MOMENTS OF VECTOR-VALUED FUNCTIONS AND MEASURES

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ABSTRACT. There are investigated conditions under which the elements of a normed vector space are the moments of a vector-valued measure, and of a Bochner integrable function, respectively, both with values in a Banach space.

Introduction

If \( g \) is a real-valued function of bounded variation on the unit interval \( I \) of the real line, the numbers

\[
a_k = \int_0^1 t^k \, dg(t) \quad (k = 0, 1, \ldots)
\]

are called the moments of \( g \), \( g \) being, e.g., distribution of mass of particle, body, or electric charge. Hausdorff [H] has shown that for a sequence \( a_k \) of real numbers to be moment sequence of some non-decreasing \( g \) (the case of particular interest in some cases), it is necessary and sufficient that \( a_k \) be totally monotone. Recall that a sequence is called totally monotone if all its difference sequences are non-negative.

We shall search conditions under which, given a sequence \( a_n, n = 0, 1, 2, \ldots \) of complex numbers,

(a) there exists a measure on \([a, b]\) such that \( a_n \) are the moments of the measure, respectively,

(b) there exists a function in \( L_p([a, b]), 1 \leq p \leq \infty \), such that \( a_n \) are the moments of this function.
Then, in Section 3, \( a_n \) are taken to be elements of a normed vector space and conditions are given under which \( a_n \) are the moments of (a) a vector-valued measure, and (b) of a Bochner integrable function, respectively.

1. Preliminaries

The Hausdorff moment problem \([H, L_0, W]\) is the following: given a prescribed set of real numbers \( \{a_n\}_0^\infty \), find a bounded non-decreasing function \( u(t) \) on the closed interval \([0, 1]\) such that its moments are equal to the prescribed values, that is,

\[
\int_{[0,1]} t^n \, du(t) = a_n, \quad n = 0, 1, 2, \ldots
\]

The integral is a Riemann-Stieltjes integral. Equivalently, find a nonnegative measure \( \mu \) on borelian subsets in \([0, 1]\) with

\[
\int_{[0,1]} t^n \, d\mu(t) = a_n, \quad n = 0, 1, 2, \ldots
\]

We shall need the operator \( \nabla^k \) \( (k = 0, 1, 2, \ldots) \) defined by

\[
\nabla^0 a_n = a_n, \quad \nabla^1 a_n = a_n - a_{n+1},
\]

\[
\nabla^k a_n = a_n - \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} - \cdots + (-1)^k a_{n+k}, \quad n = 1, 2, \ldots
\]

for any sequence of real or complex numbers \( \{a_n\}_0^\infty \).

Remark that we confine ourselves to interval \([0, 1]\) for simplicity, this is, however, no limitation of generality, for any bounded interval \([a, b]\).

**Definition.** For each \( f \in L_1([0, 1]) \), the elements

\[
a_n = \int_0^1 t^n f(t) \, dt, \quad n = 1, 2, \ldots
\]

are called the moments of \( f \). For a measure \( \mu \) on \([0, 1]\), the elements

\[
a_n = \int_0^1 t^n \mu(dt), \quad n = 0, 1, 2, \ldots
\]

are called the moments of \( \mu \).
MOMENTS OF VECTOR-VALUED FUNCTIONS AND MEASURES

For a given sequence \( a = \{a_n\}_0^\infty \), put
\[
l_{k,m} = \binom{k}{m} \nabla^{k-m}a_m, \quad (k,m = 0,1,2,\ldots).\]

Let us mention that this definition is slightly different from the definition of \( \lambda_{k,m} \) given in [W], but we obtain similar results.

In order to remain in some reasonable frame in the following we shall suppose satisfied the

**Condition A.** There is a finite positive constant \( L \) such that
\[
\sum_{k=0}^{N} |l_{N,k}| < L, \quad N = 0,1,2,\ldots
\]

Following [W] p. 107 for the sequence \( \{a_n\} \), define an operator \( L_N(t) \) by
\[
L_N(t) = (1+N)l_{N,\lfloor Nt \rfloor}, \quad N = 1,2,\ldots
\]

[\( \lfloor Nt \rfloor \)] means the largest integer contained in \( Nt \).

We have [W] p. 110 for \( p > 1 \),
\[
\int_0^1 |L_N(t)|^p \, dt = \frac{(N + 1)^p}{N} \sum_{m=0}^{N-1} |l_{N,m}|^p < 2L, \quad N = 0,1,2,\ldots
\]

If \( f \) is in \( L_p([0,1]) \) and \( a_n \) are its moments, then [W] p. 111
\[
\lim_{N \to \infty} \int_0^1 t^n L_N(t) \, dt = a_n, \quad n = 0,1,\ldots \quad (1)
\]

Hence for every continuous, and more generally, for every bounded measurable function \( g \) there exists
\[
\lim_{N \to \infty} \int_0^1 g(t)L_N(t) \, dt = A.
\]

If \( a_n \) are moments of \( f \), then for all \( t \in [0,1] \), we have
\[
\int_0^1 |L_N(s)| \, ds \leq \int_0^1 K_N(t,s)|f(s)| \, ds, \quad (2)
\]

where
\[
K_N(t,s) = (1+N)\binom{N}{\lfloor Nt \rfloor} s^{\lfloor Nt \rfloor}(1-s)^{N-\lfloor Nt \rfloor}, \quad (N = 1,2,\ldots; 0 \leq s, t \leq 1).
\]

It is easy to prove the following lemma.
Lemma 1. For every $t \in [0, 1]$

$$\int_0^1 |K_N(t, s)| \, ds \leq \frac{N+1}{N} \sum_{m=0}^{N-1} \int_0^1 \left( \frac{N}{m} \right) s^m (1-s)^{N-m} \, ds \leq M < \infty,$$

for $N = 1, 2, \ldots$

2. Scalar-valued moments

At least the part of the following theorem is already known see [W, pp. 100–114], where the $L_p$ case ($1 < p \leq \infty$) and (iii)–(iii)' are explicitly given. Recall that $L_p$, $(1 \leq p \leq \infty)$, denotes the space of all measurable functions $f$ such that $|f|^p$ is integrable on $[0, 1]$.

Theorem 2. Given a sequence $a_n, n = 0, 1, 2, \ldots$, of complex numbers, there exists

(i) a function $f \in L_1$ such that $a_n$ are the moments of $f$;
(ii) a function $f \in L_p, 1 < p \leq \infty$, such that $a_n$ are the moments of $f$;
(iii) a complex, regular Borel measure $\mu$ such that $a_n$ are the moments of $\mu$;
(iv) a nonnegative regular Borel measure $\mu$ such that $a_n$ are the moments of $\mu$;
(v) a continuous function $f$ such that $a_n$ are the moments of $f$

if and only if the functions $L_k(t)\{a_n\}$

(i)' converge in the $L_1$-norm;
(ii)' are bounded in the $L_p$-norm;
(iii)' are bounded in the $L_1$-norm;
(iv)' are nonnegative;
(v)' converge uniformly.

For completeness, we give a little different and more compact proof of all parts using the following lemmas.

Lemma 3. For each $N$, define the linear map $T_N: L_p \rightarrow L_p$ (for all $p, 1 \leq p \leq \infty$) by

$$(T_N g)(t) = \int_0^1 K_N(t, s)g(s) \, ds, \quad g \in L_p.$$ 

Then each $T_N$ is continuous for all $p, 1 \leq p \leq \infty$, and there exists a constant $D < \infty$ such that $||T_N|| \leq D$ for all $N$. 

202
Moments of Vector-Valued Functions and Measures

Proof. When \( p = 1 \), we have, by Lemma 1, for each \( g \in L_1 \),

\[
\sup_N \|T_N g\| = \sup_N \int_0^1 \left| \int_0^1 K_N(t, s) g(s) \, ds \right| \, dt \\
\leq \sup_N \int_0^1 \left( \|g(s)\| \int_0^1 |K_N(t, s)| \, dt \right) \, ds \leq M \|g\|.
\]

Therefore \( \sup \|T_N\| = D_1 < \infty \). When \( p = \infty \), for each \( N \),

\[
\|T_N\| = \sup_{\|g\|=1} \text{ess sup}_t \int_0^1 K_N(t, s) g(s) \, ds \\
\leq \sup_{\|g\|=1} \text{ess sup}_t \int_0^1 |K_N(t, s)||g(s)| \, ds = D_2 < \infty.
\]

Putting \( D = \max(D_1, D_2) \), we have that, for each \( N \), \( T_N \) is a linear mapping of each \( L_p \) space into itself \( (1 \leq p \leq \infty) \) and \( T_N \) is continuous for \( p = 1 \) and \( p = \infty \) with norm at most \( D \). The result now follows from the Riesz convexity theorem \[E\] p. 526.

Lemma 4. If \( f \) is in \( L_p([0,1]) \), \( 1 \leq p < \infty \), then \( T_N f \) converges to \( f \) in the \( L_p \)-norm.

Proof. Let \( \epsilon > 0 \) be given. Then there exists \( \psi \in C([0,1]) \) such that \( \|f - \psi\|_p < \epsilon \). So, for all sufficiently large \( N \),

\[
\|T_N f - f\| \leq \|T_N (f - \psi)\| + \|T_N \psi - \psi\| + \|\psi - f\| \\
\leq \|T_N\| \|f - \psi\| + \epsilon + \epsilon \\
< D \epsilon + 2 \epsilon.
\]

Lemma 5. If \( f \) is in \( L_\infty([0,1]) \), then \( T_N f \) is a bounded subset of \( L_\infty([0,1]) \) and so weak-star compact and (some subsequence of) \( T_N f \) converges to \( f \) in the weak-star topology of \( L_\infty \).

Proof. For all \( f \) in \( L_\infty([0,1]) \) the sequence \( (T_N f) \) is bounded in \( L_\infty([0,1]) \), and so weak-star compact.

Proof of Theorem 2. Suppose that \( f \) is in \( L_1 \) and \( a_n \) are the moments of \( f \). Then \( L_N = T_N f \) and so (i)' follows from Lemma 4. Also if \( f \) is in \( L_p \), (ii)' follows from Lemma 4 \((1 < p < \infty)\) and Lemma 5 \((p = \infty)\).
If the $a_n$ are the moments of a complex measure $\mu$, $a_n = \int_0^1 s^n \mu(ds)$ then, by Lemma 1

$$\|L_N\|_1 = \int_0^1 |L_N(t)| \, dt = \frac{N + 1}{N} \sum_{m=0}^{N-1} |\lambda_{N,m}|$$

$$\leq \frac{N + 1}{N} \sum_{m=0}^{N} \binom{N}{m} \int_0^1 s^m (1-s)^{N-m} |\mu| (ds)$$

$$\leq M|\mu|([0,1]),$$

which shows that (iii)’ follows from (iii).

We now show the converse implications. First, observe that, if $n$ is fixed and $N > n$, then

$$\int_0^1 t^n L_N(t) \, dt \to a_n, \quad (n = 0, 1, 2, \ldots), \ N \to \infty. \quad (3)$$

If the $L_N$ converge in the $L_1$-norm, they converge to an integrable function $f$. So

$$\left| \int_0^1 (f(t) - L_N(t)) t^n \, dt \right| \leq \int_0^1 |f(t) - L_N(t)||t^n| \, dt$$

$$\leq \sup_t |t^n| \|f - L_N\|_1.$$ 

Therefore,

$$\lim_{N \to \infty} \int_0^1 f(t) t^n \, dt - \int_0^1 L_N(t) t^n \, dt = 0.$$ 

Hence, by (3),

$$\int_0^1 f(t) t^n \, dt = a_n.$$

Suppose that $L_N$ are bounded in $L_p$-norm, $1 < p \leq \infty$. We may as well assume that $\|L_N\|_p \leq 1$, $N = 1, 2, \ldots$. Then the $L_N$ lie in the unit ball of the conjugate space of $L_q$ (where $p^{-1} + q^{-1} = 1$). Since this unit ball is weak-star compact, there is a function $f$ in $L_p$ with $\|f\|_p \leq 1$ such that every weak-star neighbourhood of $f$ contains $L_N$ for infinitely many values of $N$. In other words,
MOMENTS OF VECTOR-VALUED FUNCTIONS AND MEASURES

given any \( g \) in \( L_q \), the numbers

\[
\int_0^1 L_N(t)g(t)\,dt
\]

are near \( \int_0^1 f(t)g(t)\,dt \) for infinitely many values of \( N \). But each \( t^n \) is in \( L_q \) and so, by (3)

\[
\int_0^1 f(t)t^n\,dt = a_n.
\]

Suppose that

\[
\int_0^1 |L_N(t)|\,dt \leq \alpha
\]

for all \( N \). Define, for each \( N \), the scalar-valued map \( \Phi_N \) on \( C([0,1]) \) by

\[
\Phi_N(\psi) = \int_0^1 \psi(t)L_N(t)\,dt, \quad \psi \in C([0,1]).
\]

Then, for all \( N \), \( ||\Phi_N|| \leq \alpha \). That is, all the \( \Phi_N \) are in the (weak-star compact) ball of radius \( \alpha \) of the dual space of \( C([0,1]) \). Hence there exists a regular Borel measure \( \mu \) such that, for all \( \epsilon > 0 \) and all \( \psi \in C([0,1]) \), there exists \( N \) such that

\[
\left| \int_0^1 \psi(t)L_N(t)\,dt - \int_0^1 \psi(t)\mu(dt) \right| < \epsilon.
\]

(4)

Since each \( t^n \) is in \( C([0,1]) \), we have, by (3),

\[
\int_0^1 t^n\mu(dt) = a_n.
\]

\[ \square \]

Remark. By using \( \lim_{N \to \infty} ||L_N - f||_1 = 0 \), in conjunction with results in general integration theory referring to criteria for weak compactness for subsets of \( L_1 \) (see [DS, pp. 294–295] and [E, pp. 274–276]) the following characterization of \( L_1 \) moments can be established.

Recalling an operator \( L_N(t) \) defined by

\[
L_N(t) = (N + 1)l_{N,[Nt]} \quad (k = 1, 2, \ldots; 0 \leq t \leq 1)
\]

the following four conditions are equivalent:
(a) a subsequence \( (L_{N_k})_{k=1}^\infty \) exists such that the set functions \( E \to \int_E L_{N_k}(t) \, dt \) \((k = 1, 2, \ldots)\) are uniformly (or equi-) absolutely continuous, that is, to each \( e > 0 \) there corresponds \( d > 0 \) such that

\[
\sup_k \left| \int_E L_{N_k}(t) \, dt \right| \leq e
\]

for all measurable sets \( E \) satisfying \( m(E) < d \);

(b) some subsequence \( (L_{N_k})_{k=1}^\infty \) converges weakly in \( L^1 \);

(c) some subsequence \( (L_{N_k})_{k=1}^\infty \) converges in \( L^1 \);

(d) \( (a_n)_{n \in \mathbb{Z}} \in Mo(Z) \), i.e., \( a = Mo f \in L^1 \), \( Mo \) denotes the moment.

If any one of these conditions holds, and if \( f = \lim_k L_{N_k} \) weakly or strongly in \( L^1 \), then \( a = Mo f \), \( L_N = L_N f \), and so \( \lim_{N \to \infty} \|L_N - f\|_1 = 0 \).

**Remark.** The uniform absolute continuity of the set functions can be expressed as uniform absolute continuity of the point functions

\[
F_k(x) = \int_0^x L_{N_k}(t) \, dt
\]

the condition being precisely that to each \( e > 0 \) there corresponds a number \( d = d(e) > 0 \) such that

\[
\sup_k \sum_{j=1}^r |F_k(b_j) - F_k(a_j)| \leq e
\]

for any finite sequence \( ((a_j, b_j))_{j=1}^r \) of disjoint open intervals \((a_j, b_j)\) the sum of whose lengths does not exceed \( d \). In this connection see, for example, [HS] Theorem (19.53) and its proof.

### 3. Vector-valued moments

Let \( X \) be a Banach space. Note that at some places, e.g., in Condition A the absolute value will be replaced by norm in \( X \).

**Theorem 6.** Given a sequence \( a_n, n = 0, 1, 2, \ldots \) of elements of \( X \), there exists a regular measure \( \mu: B([0, 1]) \to X \) of finite total variation such that \( a_n \) are the moments of \( \mu \) if and only if there exists a constant \( 0 < H < \infty \) such that

\[
\int_0^1 \|L_N(t)\| \, dt \leq H, \quad N = 1, 2, \ldots
\]
MOMENTS OF VECTOR-VALUED FUNCTIONS AND MEASURES

Proof. Suppose that such a measure exists. Then, for each $N$, by (1)

$$
\int_0^1 \left\| L_N \right\| \, dt = \int_0^1 \left\| \int_0^1 K_N(t, s) \mu(ds) \right\| \, dt
$$

$$
\leq \int_0^1 \int_0^1 |K_N(t, s)| \, dt |\mu|(ds)
$$

$$
\leq M|\mu|([0, 1]).
$$

Conversely, suppose that $\int_0^1 \left\| L_N \right\| \, dt \leq H$ for all $N$. If we define

$$
\Phi_N(\psi) = \int_0^1 \psi(t) L_N(t) \, dt, \quad \psi \in C([0, 1]),
$$

then $\left\| \Phi_N \right\| \leq H$ for all $N$. Since, for each $n$, $\lim_N \Phi_N(t^n) = a_n$, we have that $\lim_N \Phi_N(\psi)$ exists for all $\psi$ which are linear combinations of the $t^n$ and so, as $\left\| \Phi_N \right\| \leq H$ for all $N$, we conclude that $\lim_N \Phi_N(\psi)$ exists for all $\psi \in C([0, 1])$. Denote this limit by $\Phi$. To obtain our required measure, we use the following lemma (see [D, § 19, pp. 380, 383]). For each subset $A$ of $[0, 1]$, let $C([0, 1], A)$ denote the space of continuous functions on $[0, 1]$ vanishing outside $A$. If $F: C([0, 1]) \to X$ is a linear mapping, define for each $A$

$$
|||F_A||| = \sup \sum \left\| F(\psi_i) \right\|,
$$

where supremum is over all finite families $\psi_i$ in $C([0, 1])$ with $\sum |\psi_i(t)| \leq \chi_A(t)$ for all $t \in [0, 1]$.

Lemma. If $F: C([0, 1]) \to X$ is a linear mapping, then there exists a regular measure $\mu: B([0, 1]) \to X$ with finite variation such that

$$
F(\psi) = \int_0^1 \psi(t) \mu(dt), \quad \psi \in C([0, 1]),
$$

if and only if $|||F_A||| < \infty$ for all $A \in B([0, 1])$.

Let $A$ be in $B([0, 1])$ and let $\{\psi_i; i = 1, 2, \ldots, n\}$ be a finite family of functions in $C([0, 1], A)$ with

$$
\sum_{i=1}^n |\psi_i(t)| \leq \chi_A(t), \quad t \in [0, 1].
$$

Then for each $N$, 

$$
\sum_{i=1}^n |\psi_i(t)| \leq \chi_A(t), \quad t \in [0, 1].
$$
So for each \( N \),
\[
\sum_{i=1}^{n} ||\Phi_N(\psi_i)|| = \sum_{i=1}^{n} \left| \left| \int_0^1 \psi_i(t) L_N(t) \, dt \right| \right|
\leq \sum_{i=1}^{n} \int_0^1 |\psi_i(t)||L_N(t)|| \, dt
\leq \int_0^1 \chi_A(t)||L_N(t)|| \, dt \leq H.
\]

Hence \( \sum_{i=1}^{n} ||\Phi(\psi_i)|| \leq H \) and so \( ||\Phi_A|| \leq H \). Therefore, there exists a regular measure \( \mu : B([0,1]) \to X \) with finite variation such that
\[
\Phi(\psi) = \int_0^1 \psi(t) \mu(dt), \quad \psi \in C([0,1]).
\]

But each \( t^n \) is in \( C([0,1]) \) and so,
\[
a_n = \int_0^1 t^n \mu(dt).
\]

and thus theorem is proved. \( \square \)

Recall a definition of a Bochner integrable function. If \( f : [a,b] \to X \) is simple, i.e., \( f(s) = \sum_{i=1}^{n} \chi_{E_i}(s)x_i \), where \( \chi_E \) denotes the indicator function of the set \( E \subset [a,b] \), \( x_i \in X \), then for any \( E \in B([a,b]) \)
\[
\int_E f \, d\lambda = \sum_{i=1}^{n} \lambda(E \cap E_i)x_i,
\]
where \( \lambda \) is a probability measure on \([a,b] \). Such functions are \( \lambda \)-measurable. Any function \( f : [a,b] \to X \) which is the \( \lambda \)-almost everywhere limit of a sequence of simple functions is (called) \( \lambda \)-measurable.

A \( \lambda \)-measurable function \( f : [a,b] \to X \) is called Bochner integrable if there exists a sequence of simple functions \( (f_n) \) such that
\[
\lim_{[a,b]} \int ||f_n(s) - f(s)|| \, d\lambda(s) = 0.
\]
In this case \( \int_E f \, d\lambda \) is defined for each measurable set \( E \) in \([a, b]\) by

\[
\int_E f \, d\lambda = \lim_n \int_E f_n \, d\lambda.
\]

A \( \lambda \)-measurable function \( f: [a, b] \to X \) is Bochner integrable if and only if \( \int_{[a, b]} ||f|| \, d\lambda < \infty \).

The following theorem concerns moments of Bochner integrable function.

**Theorem 7.** Given a sequence \( a_n, n = 0, 1, 2, \ldots \), of elements of \( X \), there exists an \( X \)-valued Bochner integrable function \( f \) on \([0, 1]\) such that \( a_n \) are the moments of \( f \) if and only if

\[
\lim_{N,J \to \infty} \int_0^1 ||L_N(t) - L_J(t)|| \, dt = 0.
\]

**Proof.** Suppose that \( f \) is Bochner integrable and \( a_n \) are the moments of \( f \). Let \( \{A_i\}_{i=1}^n \) be a finite family in \( B([0, 1]) \) and \( \{\beta^n_i\} \) a finite family of vectors in \( X \) and define \( g: [0, 1] \to X \) by

\[
g(t) = \sum_{i=1}^n \beta_i \chi_i(t).
\]

Then

\[
\int_0^1 \left| \int_0^1 K_N(t, s) g(s) \, ds - g(t) \right| \, dt \\
\leq \sum_{i=1}^n \left( ||\beta_i|| \int_0^1 \left| \int_0^1 K_N(t, s) \chi_{A_i}(s) \, ds - \chi_{A_i}(t) \right| \, dt \right)
\]

which, by Lemma 4, tends to 0 as \( N \to \infty \). Therefore, since the set of all such \( g \) is dense in the space of all Bochner integrable functions and as

\[
L_N(t) = \int_0^1 K_N(t, s) f(s) \, ds,
\]

we have

\[
\lim_N \int_0^1 ||L_N(t) - f(t)|| \, dt = 0.
\]

Conversely, suppose that the sequence \( L_N \) is Cauchy in the norm of the space of all Bochner integrable functions. Since this space is Banach, \( L_N \) converges in the Bochner space norm to a Bochner integrable function \( f \).
So, for each $n$, 
\[ \left\| \int_{0}^{1} (f(t) - L_N(t)) t^n \, dt \right\| \leq \int_{0}^{1} \|f(t) - L_N(t)\| |t^n| \, dt \leq \sup_t |t^n| ||L_N - f||_B \to 0, \quad N \to \infty. \]

Thus, by (3), 
\[ a_n = \int_{0}^{1} t^n f(t) \, dt. \]

So the theorem is proved. \hfill \Box

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Received December 17, 2008

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