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MOMENTS OF VECTOR-VALUED FUNCTIONS AND MEASURES

Miloslav Duchoň — Camille Debiève

ABSTRACT. There are investigated conditions under which the elements of a normed vector space are the moments of a vector-valued measure, and of a Bochner integrable function, respectively, both with values in a Banach space.

Introduction

If g is a real-valued function of bounded variation on the unit interval I of the real line, the numbers

$$a_k = \int_0^1 t^k \mathrm{d}g(t) \qquad (k = 0, 1, \dots)$$

are called the moments of g, g being, e.g., distribution of mass of particle, body, or electric charge. Hausdorff [H] has shown that for a sequence a_k of real numbers to be moment sequence of some non-decreasing g (the case of particular interest in some cases), it is necessary and sufficient that a_k be totally monotone. Recall that a sequence is called totally monotone if all its difference sequences are non-negative.

We shall search conditions under which, given a sequence a_n, $n = 0, 1, 2, \dots$ of complex numbers,

(a) there exists a measure on $[a, b]$ such that a_n are the moments of the measure, respectively,

(b) there exists a function in $L_p\big([a, b]\big), 1 \leq p \leq \infty$, such that a_n are the moments of this function.

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Then, in Section 3, a_n are taken to be elements of a normed vector space and conditions are given under which a_n are the moments of (a) a vector-valued measure, and (b) of a Bochner integrable function, respectively.

1. Preliminaries

The Hausdorff moment problem [H, Lo, W] is the following: given a prescribed set of real numbers $\{a_n\}_0^\infty$, find a bounded non-decreasing function $u(t)$ on the closed interval $[0,1]$ such that its moments are equal to the prescribed values, that is,

$$\int\limits_{[0,1]} t^n \mathrm{d}u(t) = a_n, \qquad n = 0,1,2,\dots$$

The integral is a Riemann-Stieltjes integral. Equivalently, find a nonnegative measure μ on borelian subsets in $[0,1]$ with

$$\int\limits_{[0,1]} t^n \mathrm{d}\mu(t) = a_n, \qquad n = 0,1,2,\dots$$

We shall need the operator ∇^k $(k = 0,1,2,\dots)$ defined by

$$\nabla^0 a_n = a_n, \qquad \nabla^1 a_n = a_n - a_{n+1},$$

$$\nabla^k a_n = a_n - \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} - \dots + (-1)^k a_{n+k}, \qquad n = 1,2,\dots$$

for any sequence of real or complex numbers $\{a_n\}_0^\infty$.

Remark that we confine ourselves to interval $[0,1]$ for simplicity, this is, however, no limitation of generality, for any bounded interval $[a,b]$.

Definition. For each $f \in L_1\big([0,1]\big)$, the elements

$$a_n = \int\limits_0^1 t^n f(t)\,\mathrm{d}t, \qquad n = 1,2,\dots$$

are called the moments of f. For a measure μ on $[0,1]$, the elements

$$a_n = \int\limits_0^1 t^n \mu(\mathrm{d}t), \qquad n = 0,1,2,\dots$$

are called the moments of μ.

For a given sequence $a = \{a_n\}_0^\infty$, put

$$l_{k,m} = \binom{k}{m} \nabla^{k-m} a_m, \qquad (k, m = 0, 1, 2, \dots).$$

Let us mention that this definition is sligthly different from the definition of $\lambda_{k,m}$ given in [W], but we obtain similar results.

In order to remain in some reasonable frame in the following we shall suppose satisfied the

Condition A. There is a finite positive constant L such that

$$\sum_{k=0}^{N} |l_{N,k}| < L, \qquad N = 0, 1, 2, \dots$$

Following [W, p. 107] for the sequence $\{a_n\}$, define an operator $L_N(t)$ by

$$L_N(t) = (1+N) l_{N,([Nt])}, \qquad N = 1, 2, \dots$$

$[Nt]$ means the largest integer contained in Nt.

We have [W, p. 110] for $p > 1$,

$$\int_0^1 |L_N(t)|^p \, \mathrm{d}t = \frac{(N+1)^p}{N} \sum_{m=0}^{N-1} |l_{N,m}|^p < 2L, \qquad N = 0, 1, 2, \dots$$

If f is in $L_p([0,1])$ and a_n are its moments, then [W, p. 111]

$$\lim_{N \to \infty} \int_0^1 t^n L_N(t) \, \mathrm{d}t = a_n, \qquad n = 0, 1, \dots \tag{1}$$

Hence for every continuous, and more generally, for every bounded measurable function g there exists

$$\lim_{N \to \infty} \int_0^1 g(t) L_N(t) \, \mathrm{d}t = A\,.$$

If a_n are moments of f, then for all $t \in [0,1]$, we have

$$\int_0^1 |L_N(s)| \, \mathrm{d}s \le \int_0^1 K_N(t,s) |f(s)| \, \mathrm{d}s, \tag{2}$$

where

$$K_N(t,s) = (1+N) \binom{N}{[Nt]} s^{[Nt]} (1-s)^{N-[Nt]}, \quad (N = 1, 2 \dots;\ 0 \le s,\ t \le 1).$$

It is easy to prove the following lemma.

Lemma 1. *For every $t \in [0,1]$*

$$\int_0^1 |K_N(t,s)|\,\mathrm{d}s \le \frac{N+1}{N} \sum_{m=0}^{N-1} \int_0^1 \binom{N}{m} s^m (1-s)^{N-m} \mathrm{d}s \le M < \infty,$$

for $N = 1, 2, \ldots$

2. Scalar-valued moments

At least the part of the following theorem is already known see [W, pp. 100–114], where the L_p case $(1 < p \le \infty)$ and (iii)–(iii)' are explicitly given). Recall that L_p, $(1 \le p \le \infty)$, denotes the space of all measurable functions f such that $|f|^p$ is integrable on $[0,1]$.

Theorem 2. *Given a sequence $a_n, n = 0, 1, 2, \ldots,$ of complex numbers, there exists*

(i) *a function $f \in L_1$ such that a_n are the moments of f;*

(ii) *a function $f \in L_p, 1 < p \le \infty$, such that a_n are the moments of f;*

(iii) *a complex, regular Borel measure μ such that a_n are the moments of μ;*

(iv) *a nonnegative regular Borel measure μ such that a_n are the moments of μ;*

(v) *a continuous function f such that a_n are the moments of f if and only if the functions $L_k(t)\{a_n\}$*

(i)' *converge in the L_1-norm;*

(ii)' *are bounded in the L_p-norm;*

(iii)' *are bounded in the L_1-norm;*

(iv)' *are nonnegative;*

(v)' *converge uniformly.*

For completeness, we give a little different and more compact proof of all parts using the following lemmas.

Lemma 3. *For each N, define the linear map $T_N \colon L_p \to L_p$ (for all p, $1 \le p \le \infty$) by*

$$(T_N g)(t) = \int_0^1 K_N(t,s) g(s)\,\mathrm{d}s, \qquad g \in L_p\,.$$

Then each T_N is continuous for all $p, 1 \le p \le \infty$, and there exists a constant $D < \infty$ such that $||T_N|| \le D$ for all N.

P r o o f. When $p=1$, we have, by Lemma l, for each $g \in L_1$,

$$\begin{aligned} \sup_N ||T_N g|| &= \sup_N \int_0^1 \left| \int_0^1 K_N(t,s) g(s)\,\mathrm{d}s \right| \mathrm{d}t \\ &\le \sup_N \int_0^1 \left(|g(s)| \int_0^1 |K_N(t,s)| \mathrm{d}t \right) \mathrm{d}s \le M||g||. \end{aligned}$$

Therefore $\sup ||T_N|| = D_1 < \infty$. When $p=\infty$, for each N,

$$\begin{aligned} ||T_N|| &= \sup_{||g||=1} \operatorname{ess\,sup}_t \left| \int_0^1 K_N(t,s) g(s)\,\mathrm{d}s \right| \\ &\le \sup_{||g||=1} \operatorname{ess\,sup}_t \int_0^1 |K_N(t,s)||g(s)|\,\mathrm{d}s = D_2 < \infty. \end{aligned}$$

Putting $D = \max(D_1, D_2)$, we have that, for each N, T_N is a linear mapping of each L_p space into itself ($1 \le p \le \infty$) and T_N is continuous for $p=1$ and $p=\infty$ with norm at most D. The result now follows from the Riesz convexity theorem [E, p. 526]. □

Lemma 4. *If f is in $L_p\big([0,1]\big)$, $1 \le p < \infty$, then $T_N f$ converges to f in the L_p-norm.*

P r o o f. Let $\epsilon > 0$ be given. Then there exists $\psi \in C([0,1])$ such that $||f-\psi||_p < \epsilon$. So, for all sufficiently large N,

$$\begin{aligned} ||T_N f - f|| &\le ||T_N(f-\psi)|| + ||T_N\psi - \psi|| + ||\psi - f|| \\ &\le ||T_N|| ||f-\psi|| + \epsilon + \epsilon \\ &< D\epsilon + 2\epsilon. \end{aligned}$$

□

Lemma 5. *If f is in $L_\infty\big([0,1]\big)$, then $T_N f$ is a bounded subset of $L_\infty\big([0,1]\big)$ and so weak-star compact and (some subsequence of) $T_N f$ converges to f in the weak-star topology of L_∞.*

P r o o f. For all f in $L_\infty\big([0,1]\big)$ the sequence $(T_N f)$ is bounded in $L_\infty\big([0,1]\big)$, and so weak-star compact. □

P r o o f o f T h e o r e m 2. Suppose that f is in L_1 and a_n are the moments of f. Then $L_N = T_N f$ and so (i)' follows from Lemma 4. Also if f is in L_p , (ii)' follows from Lemma 4 ($1 < p < \infty$) and Lemma 5 ($p = \infty$).

If the a_n are the moments of a complex measure μ, $a_n = \int_0^1 s^n \mu(\mathrm{d}s)$ then, by Lemma 1

$$\begin{aligned} \|L_N\|_1 = \int_0^1 |L_N(t)| \,\mathrm{d}t &= \frac{N+1}{N} \sum_{m=0}^{N-1} |\lambda_{N,m}| \\ &\le \frac{N+1}{N} \sum_{m=0}^{N} \binom{N}{m} \int_0^1 s^m (1-s)^{N-m} |\mu|(\mathrm{d}s) \\ &\le M|\mu|\big([0,1]\big), \end{aligned}$$

which shows that (iii)' follows from (iii).

We now show the converse implications. First, observe that, if n is fixed and $N > n$, then

$$\int_0^1 t^n L_N(t) \,\mathrm{d}t \to a_n, \qquad (n = 0, 1, 2, \dots),\ N \to \infty. \tag{3}$$

If the L_N converge in the L_1-norm, they converge to an integrab1e function f. So

$$\begin{aligned} \left| \int_0^1 \big(f(t) - L_N(t)\big) t^n \,\mathrm{d}t \right| &\le \int_0^1 |f(t) - L_N(t)| |t^n| \,\mathrm{d}t \\ &\le \sup_t |t^n| \, \|f - L_N\|_1. \end{aligned}$$

Therefore,

$$\lim_N \left| \int_0^1 f(t) t^n \,\mathrm{d}t - \int_0^1 L_N(t) t^n \,\mathrm{d}t \right| = 0.$$

Hence, by (3),

$$\int_0^1 f(t) t^n \,\mathrm{d}t = a_n.$$

Suppose that L_N are bounded in L_p-norm, $1 < p \le \infty$. We may as well assume that $\|L_N\|_p \le 1$, $N = 1, 2, \dots$ Then the L_N lie in the unit ball of the conjugate space of L_q (where $p^{-1} + q^{-1} = 1$). Since this unit ball is weak-star compact, there is a function f in L_p with $\|f\|_p \le 1$ such that every weak-star neighbourhood of f contains L_N for infinitely many values of N. In other words, given any g in L_q, the numbers

$$\int_0^1 L_N(t)g(t)\,\mathrm{d}t$$

are near $\int_0^1 f(t)g(t)\,\mathrm{d}t$ for infinitely many values of N. But each t^n is in L_q and so, by (3)

$$\int_0^1 f(t)t^n\,\mathrm{d}t = a_n.$$

Suppose that

$$\int_0^1 |L_N(t)|\,\mathrm{d}t \le \alpha$$

for all N. Define, for each N, the scalar-valued map Φ_N on $C\big([0,1]\big)$ by

$$\Phi_N(\psi) = \int_0^1 \psi(t)L_N(t)\,\mathrm{d}t, \qquad \psi \in C\big([0,1]\big).$$

Then, for all N, $||\Phi_N|| \le \alpha$. That is, all the Φ_N are in the (weak-star compact) ball of radius α of the dual space of $C\big([0,1]\big)$. Hence there exists a regular Borel measure μ such that, for all $\epsilon > 0$ and all $\psi \in C\big([0,1]\big)$, there exists N such that

$$\left| \int_0^1 \psi(t)L_N(t)\,\mathrm{d}t - \int_0^1 \psi(t)\mu(\mathrm{d}t) \right| < \epsilon. \tag{4}$$

Since each t^n is in $C\big([0,1]\big)$, we have, by (3),

$$\int_0^1 t^n \mu(\mathrm{d}t) = a_n. \qquad \square$$

Remark. By using $\lim_{N\to\infty} ||L_N - f||_1 = 0$, in conjunction with results in general integration theory referring to criteria for weak compactness for subsets of L_1 (see [DS, pp. 294–295] and [E, pp. 274–276]) the following characterization of L_1 moments can be established.

Recalling an operator $L_N(t)$ defined by

$$L_N(t) = (N+1)l_{N,[Nt]} \qquad (k = 1, 2, \ldots;\ 0 \le t \le 1)$$

the following four conditions are equivalent:

(a) a subsequence $(L_{N_k})_{k=1}^{\infty}$ exists such that the set functions $E \to \int_E L_{N_k}(t)\,\mathrm{d}t$ $(k = 1, 2, \dots)$ are uniformly (or equi-) absolutely continuous, that is, to each $e > 0$ there corresponds $d > 0$ such that

$$\sup_k \left| \int_E L_{N_k}(t)\,\mathrm{d}t \right| \le e$$

for all measurable sets E satisfying $m(E) < d$;

(b) some subsequence $(L_{N_k})_{k=1}^{\infty}$ converges weakly in L^l;

(c) some subsequence $(L_{N_k})_{k=1}^{\infty}$ converges in L^1;

(d) $(a_n)_{n\in Z} \in Mo(Z)$, i.e., $a = Mof, f \in L^1$, Mo denotes the moment.

If any one of these conditions holds, and if $f = \lim_k L_{N_k}$ weakly or strongly in L^1, then $a = Mof$, $L_N = L_N f$, and so $\lim_{N\to\infty} ||L_N - f||_1 = 0$.

Remark. The uniform absolute continuity of the set functions can be expressed as uniform absolute continuity of the point functions

$$F_k(x) = \int_0^x L_{N_k}(t)\,\mathrm{d}t$$

the condition being precisely that to each $e > 0$ there corresponds a number $d = d(e) > 0$ such that

$$\sup_k \sum_{j=1}^{r} |F_k(b_j) - F_k(a_j)| \le e$$

for any finite sequence $\big((a_j, b_j)\big)_{j=1}^{r}$ of disjoint open intervals (a_j, b_j) the sum of whose lengths does not exceed d. In this connection see, for example, [HS, Theorem (19.53) and its proof].

3. Vector-valued moments

Let X be a Banach space. Note that at some places, e.g., in Condition A the absolute value will be replaced by norm in X.

Theorem 6. *Given a sequence $a_n, n = 0, 1, 2, \dots$ of elements of X, there exists a regular measure $\mu\colon \mathcal{B}\big([0,1]\big) \to X$ of finite total variation such that a_n are the moments of μ if and only if there exists a constant $0 < H < \infty$ such that*

$$\int_0^1 ||L_N(t)||\,\mathrm{d}t \le H, \qquad N = 1, 2, \dots$$

P r o o f. Suppose that such a measure exists. Then, for each N, by (1)

$$\begin{aligned}\int_0^1 \|L_N\|\,\mathrm{d}t &= \int_0^1 \left\| \int_0^1 K_N(t,s)\mu(\mathrm{d}s) \right\| \mathrm{d}t \\ &\le \int_0^1\int_0^1 |K_N(t,s)|\,\mathrm{d}t|\mu|(\mathrm{d}s) \\ &\le M|\mu|\big([0,1]\big).\end{aligned}$$

Converse1y, suppose that $\int_0^1 \|L_N\|\,\mathrm{d}t \le H$ for all N. If we define

$$\Phi_N(\psi) = \int_0^1 \psi(t)L_N(t)\,\mathrm{d}t, \qquad \psi \in \mathcal{C}\big([0,1]\big),$$

then $\|\Phi_N\| \le H$ for all N. Since, for each n, $\lim_N \Phi_N(t^n) = a_n$, we have that $\lim_N \Phi_N(\psi)$ exists for all ψ which are linear combinations of the t^n and so, as $\|\Phi_N\| \le H$ for all N, we conclude that $\lim_N \Phi_N(\psi)$ exists for all $\psi \in C\big([0,1]\big)$. Denote this limit by Φ. To obtain our required measure, we use the following lemma (see [D, § 19, pp. 380, 383]). For each subset A of $[0,1]$, let $C\big([0,1],A\big)$ denote the space of continuous functions on $[0,1]$ vanishing outside A. If $F\colon C\big([0,1]\big) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum \|F(\psi_i)\|,$$

where supremum is over all finite families ψ_i in $C\big([0,1]\big)$ with $\sum |\psi_i(t)| \le \chi_A(t)$ for all $t \in [0,1]$.

Lemma. *If $F\colon C\big([0,1]\big) \to X$ is a linear mapping, then there exists a reqular measure $\mu\colon \mathcal{B}\big([0,1]\big) \to X$ with finite variation such that*

$$F(\psi) = \int_0^1 \psi(t)\mu(\mathrm{d}t), \qquad \psi \in C\big([0,1]\big),$$

if and only if $|||F_A||| < \infty$ for all $A \in \mathcal{B}\big([0,1]\big)$.

Let A be in $\mathcal{B}\big([0,1]\big)$ and let $\{\psi_i; i = 1,2,\ldots,n\}$ be a finite family of functions in $C\big([0,1],A\big)$ with

$$\sum_{i=1}^n |\psi_i(t)| \le \chi_A(t), \qquad t \in [0,1].$$

Then for each N,

$$\sum_{i=1}^n |\psi_i(t)| \le \chi_A(t), \qquad t \in [0,1].$$

So for each N,

$$\begin{aligned}\sum_{i=1}^{n} ||\Phi_N(\psi_i)|| &= \sum_{i=1}^{n} \left|\left| \int_0^1 \psi_i(t) L_N(t)\,\mathrm{d}t \right|\right| \\ &\leq \sum_{i=1}^{n} \int_0^1 |\psi_i(t)| ||L_N(t)||\,\mathrm{d}t \\ &\leq \int_0^1 \chi_A(t) ||L_N(t)||\,\mathrm{d}t \leq H.\end{aligned}$$

Hence $\sum_1^n ||\Phi(\psi_i)|| \leq H$ and so $|||\Phi_A||| \leq H$. Therefore, there exists a regular measure $\mu\colon \mathcal{B}([0,1]) \to X$ with finite variation such that

$$\Phi(\psi) = \int_0^1 \psi(t)\mu(\mathrm{d}t), \qquad \psi \in C([0,1]).$$

But each t^n is in $C([0,1])$ and so,

$$a_n = \int_0^1 t^n \mu(\mathrm{d}t).$$

and thus theorem is proved. □

Recall a definition of a Bochner integrable function. If $f\colon [a,b] \to X$ is simple, i.e., $f(s) = \sum_{i=1}^n \chi_{E_i}(s)x_i$, where χ_E denotes the indicator function of the set $E \subset [a,b]$, $x_i \in X$, then for any $E \in B([a,b])$

$$\int_E f\mathrm{d}\lambda = \sum_{i=1}^{n} \lambda(E \cap E_i)x_i,$$

where λ is a probability measure on $[a,b]$. Such functions are λ-measurable. Any function $f\colon [a,b] \to X$ which is the λ-almost everywhere limit of a sequence of simple functions is (called) λ-measurable.

A λ-measurable function $f\colon [a,b] \to X$ is called Bochner integrable if there exists a sequence of simple functions (f_n) such that

$$\lim \int_{[a,b]} ||f_n(s) - f(s)||\,\mathrm{d}\lambda(s) = 0.$$

In this case $\int_E f\mathrm{d}\lambda$ is defined for each measurable set E in $[a,b]$ by

$$\int_E f\mathrm{d}\lambda = \lim_n \int_E f_n\mathrm{d}\lambda.$$

A λ-measurable function $f\colon [a,b] \to X$ is Bochner integrable if and only if $\int_{[a,b]} ||f||\,\mathrm{d}\lambda < \infty$.

The following theorem concerns moments of Bochner integrable function.

Theorem 7. *Given a sequence $a_n, n = 0,1,2,\ldots,$ of elements of X, there exists an X-valued Bochner integrable function f on $[0,1]$ such that a_n are the moments of f if and only if*

$$\lim_{N,J\to\infty} \int_0^1 ||L_N(t) - L_J(t)||\,\mathrm{d}t = 0.$$

P r o o f. Suppose that f is Bochner integrable and a_n are the moments of f. Let $\{A_i\}_1^n$ be a finite family in $\mathcal{B}\big([0,1]\big)$ and $\{\beta_i^n\}$ a finite family of vectors in X and define $g\colon [0,1] \to X$ by

$$g(t) = \sum_{i=1}^n \beta_i \chi_i(t).$$

Then

$$\int_0^1 \left|\left| \int_0^1 K_N(t,s)g(s)\,\mathrm{d}s - g(t) \right|\right| \mathrm{d}t = \int_0^1 \left|\left| \sum_{i=1}^n \beta_i \left(\int_0^1 K_N(t,s)\chi_{A_i}(s)\,\mathrm{d}s - \chi_{A_i}(t) \right) \right|\right| \mathrm{d}t$$

$$\le \sum_{i=1}^n \left(||\beta_i|| \int_0^1 \left| \int_0^1 K_N(t,s)\chi_{A_i}(s)\,\mathrm{d}s - \chi_{A_i}(t) \right| \mathrm{d}t \right)$$

which, by Lemma 4, tends to 0 as $N \to \infty$. Therefore, since the set of all such g is dense in the space of all Bochner integrable functions and as

$$L_N(t) = \int_0^1 K_N(t,s)f(s)\,\mathrm{d}s,$$

we have

$$\lim_N \int_0^1 ||L_N(t) - f(t)||\,\mathrm{d}t = 0.$$

Conversely, suppose that the sequence L_N is Cauchy in the norm of the space of all Bochner integrable functions. Since this space is Banach, L_N converges in the Bochner space norm to a Bochner integrable function f.

So, for each n,

$$\left\| \int_0^1 \big(f(t) - L_N(t)\big) t^n \, \mathrm{d}t \right\| \leq \int_0^1 \|f(t) - L_N(t)\| \, |t^n| \, \mathrm{d}t$$
$$\leq \sup_t |t^n| \, \|L_N - f\|_B \to 0, \qquad N \to \infty.$$

Thus, by (3),

$$a_n = \int_0^1 t^n f(t) \, \mathrm{d}t.$$

So the theorem is proved. □

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Miloslav Duchoň
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814-73 Bratislava
SLOVAKIA
E-mail: duchon@mat.savba.sk

Camille Debiève
Université Catholique de Louvain
Département de Mathématiques
2, Chemin du Cyclotron
B-1348 Louvain-la-Neuve
BELGIUM
E-mail: camille.debieve@uclouvain.be