dynamical characterisitcs—such as the sets of equilibria and
their stability characteristics—of the networks are preserved. The successful im-
plementation of the analogues is largely due to the presence of feedback terms
and denominator functions that correspond to the linear stabilizing feedback
terms of the continuous-time networks. With a similar architectural design, the
method has been applied further by A k ç a *et al.* [1], C o v a c h e v *et al.* [7],

G o p a l s a m y [11] and M o h a m a d [22] for simulating the dynamical charac-
teristics of continuous-time neural networks that are subject to impulsive state
displacements at fixed moments of time.

C o h e n- G r o s s b e r g neural network [6] and its various generalizations with
or without transmission delays and impulsive state displacements have been the
subject of intense investigation recently [2, 4, 5, 27, 31, 32]. In a Cohen-Grossberg
neural network model, the feedback terms consist of amplification and stabiliz-
ing functions which are generally nonlinear. These terms provide the model with
a special kind of generalization wherein many neural network models that are ca-
pable for content addressable memory such as additive neural networks, cellular
neural networks and bidirectional associative memory networks and also biolog-
ical models such as Lotka-Volterra models of population dynamics are included
as special cases.

Despite the generality, the nonlinear feedback terms of the networks are not
cohesive with a direct extension of the existing semi-discretization method for
obtaining the corresponding discrete-time analogues. In pursuance of the exten-
sion, a systematic approach of the semi-discretization method is given in this
paper for obtaining the corresponding analogue of a Cohen-Grossberg neural
network with distributed transmission delays and impulsive state displacements
at fixed moments of time. The dynamical characteristics of the analogue in the
general norm $\|\cdot\|_p$ ($p \geq 1$ is a real number) are subsequently investigated by ex-
ploiting an appropriate Lyapunov sequence and properties of a monotone matrix
(i.e., M-matrix), and this leads to obtaining a Lyapunov exponent expressed in
the form of a relation involving the magnitude and frequency of the impulses.
A corollary is given to demonstrate the effectiveness of the main result.

2. Impulsive continuous-time Cohen-Grossberg neural network

The impulsive continuous time neural network consists of m elementary pro-
cessing units (or neurons) whose state variables x_i ($i = \overline{1, m}$) are governed by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = a_i\big(x_i(t)\big)\Bigg[-b_i\big(x_i(t)\big) + \sum_{j=1}^{m} c_{ij} f_j\big(x_j(t - \tau_{ij})\big)$$

$$+ \sum_{j=1}^{m} d_{ij} \int_0^\infty K_{ij}(s) g_j\big(x_j(t - s)\big)\,\mathrm{d}s + I_i\Bigg], \qquad t > t_0,\ t \neq t_k, \quad (1)$$

$$\Delta x_i(t_k) = r_{ik}\big(x_i(t_k^-)\big), \qquad i = \overline{1, m}, \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ which are bounded for $s \in (-\infty, t_0]$. In (1) $a_i(x_i)$ denotes an amplification function; $b_i(x_i)$ denotes an appropriate function which supports the stabilizing (or negative) feedback term $-a_i(x_i)b_i(x_i)$ of the unit i; $f_j(x_j)$, $g_j(x_j)$ denote activation functions; the parameters c_{ij}, d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the jth unit and the ith unit; the real constant I_i represents an input signal introduced from outside the network to the ith unit; τ_{ij} are nonnegative real numbers whose presence indicates the delayed transmission of signals at time $t - \tau_{ij}$ from the jth unit to the unit i; and the delay kernels K_{ij} incorporate the fading past effects (or fading memories) of the jth unit on the ith unit; $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ denote impulsive state displacements characterized by the nonlinear functions $r_{ik}(x_i(t_k^-))$ at fixed moments of time t_k, $k \in \mathbb{N}$. Here it is assumed that $x_i(t_k^+) = \lim_{t \to t_k^+} x_i(t)$ and $x_i(t_k^-) = \lim_{t \to t_k^-} x_i(t)$, and the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies $t_0 < t_1 < t_2 < \cdots < t_k \to \infty$ as $k \to \infty$ and $\Delta t_k = t_k - t_{k-1} \geq \theta$, where $\theta > 0$ denotes the minimum time interval between successive impulses. In other words, the value $\theta > 0$ means that the impulses do not occur too often, but $\theta = \infty$ means that the network (1) is free of impulses.

The assumptions that accompany the impulsive network (1) are given as follows:

A$_1$: The amplification functions $a_i \colon \mathbb{R} \to \mathbb{R}^+$ are continuous and bounded in the sense that

$$0 < \underline{a_i} \leq a_i(u) \leq \overline{a}_i \qquad \text{for} \quad u \in \mathbb{R}, \quad i = \overline{1, m}.$$

A$_2$: The stabilizing functions $b_i \colon \mathbb{R} \to \mathbb{R}$ are continuous and monotone increasing, namely,

$$0 < \underline{b}_i \leq \frac{b_i(u) - b_i(v)}{u - v} \qquad \text{for} \quad u \neq v, \quad u, v \in \mathbb{R}, \quad i = \overline{1, m}.$$

A$_3$: The activation functions $f_j, g_j \colon \mathbb{R} \to \mathbb{R}$ with $f_j(0) = g_j(0) = 0$ are Lipschitz continuous in the sense of

$$L_j^f = \sup_{u \neq v} \left| \frac{f_j(u) - f_j(v)}{u - v} \right|, \; L_j^g = \sup_{u \neq v} \left| \frac{g_j(u) - g_j(v)}{u - v} \right| \quad \text{for} \quad u, v \in \mathbb{R}, \quad i = \overline{1, m},$$

where L_j^f, L_j^g denote positive constants.

A$_4$: For the impulse functions $r_{ik} \colon \mathbb{R} \to \mathbb{R}$ there exist positive numbers γ_{ik} such that

$$\gamma_{ik} = \sup_{u \neq v} \left| \frac{r_{ik}(u) - r_{ik}(v)}{u - v} \right| \qquad \text{for} \quad u, v \in \mathbb{R}, \quad k \in \mathbb{N}, \quad i = \overline{1, m}.$$

A$_5$: The delay kernels $K_{ij}\colon [0,\infty) \to [0,\infty)$ are piecewise continuous functions that satisfy

$$\int\limits_0^\infty K_{ij}(s)\,\mathrm{d}s = \kappa_{ij} \quad \text{and} \quad \int\limits_0^\infty K_{ij}(s)e^{\nu_0 s}\,\mathrm{d}s < \infty,$$

where κ_{ij} denote nonnegative constants and ν_0 is some positive number.

Under these assumptions and the given initial conditions, there is a unique solution of the impulsive network (1). The solution is a vector

$$x(t) = \big(x_1(t), x_2(t), \ldots, x_m(t)\big)^T$$

in which $x_i(t)$ are piecewise continuous for $t \in (t_0, \beta)$, where $\beta > t_0$ is some positive number, possibly ∞, such that the limits $x_i\big(t_k^+\big)$ and $x_i\big(t_k^-\big)$ exist and $x_i(t)$ are differentiable for $t \in (t_{k-1}, t_k) \subset (t_0, \beta)$. An equilibrium point of the impulsive network (1) is denoted by $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^m c_{ij} f_j(x_j^*) + \sum_{j=1}^m d_{ij}\kappa_{ij}g_j(x_j^*) + I_i, \qquad i = \overline{1,m}, \tag{2}$$

and satisfy the equalities

$$r_{ik}(x_i^*) = 0, \qquad k \in \mathbb{N}, \quad i = \overline{1,m}. \tag{3}$$

3. Formulation of an impulsive discrete-time analogue

For a non-impulsive neural network model with linear stabilizing feedback terms given by $-\hat{b}_i x_i(t)$, that is, the terms are obtained from (1) by letting $a_i\big(x_i(t)\big) = 1$ and $b_i\big(x_i(t)\big) = \hat{b}_i x_i(t)$, where $\hat{b}_i$ are positive constants, the application of the existing semi-discretization method [25] yields a discrete-time analogue with corresponding feedback terms given by $e^{-\hat{b}_i h}x_i(n)$, where h denotes a fixed time-step of the discretization. These terms together with the associated denominator functions given by $\hat{\psi}_i(h) = \frac{1-e^{-\hat{b}_i h}}{\hat{b}_i}$ provide the analogue with features that are conducive for preserving the dynamical characteristics of the continuous-time model, without requiring to restrict the magnitude of the time-step and introduce additional restrictions not envisaged in the continuous-time models.

Till now, the semi-discretization model has not been exploited for obtaining a discrete-time analogue of Cohen-Grossberg neural network mainly due to the nonlinearity of the feedback terms $-a_i(x_i)b_i(x_i)$. An appropriate extension of the

method is presented here. We begin by rewriting the differential system in (1) as

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\beta_i x_i(t) + \beta_i x_i(t) + a_i\big(x_i(t)\big)\Bigg[-b_i\big(x_i(t)\big) + \sum_{j=1}^{m} c_{ij} f_j\big(x_j(t-\tau_{ij})\big)$$

$$+ \sum_{j=1}^{m} d_{ij} \int_{0}^{\infty} K_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s + I_i\Bigg], \qquad i = \overline{1,m}, \ t > t_0, \ t \neq t_k, \tag{4}$$

where $\beta_i = \underline{a}_i\underline{b}_i > 0$. Let the value $h \in (0,\theta)$ be fixed, and $n_0 = \big[\frac{t_0}{h}\big]$, $n = \big[\frac{t}{h}\big]$, $\sigma_{ij} = \big[\frac{\tau_{ij}}{h}\big]$ and $\ell = \big[\frac{s}{h}\big]$, where $[r]$ denotes the greatest integer contained in the real number r. On any interval $\big[nh, (n+1)h\big)$ not containing a moment of impulse effect t_k the equation (4) can be approximated by equations with constant arguments of the form

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\beta_i x_i(t) + \beta_i x_i\left(\left[\frac{t}{h}\right]h\right)$$

$$+ a_i\left(x_i\left(\left[\frac{t}{h}\right]h\right)\right)\Bigg[-b_i\left(x_i\left(\left[\frac{t}{h}\right]h\right)\right) + \sum_{j=1}^{m} c_{ij} f_j\left(x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{\tau_{ij}}{h}\right]h\right)\right)$$

$$+ \sum_{j=1}^{m} d_{ij} \sum_{[\frac{s}{h}]=1}^{\infty} \mathcal{K}_{ij}\left(\left[\frac{s}{h}\right]h\right) g_j\left(x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\right) + I_i\Bigg], \qquad i = \overline{1,m}, \tag{5}$$

with

$$\left[\frac{t}{h}\right]h = nh \to t, \qquad \left[\frac{\tau_{ij}}{h}\right]h = \sigma_{ij}h \to \tau_{ij}, \qquad \left[\frac{s}{h}\right]h = \ell h \to s,$$

$$\sum_{[\frac{s}{h}]=1}^{\infty} \mathcal{K}_{ij}\left(\left[\frac{s}{h}\right]h\right) g_j\left(x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\right) \longrightarrow \int_{0}^{\infty} K_{ij}(s) g_j\big(x_j(t-s)\big)\,\mathrm{d}s$$

for a fixed time t as $h \to 0$. Moreover,

$\mathbf{A}_5'$: The delay kernels $\mathcal{K}_{ij} \colon \mathbb{N} \to [0,\infty)$ $(i,j = \overline{1,m})$ satisfy

$$\sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) = \kappa_{ij} \quad \text{and} \quad \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\nu\ell h} < \infty,$$

where the positive real number $\nu = \nu(h)$ for a given $h \in (0,\theta)$ is related to the positive number ν_0 (cf. assumption $\mathbf{A}_5$) by $\nu \to \nu_0$ (from below) as $h \to 0$.

For simplicity, we write system (5) as

$$
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = - \beta_i x_i(t) + \beta_i x_i(n)
$$

$$
+ a_i\big(x_i(n)\big)\Bigg[-b_i\big(x_i(n)\big)\sum_{j=1}^{m} + \sum_{j=1}^{m} c_{ij} f_j\big(x_j(n-\sigma_{ij})\big) \qquad (6)
$$

$$
+ \sum_{j=1}^{m} d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) g_j\big(x_j(n-\ell)\big) + I_i\Bigg]
$$

for $i = \overline{1,m}$, $t \in \big[nh, (n+1)h\big)$, $t \neq t_k$, $n = n_0, n_0 + 1, \ldots$, wherein the notation $w(n) \equiv w(nh)$ has been adopted. Upon integrating (6) over the interval $\big[nh, (n+1)h\big)$, one obtains a discrete analogue of the differential system in (1) given by

$$
x_i(n+1) = e^{-\beta_i h} x_i(n) + \psi_i(h)\Bigg\{\beta_i x_i(n) + a_i\big(x_i(n)\big)\Bigg[-b_i\big(x_i(n)\big) \qquad (7)
$$

$$
+ \sum_{j=1}^{m} c_{ij} f_j\big(x_j(n-\sigma_{ij})\big) + \sum_{j=1}^{m} d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) g_j\big(x_j(n-\ell)\big) + I_i\Bigg]\Bigg\}
$$

for $i = \overline{1,m}$, $n \geq n_0$, $n \neq n_k$, where $\psi_i(h) = \frac{1-e^{-\beta_i h}}{\beta_i}$ denotes the associated denominator function. Observe that $0 < \psi_i(h) < \frac{1}{\beta_i}$ for $0 < h < \theta$ and $\psi_i(h) \approx h + O(h^2)$ for small $h > 0$.

The analogue (7) is supplemented with an initial vector sequence $\phi(\ell) = \big(\phi_1(\ell), \phi_2(\ell), \ldots, \phi_m(\ell)\big)^T$ for $\ell = n_0, n_0-1, n_0-2, \ldots$ and is subject to impulsive state displacements characterized by the map

$$
x_i(n_k^+) = x_i(n_k^-) + r_{ik}\big(x_i(n_k^-)\big), \qquad i = \overline{1,m}, \quad k \in \mathbb{N}. \qquad (8)
$$

The iterations involved in (7) and (8) are described as follows: The notations n_k^+ and n_k^- denote the same integer n_k for which the values $x_i(n_k^-)$ generated by system (7) at time $n = n_k - 1$ are mapped impulsively by (8) to give the values $x_i(n_k^+)$. The mapped values $x_i(n_k^+)$ together with the past values $x_i(\ell)$ for $\ell = n_k - 1, n_k - 2, \ldots$ are then supplied back to system (7) as initial values required for the next successive iterations of $x_i(n+1)$ for $n = n_k, n_k+1, \ldots, n_{k+1}-1$. The existence of a unique solution $x(n) = \big(x_1(n), x_2(n), \ldots, x_m(n)\big)^T$ of the impulsive analogue (7), (8) for $n > n_0$ is therefore justified.

The impulsive map (8) was introduced in [22], wherein the impulse functions r_{ik} were defined linearly by $r_{ik}\big(x_i(n_k^-)\big) = -\delta_{ik}\big(x_i(n_k^-)-x_i^*\big)$ for $i = \overline{1,m}$, $k \in \mathbb{N}$ with δ_{ik} denoting real numbers. An iteration scheme similar to (7), (8) was

introduced in [8]. This map, for a given $h \in (0, \theta)$, differs from the impulsive state displacements described by difference equations of the form

$$x_i(n_k + 1) = x_i(n_k) + r_{ik}(x_i(n_k)), \qquad i = \overline{1, m}, \quad k \in \mathbb{N},$$

considered in [1, 7, 11]. However, in the limit $h \to 0$, both characterizations operate in a similar manner in accordance with the continuous-time characterization (1) at the impulse moment $t = t_k$.

4. Exponential stability of equilibria

One can verify that an equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$ of the impulsive analogue (7), (8) satisfies the same algebraic system (2) and (3) under the assumptions $\mathbf{A}_1$–$\mathbf{A}_4$, $\mathbf{A}_5'$ for any given value $h \in (0, \theta)$. To prove the global exponential stability of the point x^* we will need the following definition and lemmas.

DEFINITION 1 ([33]). A real matrix $A = (a_{ij})_{m \times m}$ is said to be an M-matrix if $a_{ij} \leq 0$ for $i, j = \overline{1, m}$, $i \neq j$ and all successive principle minors of A are positive.

LEMMA 1 ([3]). *Let $A = (a_{ij})_{m \times m}$ be a real matrix with non-positive off-diagonal elements. Then A is an M-matrix if and only if one of the following conditions holds:*

(1) *There exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ with $\xi_i > 0$ such that every component of $\xi^T A$ is positive—that is, $\sum_{i=1}^m \xi_i a_{ij} > 0$, $j = \overline{1, m}$.*

(2) *There exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)^T$ with $\xi_i > 0$ such that every component of $A\xi$ is positive—that is, $\sum_{j=1}^m a_{ij} \xi_j > 0$, $i = \overline{1, m}$.*

For more details about M-matrices the reader is referred to [9, 13].

LEMMA 2 ([10]). *A locally invertible C^0 map $H \colon \mathbb{R}^m \to \mathbb{R}^m$ is a homeomorphism of $\mathbb{R}^m$ onto itself if and only if it is proper.*

In fact, this assertion is due to H a d a m a r d [12]. A mapping is proper if the pre-image of every compact is compact. In the finite-dimensional case it suffices to show that $\|H(x)\| \to \infty$ as $\|x\| \to \infty$.

Our first task is to prove the existence and uniqueness of the solution x^* of the algebraic system (2).

THEOREM 1. *Let $p \geq 1$ be a real number, the value $h \in (0, \theta)$ be fixed and the assumptions $\mathbf{A}_1$–$\mathbf{A}_4$, $\mathbf{A}_5'$ hold. Suppose the matrix*

$$\Xi_0 = B_0 - \frac{p-1}{p}(C_0^* + D_0^*) - \frac{1}{p}(C_0 L^f + D_0 L^g) \tag{9}$$

is an M-matrix, where

$$
\begin{aligned}
B_0 &= \operatorname{diag}\left(\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m\right), \quad C_0 = (|c_{ij}|)_{m \times m}, \quad D_0 = (|d_{ij}|\kappa_{ij})_{m \times m}, \\
C_0^* &= \operatorname{diag}\left(\sum_{j=1}^m |c_{1j}|L_j^f, \sum_{j=1}^m |c_{2j}|L_j^f, \ldots, \sum_{j=1}^m |c_{mj}|L_j^f\right), \\
D_0^* &= \operatorname{diag}\left(\sum_{j=1}^m |d_{1j}|\kappa_{1j}L_j^g, \sum_{j=1}^m |d_{2j}|\kappa_{2j}L_j^g, \ldots, \sum_{j=1}^m |d_{mj}|\kappa_{mj}L_j^g\right), \\
L^f &= \operatorname{diag}\left(L_1^f, L_2^f, \ldots, L_m^f\right), \qquad L^g = \operatorname{diag}\left(L_1^g, L_2^g, \ldots, L_m^g\right).
\end{aligned}
$$

Then the algebraic system (2) *has a unique solution* $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T$.

I d e a   o f   t h e   p r o o f. Define a mapping $F \colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$
F(x) = \big(F_1(x), F_2(x), \ldots, F_m(x)\big)^T \qquad \text{for} \quad x \in \mathbb{R}^m,
$$

where

$$
F_i(x) = -b_i(x_i) + \sum_{j=1}^m c_{ij} f_j(x_j) + \sum_{j=1}^m d_{ij}\kappa_{ij} g_j(x_j) + I_i, \qquad i = \overline{1, m}.
$$

The space $\mathbb{R}^m$ is endowed with the norm $\|x\|_p = \left(\sum_{i=1}^m |x_i|^p\right)^{1/p}$. Under the assumptions $\mathbf{A}_2$, $\mathbf{A}_3$, $F(x) \in C^0$. It is known that if $F(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$, then there is a unique point $x^* = (x_1^*, x_2^*, \ldots, x_m^*)^T \in \mathbb{R}^m$ such that $F(x^*) = 0$, that is, $F_i(x^*) = 0$, $i = \overline{1, m}$.

To demonstrate the one-to-one property of F, we take arbitrary vectors $x, y \in \mathbb{R}^m$ and assume that $F(x) = F(y)$. From

$$
\begin{aligned}
b_i(x_i) - b_i(y_i) = {} & \sum_{j=1}^m c_{ij}\big(f_j(x_j) - f_j(y_j)\big) \\
& + \sum_{j=1}^m d_{ij}\kappa_{ij}\big(g_j(x_j) - g_j(y_j)\big), \qquad i = \overline{1, m},
\end{aligned}
$$

one obtains

$$
\underline{b}_i |x_i - y_i| \le \sum_{j=1}^m |c_{ij}|L_j^f|x_j - y_j| + \sum_{j=1}^m |d_{ij}|\kappa_{ij}L_j^g|x_j - y_j|, \qquad i = \overline{1, m},
$$

under the given assumptions. Multiplying both sides of the last inequality by $|x_i - y_i|^{p-1}$ and applying the inequality

$$
\eta_1^{p-1}\eta_2 \le \frac{p-1}{p}\eta_1^p + \frac{1}{p}\eta_2^p, \qquad \eta_1, \eta_2 \ge 0, \qquad p \ge 1,
$$

we derive

$$\underline{b}_i |x_i - y_i|^p \le \sum_{j=1}^m |c_{ij}| L_j^f |x_i - y_i|^{p-1} |x_j - y_j|$$

$$+ \sum_{j=1}^m |d_{ij}| \kappa_{ij} L_j^g |x_i - y_i|^{p-1} |x_j - y_j|$$

$$\le \sum_{j=1}^m |c_{ij}| L_j^f \left(\frac{p-1}{p} |x_i - y_i|^p + \frac{1}{p} |x_j - y_j|^p \right)$$

$$+ \sum_{j=1}^m |d_{ij}| \kappa_{ij} L_j^g \left(\frac{p-1}{p} |x_i - y_i|^p + \frac{1}{p} |x_j - y_j|^p \right), \qquad i = \overline{1, m},$$

which can be expressed as

$$\Xi_0 \big(|x_1 - y_1|^p, |x_2 - y_2|^p, \ldots, |x_m - y_m|^p \big)^T \le 0.$$

The assertion $x_i = y_i$, $i = \overline{1, m}$, follows by virtue of Ξ_0 being an M-matrix. Thus $F(x) = F(y)$ implies $x = y$.

Finally, we show that $\|F(x)\|_p \to \infty$ as $\|x\|_p \to \infty$. According to Lemma 2, $F(x) \in C^0$ is a homeomorphism of $\mathbb{R}^m$. Thus there is a unique point $x^* \in \mathbb{R}^m$ such that $F(x^*) = 0$. The point represents a unique solution of the algebraic system (2). $\qquad \square$

The next task is to investigate the global exponential stability characteristics of the unique equilibrium point x^* of the impulse analogue (7), (8) for a fixed time-step $h \in (0, \theta)$. Upon introducing the translations

$$u_i(n) = x_i(n) - x_i^*, \quad \varphi_i(\ell) = \phi_i(\ell) - x_i^*, \quad \tilde{a}_i\big(u_i(n)\big) = a_i\big(u_i(n) + x_i^*\big),$$

$$\tilde{b}_i\big(u_i(n)\big) = b_i\big(u_i(n) + x_i^*\big) - b_i(x_i^*), \qquad \tilde{f}_j\big(u_j(n)\big) = f_j\big(u_j(n) + x_j^*\big) - f_j(x_j^*),$$

$$\tilde{g}_j\big(u_j(n)\big) = g_j\big(u_j(n) + x_j^*\big) - g_j(x_j^*), \quad \tilde{r}_{ik}\big(u_i(n_k^-)\big) = r_{ik}\big(u_i(n_k^-) + x_i^*\big) + u_i(n_k^-),$$

one obtains

$$u_i(n+1) = e^{-\beta_i h} u_i(n) \tag{10}$$

$$+ \psi_i(h) \Bigg\{ \beta_i u_i(n) + \tilde{a}_i\big(u_i(n)\big) \Bigg[-\tilde{b}_i\big(x_i(n)\big) + \sum_{j=1}^m c_{ij} \tilde{f}_j\big(u_j(n - \sigma_{ij})\big)$$

$$+ \sum_{j=1}^m d_{ij} \sum_{\ell=1}^\infty \mathcal{K}_{ij}(\ell) \tilde{g}_j\big(x_j(n - \ell)\big) \Bigg] \Bigg\}, \quad i = \overline{1, m}, \ n \ge n_0, \ n \ne n_k,$$

$$u_i(n_k^+) = \tilde{r}_{ik}\big(u_i(n_k^-)\big), \qquad i = \overline{1, m}, \quad k \in \mathbb{N}.$$

This system inherits the assumptions $\mathbf{A_1}$–$\mathbf{A_4}$, $\mathbf{A_5'}$ given before. In particular,

$$\tilde{r}_{ik}(0) = 0 \qquad \text{for} \quad i = \overline{1, m}, \quad k \in \mathbb{N}$$

and

$$|\tilde{r}_{ik}(u)| \leq (1 + \gamma_{ik})|u| \qquad \text{for} \quad i = \overline{1, m}, \quad k \in \mathbb{N}, \quad u \in \mathbb{R}, \tag{11}$$

where γ_{ik} denote positive real numbers.

Due to the equivalence between the systems (10) and (7), (8), it suffices to examine the exponential stability characteristics of the trivial equilibrium point $u^* = 0$ of the impulsive analogue (10). The main result is given by the following theorem.

THEOREM 2. *Let $p \geq 1$ be a real number, the value $h \in (0, \theta)$ be fixed and the assumptions $\mathbf{A_1}$–$\mathbf{A_4}$, $\mathbf{A_5'}$ hold. Suppose the matrix*

$$\Xi_1 = B_1 - \frac{p-1}{p}(C_1^* + D_1^*) - \frac{1}{p}(C_1 L^f + D_1 L^g) \tag{12}$$

is an M-matrix, where

$$B_1 = \text{diag}\,(\underline{a}_1\underline{b}_1, \underline{a}_2\underline{b}_2, \ldots, \underline{a}_m\underline{b}_m), \quad C_1 = (\overline{a}_i|c_{ij}|)_{m \times m}, \quad D_1 = (\overline{a}_i|d_{ij}|\kappa_{ij})_{m \times m},$$

$$C_1^* = \text{diag}\left(\sum_{j=1}^m \overline{a}_1|c_{1j}|L_j^f, \sum_{j=1}^m \overline{a}_2|c_{2j}|L_j^f, \ldots, \sum_{j=1}^m \overline{a}_m|c_{mj}|L_j^f\right),$$

$$D_1^* = \text{diag}\left(\sum_{j=1}^m \overline{a}_1|d_{1j}|\kappa_{1j}L_j^g, \sum_{j=1}^m \overline{a}_2|d_{2j}|\kappa_{2j}L_j^g, \ldots, \sum_{j=1}^m \overline{a}_m|d_{mj}|\kappa_{mj}L_j^g\right).$$

Suppose, further, that there exist positive numbers $\Lambda > 1$ and μ satisfying $\frac{\ln \Lambda}{\theta} < \mu < \nu$ for which

$$1 + \gamma_{ik} \leq \Lambda \qquad \text{for} \quad i = \overline{1, m}, \quad k \in \mathbb{N}. \tag{13}$$

Then the impulsive analogue (10) is globally exponentially stable with a Lyapunov exponent $\mu - \frac{\ln \Lambda}{\theta}$, namely,

$$\left[\sum_{i=1}^m |u_i(n)|^p\right]^{1/p} \leq \Omega e^{-\left(\mu - \frac{\ln \Lambda}{\theta}\right)h(n-n_0)} \left[\sum_{i=1}^m \sup_{\ell \leq n_0} |\varphi_i(\ell)|^p\right]^{1/p} \tag{14}$$

for $n \geq n_0$, where $\Omega \geq 1$ denotes a constant.

S k e t c h   o f   t h e   p r o o f. The property of the M-matrix Ξ_1 lends itself towards ascertaining that Ξ_0 is an M-matrix. Thus the existence and uniqueness of the equilibrium point x^* is assured by Theorem 1.

On applying the given assumptions to (10),

$$
\begin{aligned}
|u_i(n+1)| \leq\; & e^{-\beta_i h}|u_i(n)| \\
& + \psi_i(h)\Bigg\{\sum_{j=1}^{m}\overline{a}_i|c_{ij}|L_j^f|u_j(n-n_{ij})| \\
& + \sum_{j=1}^{m}\overline{a}_i|d_{ij}|\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)L_j^g|u_j(n-\ell)|\Bigg\}, \quad i=\overline{1,m},\ n\geq n_0,\ n\neq n_k.
\end{aligned}
\tag{15}
$$

Recall that $\beta_i = \underline{a}_i\underline{b}_i > 0$ and $\psi_i(h) = \frac{1-e^{-\beta_i h}}{\beta_i} > 0$. From the M-matrix Ξ_1 and the property (1) of Lemma 1 one has that there is a positive vector $\xi = (\xi_1,\xi_2,\ldots,\xi_m)^T$ for which

$$
\begin{aligned}
& \xi_i\underline{a}_i\underline{b}_i - \frac{p-1}{p}\sum_{j=1}^{m}\xi_i\overline{a}_i|c_{ij}|L_j^f - \frac{p-1}{p}\sum_{j=1}^{m}\xi_i\overline{a}_i|d_{ij}|\kappa_{ij}L_j^g \\
& - \frac{1}{p}\sum_{j=1}^{m}\xi_j\overline{a}_j|c_{ji}|L_i^f - \frac{1}{p}\sum_{j=1}^{m}\xi_j\overline{a}_j|d_{ji}|\kappa_{ji}L_i^g > 0, \qquad i=\overline{1,m}.
\end{aligned}
$$

Let us introduce a perturbation $\mu = \mu(h)$ for a fixed value $h \in (0,\theta)$ such that $0 < \mu < \min\left\{\min_{i=\overline{1,m}}\beta_i,\ \nu\right\}$ and

$$
\begin{aligned}
& \frac{e^{(\mu-\beta_i)h}-1}{\psi_i(h)} + \frac{p-1}{p}\sum_{j=1}^{m}\overline{a}_i|c_{ij}|L_j^f e^{\mu(\sigma_{ij}+1)h} \\
& \qquad + \frac{p-1}{p}\sum_{j=1}^{m}\overline{a}_i|d_{ij}|L_j^g\sum_{\ell=1}^{\infty}\mathcal{K}_{ij}(\ell)e^{\mu(\ell+1)h} \\
& \qquad + \frac{1}{p}\sum_{j=1}^{m}\frac{\xi_j}{\xi_i}\overline{a}_j|c_{ji}|L_i^f e^{\mu(\sigma_{ji}+1)h} \\
& \qquad + \frac{1}{p}\sum_{j=1}^{m}\frac{\xi_j}{\xi_i}\overline{a}_j|d_{ji}|L_i^g\sum_{\ell=1}^{\infty}\mathcal{K}_{ji}(\ell)e^{\mu(\ell+1)h} \leq 0, \qquad i=\overline{1,m}.
\end{aligned}
\tag{16}
$$

Further, let

$$
X_i(n) = e^{\mu(n-n_0)h}|u_i(n)| \qquad \text{for} \quad i=\overline{1,m}, \quad n\in\mathbb{Z},
\tag{17}
$$

into the system (15). We obtain

$$X_i(n+1) \leq e^{(\mu-\beta_i)h} X_i(n)$$
$$+ \psi_i(h)\left\{\sum_{j=1}^{m} \overline{a}_i|c_{ij}|L_j^f e^{\mu(\sigma_{ij}+1)h} X_j(n-\sigma_{ij})\right.$$
$$\left.+ \sum_{j=1}^{m} \overline{a}_i|d_{ij}|\sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) L_j^g e^{\mu(\ell+1)h} X_j(n-\ell)\right\},$$
$$i = \overline{1,m}, \quad n \geq n_0, \quad n \neq n_k,$$

which can be rearranged as

$$\frac{\Delta X_i(n)}{\psi_i(h)} \leq \frac{e^{(\mu-\beta_i)h}-1}{\psi_i(h)} X_i(n)$$
$$+ \sum_{j=1}^{m} \overline{a}_i|c_{ij}|L_j^f e^{\mu(\sigma_{ij}+1)h} X_j(n-\sigma_{ij}) \qquad (18)$$
$$+ \sum_{j=1}^{m} \overline{a}_i|d_{ij}|\sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) L_j^g e^{\mu(\ell+1)h} X_j(n-\ell),$$
$$i = \overline{1,m}, \quad n \geq n_0, \quad n \neq n_k.$$

Define a Lyapunov sequence by

$$V(n) = \sum_{i=1}^{m} \xi_i \psi_i^{-1}(h) X_i^p(n)$$
$$+ \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i|c_{ij}|L_j^f e^{\mu(\sigma_{ij}+1)h} \sum_{\ell=n-\sigma_{ij}}^{n-1} X_j^p(\ell) \qquad (19)$$
$$+ \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i|d_{ij}|L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \sum_{r=n-\ell}^{n-1} X_j^p(r), \qquad n \in \mathbb{Z}.$$

One observes that the value

$$V(n_0) \leq \max_{i=\overline{1,m}}\left\{\xi_i \psi_i^{-1}(h) + \sum_{j=1}^{m} \xi_j \overline{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \sigma_{ji}\right.$$
$$\left.+ \sum_{j=1}^{m} \xi_j \overline{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell\right\} \sum_{i=1}^{m} \sup_{r \leq n_0} X_i^p(r)$$

is finite since $\sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell < \infty$ by virtue of the assumption $\mathbf{A}_5'$ in which $0 < \mu < \nu$ for a fixed $h \in (0,\theta)$ and $\sup_{r \leq n_0} X_i^p(r) < \infty$ due to the boundedness of the initial sequence $\varphi(r) = \phi(r) - x^*$ for $r = n_0, n_0 - 1, n_0 - 2, \ldots$

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Now we estimate the forward difference $\Delta V(n) = V(n+1) - V(n)$ along the solutions of inequality (18), thus

$$
\begin{aligned}
\Delta V(n) = {} & \sum_{i=1}^{m} \xi_i \psi_i^{-1}(h) \Delta X_i^p(n) \\
& + \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \left(X_j^p(n) - X_j^p(n-\sigma_{ij}) \right) \\
& + \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(X_j^p(n) - X_j^p(n-\ell) \right) \\
\leq {} & \sum_{i=1}^{m} \xi_i p X_i^{p-1}(n) \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i(n) + \sum_{j=1}^{m} \overline{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} X_j(n-\sigma_{ij}) \right. \\
& \left. + \sum_{j=1}^{m} \overline{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} X_j(n-\ell) \right\} \\
& + \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \left(X_j^p(n) - X_j^p(n-\sigma_{ij}) \right) \\
& + \sum_{i=1}^{m} \xi_i \sum_{j=1}^{m} \overline{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(X_j^p(n) - X_j^p(n-\ell) \right) \\
\leq {} & \sum_{i=1}^{m} \xi_i p \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i^p(n) \right. \\
& + \sum_{j=1}^{m} \overline{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \left(\frac{p-1}{p} X_i^p(n) + \frac{1}{p} X_j^p(n-\sigma_{ij}) \right) \\
& + \sum_{j=1}^{m} \overline{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(\frac{p-1}{p} X_i^p(n) + \frac{1}{p} X_j^p(n-\ell) \right) \\
& + \sum_{j=1}^{m} \overline{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \left(\frac{1}{p} X_j^p(n) - \frac{1}{p} X_j^p(n-\sigma_{ij}) \right) \\
& \left. + \sum_{j=1}^{m} \overline{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(\frac{1}{p} X_j^p(n) - \frac{1}{p} X_j^p(n-\ell) \right) \right\} \\
= {} & \sum_{i=1}^{m} \xi_i p \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} + \frac{p-1}{p} \sum_{j=1}^{m} \overline{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \right. \\
& + \frac{p-1}{p} \sum_{j=1}^{m} \overline{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} + \frac{1}{p} \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} \overline{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \\
& \left. + \frac{1}{p} \sum_{j=1}^{m} \frac{\xi_j}{\xi_i} \overline{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \right\} X_i^p(n)
\end{aligned}
$$

for $n \geq n_0$, $n \neq n_k$. By virtue of the condition (16) we find that $\Delta V(n) \leq 0$ for $n \geq n_0$, $n \neq n_k$. This means that

$$V(n) \leq V\left(n_{k-1}^+\right) \qquad \text{for} \quad n_{k-1} \leq n < n_k, \quad k \in \mathbb{N}. \qquad (20)$$

Next,

$$X_i^p\left(n_{k-1}^+\right) \leq \left(1 + \gamma_{i(k-1)}\right)^p X_i^p\left(n_{k-1}^-\right) \leq \Lambda^p X_i^p\left(n_{k-1}^-\right)$$

for $k = 1, 2, 3, \ldots$, which implies

$$V\left(n_{k-1}^+\right) \leq \Lambda^p V\left(n_{k-1}^-\right) \qquad \text{for} \quad k \in \mathbb{N}. \qquad (21)$$

From the statements (20) and (21) it follows that

$$V(n) \leq \Lambda^{(k-1)p} V(n_0) \qquad \text{for} \quad n_0 \leq n < n_k, \quad k \in \mathbb{N}. \qquad (22)$$

Since

$$h(n - n_0) \geq (k-1)\theta \qquad \text{for} \quad n_0 \leq n < n_k, \quad k \in \mathbb{N},$$

from (22) we derive

$$V(n) \leq \Lambda^{p\frac{h(n-n_0)}{\theta}} V(n_0) = e^{p\frac{\ln \Lambda}{\theta} h(n-n_0)} V(n_0) \qquad \text{for} \quad n \geq n_0.$$

This estimate together with (17) and (19) leads to

$$\sum_{i=1}^m \xi_i \psi_i^{-1} e^{\mu p h(n-n_0)} |u_i(n)|^p \leq V(n) \leq e^{p\frac{\ln \Lambda}{\theta} h(n-n_0)} V(n_0)$$

for $n \geq n_0$, from which

$$\sum_{i=1}^m |u_i(n)|^p \leq \Omega_1 e^{-p\left(\mu - \frac{\ln \Lambda}{\theta}\right) h(n-n_0)} \sum_{i=1}^m \sup_{\ell \leq n_0} |\varphi_i(\ell)|^p$$

for $n \geq n_0$, where the constant

$$\Omega_1 = \max_{i=\overline{1,m}} \left\{ \xi_i \psi_i^{-1}(h) + \sum_{j=1}^m \xi_j \overline{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \sigma_{ji} \right.$$
$$\left. + \sum_{j=1}^m \xi_j \overline{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^\infty \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell \right\} \Big/ \min_{i=\overline{1,m}} \{\xi_i \psi_i^{-1}(h)\}$$

satisfies $1 \leq \Omega_1 < \infty$. The statement (14) will follow subsequently and this completes the proof. $\qquad \square$

COROLLARY 1. *Suppose the assumptions and conditions in Theorem 2 are satisfied with* (12) *replaced by*

$$\underline{a}_i\underline{b}_i - \frac{p-1}{p}\sum_{j=1}^{m}\overline{a}_i|c_{ij}|L_j^f - \frac{p-1}{p}\sum_{j=1}^{m}\overline{a}_i|d_{ij}|\kappa_{ij}L_j^g$$

$$-\frac{1}{p}\sum_{j=1}^{m}\overline{a}_j|c_{ji}|L_i^f - \frac{1}{p}\sum_{j=1}^{m}\overline{a}_j|d_{ji}|\kappa_{ji}L_i^g > 0, \qquad i = \overline{1,m}. \qquad (23)$$

Then the impulsive analogue (10) *is globally exponentially stable in the sense of* (14).

In fact, the inequalities (23) imply that Ξ_1 defined in (12) becomes an M-matrix.

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