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# CAN BUSINESS CYCLES ARISE IN A TWO-REGIONAL MODEL WITH FIXED EXCHANGE RATES?

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ABSTRACT. The two-regional five dimensional model describing the development of income, capital stock and money stock, which was introduced by T. Asada in [*A Two-regional Model of Business Cycles with Fixed Exchange Rates: A Kaldorian Approach*, Discuss. Paper Ser., No. 44, Chuo University, Tokyo, Japan, 2003] is analysed. Sufficient conditions for the existence of one pair of purely imaginary eigenvalues and three eigenvalues with negative real parts in the linear approximation matrix of the model are found. Theorem on the existence of business cycles is presented.

## 1. Introduction

T. Asada, T. Inaba and T. Misawa developed and studied in [2] a two regional model of business cycles with fixed exchange rates, which consists of five dimensional discrete time system. T. Asada introduced and analysed in [1] the continuous time version of this model describing the dynamic interaction of two regions which are connected through interregional trade and interregional capital movement. In the whole paper we preserve the notations of [1] with minor changes.

$$\begin{aligned} \dot{Y}_i &= \alpha_i(C_i + I_i + G_i + J_i - Y_i), \qquad \alpha_i > 0, \\ \dot{K}_i &= I_i, \\ \dot{M}_1 &= p_1 A_1, \qquad\qquad i = 1, 2, \end{aligned} \tag{1}$$

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where

$$
\begin{aligned}
C_i &= c_i(Y_i - T_i) + C_{0i}, && 0 < c_i < 1, \quad C_{0i} \geq 0, \\
T_i &= \tau_i Y_i - T_{0i}, && 0 < \tau_i < 1, \quad T_{0i} \geq 0, \\
I_i &= I_i(Y_i, K_i, r_i), && \frac{\partial I_i}{\partial Y_i} > 0, \quad \frac{\partial I_i}{\partial K_i} < 0, \quad \frac{\partial I_i}{\partial r_i} < 0, \\
\frac{M_i}{p_i} &= L_i(Y_i, r_i), && \frac{\partial L_i}{\partial Y_i} > 0, \quad \frac{\partial L_i}{\partial r_i} < 0, \\
J_1 &= J_1(Y_1, Y_2, E), && \frac{\partial J_1}{\partial Y_1} < 0, \quad \frac{\partial J_1}{\partial Y_2} > 0, \quad \frac{\partial J_1}{\partial E} > 0, \qquad (2) \\
Q_1 &= \beta\left(r_1 - r_2 - \frac{E^e - E}{E}\right), && \beta > 0, \\
A_1 &= J_1 + Q_1, \\
0 &= p_1 J_1 + E p_2 J_2, \\
0 &= p_1 Q_1 + E p_2 Q_2, \\
\bar{M} &= M_1 + E M_2,
\end{aligned}
$$

and the subscript $i$, $i = 1, 2$, is the index number of a region. The meanings of symbols in (1) and (2) are as follows: $Y_i$—real regional income, $K_i$—real physical capital stock, $M_i$—nominal money stock, $L_i$—money demand, $C_i$—consumption, $T_i$—taxes, $I_i$—net real private investment expenditure on physical capital, $G_i$—real government expenditure (fixed), $p_i$—price level, $r_i$—nominal rate of interest, $E$—exchange rate, $E^e$—expected exchange rate, $J_i$—balance of current account (net export) in real terms, $Q_i$—capital account in real terms (net capital inflow), $A_i$—total balance of payments in real terms, $\alpha_i$—adjustment speed in goods market and $\beta$—degree of capital mobility.

In this paper we assume as well as Asada in [1] fixed price economy with fixed exchange rates. Therefore we can suppose, normalizing the price levels of two regions, that $p_1 = p_2 = 1$, $E = E^e$. Further we suppose that nominal interest rate $r_i$, $i = 1, 2$, adjusts instantaneously to keep money stock $M_i$ and money demand $L_i$ in equilibrium. Under these assumptions taking into account (2) and supposing that $r_i$ is implicitly determined by the relations $M_i = L_i(Y_i, r_i)$, i=1,2, model (1) takes the form

$$
\begin{aligned}
\dot{Y}_1 &= \alpha_1 \Big[ c_1(1 - \tau_1)Y_1 + c_1 T_{01} + C_{01} + G_1 \\
&\qquad + I_1\big(Y_1, K_1, r_1(Y_1, M_1)\big) + J_1(Y_1, Y_2, E) - Y_1 \Big], \\
\dot{K}_1 &= I_1\big(Y_1, K_1, r_1(Y_1, M_1)\big),
\end{aligned}
$$

$$\begin{aligned}
\dot{Y}_2 &= \alpha_2 \Bigg[ c_2(1-\tau_2)Y_2 + c_2T_{02} + C_{02} + G_2 \\
&\qquad + I_2\left(Y_2, K_2, r_2\left(Y_2, \frac{\bar{M}-M_1}{E}\right)\right) - \frac{1}{E}J_1(Y_1, Y_2, E) - Y_2 \Bigg], \\
\dot{K}_2 &= I_2\left(Y_2, K_2, r_2\left(Y_2, \frac{\bar{M}-M_1}{E}\right)\right), \qquad (3) \\
\dot{M}_1 &= J_1(Y_1, Y_2, E) + \beta\left[r_1(Y_1, M_1) - r_2\left(Y_2, \frac{\bar{M}-M_1}{E}\right)\right].
\end{aligned}$$

In the whole article we suppose that:

(i) model (3) has a unique equilibrium point $(Y_{10}, K_{10}, Y_{20}, K_{20}, M_{10})$ with positive coordinates, to an arbitrary triple of positive parameters $(\alpha_1, \alpha_2, \beta)$.

(ii) all functions in model (3) are linear with respect to their variables except the functions $I_i$ and $r_i$, $i = 1, 2$, which are nonlinear in $Y_i$ of type $C^6$ in a small neighbourhood of the equilibrium point.

**Remark 1.** The analysis of the existence of an equilibrium of model (3) was performed by Asada in [1]. The requirement on the functions in (3) to be of type $C^6$ with respect to $Y_i$ enables to transform model (3) to its partial normal form on invariant surface and to use theorem on the existence of closed cycles (see, e.g., [4]).

Asada found in [1] sufficient conditions for local stability and instability of the equilibrium point. He also discussed the existence of closed orbits around the equilibrium. His results about stability enabled him to give verbal reasons for the existence of such a value $\alpha_{10}$ of parameter $\alpha_1$ at an arbitrary fixed value $\beta_0$ of parameter $\beta$ and a sufficiently small fixed value $\alpha_{20}$ of parameter $\alpha_2$, that the linear approximation matrix of the model has at the triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ either one pair of pure imaginary eigenvalues and three eigenvalues with negative real parts (Case 1) or two pairs of pure imaginary eigenvalues and one negative eigenvalue (Case 2). Asada came to a conclusion that in the Case 1 the existence of such a value $\alpha_{10}$ guarantees the existence of closed orbits around the equilibrium at some values $\alpha_1$ which are close enough to the value $\alpha_{10}$. We would like to add to this assertion the following. The existence of a triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ with above mentioned properties does not guarantee in both cases the existence of closed orbits. Its existence is only necessary condition for the arising of closed orbits. If they arise or not at some value $\alpha_1$ from a small neighbourhood of the value $\alpha_{10}$, it depends on the smoothness of the functions in the model and on the types of the first nonzero resonant coefficients in the normal form of the model.

In the present paper the question of the existence of a triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ at which the linear approximation matrix of model (3) has at the equilibrium one pair of purely imaginary eigenvalues and three eigenvalues with negative real parts is treated analytically. Theorem 1 gives sufficient conditions for the existence of such a triple. Theorem 2 says about the existence of business cycles under the assumption of the existence of the bifurcation equation of model (3).

## 2. The analysis of model (3)

Let us write model (3) in the shorten form

$$\dot{\xi} = \Xi(\xi; \alpha_1, \alpha_2, \beta),$$

where $\xi = (Y_1, K_1, Y_2, K_2, M_1)$. After the translation of the equilibrium point $\xi_0 = (Y_{10}, K_{10}, Y_{20}, K_{20}, M_{10})$ to the origin via the coordinate shift $y = \xi - \xi_0$ model (3) becomes

$$\dot{y} = \Xi(y + \xi_0; \alpha_1, \alpha_2, \beta).$$

Its Taylor expansion at $y = 0$ gives

$$\dot{y} = A(\alpha_1, \alpha_2, \beta)y + Y(y; \alpha_1, \alpha_2, \beta), \tag{4}$$

where $Y(y; \alpha_1, \alpha_2, \beta) = O\big(\|y\|^2\big)$ and the linear approximation matrix $A(\alpha_1, \alpha_2, \beta)$ is

$$A(\alpha_1, \alpha_2, \beta) = \begin{bmatrix} \alpha_1 G_{11} & \alpha_1 G_{12} & \alpha_1 G_{13} & 0 & \alpha_1 G_{15} \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ \alpha_2 G_{31} & 0 & \alpha_2 G_{33} & \alpha_2 G_{34} & \alpha_2 G_{35} \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{bmatrix}, \tag{5}$$

where

$$G_{11} = \frac{\partial I_1}{\partial Y_1} + \frac{\partial I_1}{\partial r_1}\frac{\partial r_1}{\partial Y_1} - \left[1 - c_1(1 - \tau_1) - \frac{\partial J_1}{\partial Y_1}\right],$$

$$G_{12} = \frac{\partial I_1}{\partial K_1} < 0,$$

$$G_{13} = \frac{\partial J_1}{\partial Y_2} > 0,$$

$$G_{15} = \frac{\partial I_1}{\partial r_1}\frac{\partial r_1}{\partial M_1} > 0,$$

$$
\begin{aligned}
F_{21} &= \frac{\partial I_1}{\partial Y_1} + \frac{\partial I_1}{\partial r_1}\frac{\partial r_1}{\partial Y_1},\\
G_{31} &= -\frac{1}{E}\frac{\partial J_1}{\partial Y_1} > 0,\\
G_{33} &= \frac{\partial I_2}{\partial Y_2} + \frac{\partial I_2}{\partial r_2}\frac{\partial r_2}{\partial Y_2} - \left[1 - c_2(1-\tau_2) + \frac{1}{E}\frac{\partial J_1}{\partial Y_2}\right],\\
G_{34} &= \frac{\partial I_2}{\partial K_2} < 0,\\
G_{35} &= -\frac{1}{E}\frac{\partial I_2}{\partial r_2}\frac{\partial r_2}{\partial M_2} < 0,\\
F_{43} &= \frac{\partial I_2}{\partial Y_2} + \frac{\partial I_2}{\partial r_2}\frac{\partial r_2}{\partial Y_2},\\
F_{51}(\beta) &= \frac{\partial J_1}{\partial Y_1} + \beta\frac{\partial r_1}{\partial Y_1},\\
F_{53}(\beta) &= \frac{\partial J_1}{\partial Y_2} - \beta\frac{\partial r_2}{\partial Y_2},\\
F_{55}(\beta) &= \beta\left[\frac{\partial r_1}{\partial M_1} + \frac{1}{E}\frac{\partial r_2}{\partial M_2}\right] < 0,
\end{aligned}
$$

while all derivatives are evaluated at the equilibrium $\xi_0$.

**Assumption 1.** *As in [1] we assume that the values of* $\frac{\partial I_1(\xi_0)}{\partial Y_1}$ *and* $\frac{\partial I_2(\xi_0)}{\partial Y_2}$ *are so large that* $G_{11} > 0$ *and* $G_{33} > 0$ *at the equilibrium* $\xi_0$.

**Assumption 2.** *Parameter* $\beta$ *can be considered arbitrary large comparing its value with the values of parameters* $\alpha_1$ *and* $\alpha_2$.

**Remark 2.** Assumption 1 coincides with the standard hypothesis of Kaldorian business cycle model (see [5]). This assumption automatically implies that $F_{21} > 0$ and $F_{43} > 0$. Assumption 2 also coincides with economic theory.

**Definition 1.** A triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ of parameters $\alpha_1$, $\alpha_2$ and $\beta$ in (4) is called the critical triple of model (4) if the matrix $A(\alpha_{10}, \alpha_{20}, \beta_0)$ has the eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\,\omega_1$, $\mathrm{i} = \sqrt{-1}$, $\operatorname{Re}\lambda_{3,4,5} < 0$.

The characteristic equation of $A(\alpha_1, \alpha_2, \beta)$ is

$$
\lambda^5 + a_1\lambda^4 + a_2\lambda^4 + a_3\lambda^2 + a_4\lambda + a_5 = 0, \tag{6}
$$

where

$$
\begin{aligned}
a_1 &= a_1(\alpha_1,\alpha_2,\beta) = -\text{trace } A = -\alpha_1 G_{11} - G_{12} - \alpha_2 G_{33} - G_{34} - F_{55}(\beta),\\
a_2 &= a_2(\alpha_1,\alpha_2,\beta) = \text{sum of all principal second-order minors of matrix } A\\
&= \alpha_1 \begin{vmatrix} G_{11} & G_{12} \\ F_{21} & G_{12} \end{vmatrix} + \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{vmatrix} + \alpha_1 \begin{vmatrix} G_{11} & 0 \\ 0 & G_{34} \end{vmatrix}\\
&\quad + \alpha_1 \begin{vmatrix} G_{11} & G_{15} \\ F_{51}(\beta) & F_{55}(\beta) \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{12} & 0 \\ 0 & G_{33} \end{vmatrix} + \begin{vmatrix} G_{12} & 0 \\ 0 & G_{34} \end{vmatrix}\\
&\quad + \begin{vmatrix} G_{12} & G_{15} \\ 0 & F_{55}(\beta) \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{33} & G_{34} \\ F_{43} & G_{34} \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{33} & G_{35} \\ F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}\\
&\quad + \begin{vmatrix} G_{34} & G_{35} \\ 0 & F_{55}(\beta) \end{vmatrix},
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= a_3(\alpha_1,\alpha_2,\beta) = -\text{sum of all principal third-order minors of matrix } A\\
&= -\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ F_{21} & G_{12} & 0 \\ G_{31} & 0 & G_{33} \end{vmatrix} - \alpha_1 \begin{vmatrix} G_{11} & G_{12} & 0 \\ F_{21} & G_{12} & 0 \\ 0 & 0 & G_{34} \end{vmatrix}\\
&\quad -\alpha_1 \begin{vmatrix} G_{11} & G_{12} & G_{15} \\ F_{21} & G_{12} & G_{15} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} - \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} & 0 \\ G_{31} & G_{33} & G_{34} \\ 0 & F_{43} & G_{34} \end{vmatrix}\\
&\quad -\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} & G_{15} \\ G_{31} & G_{33} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix} - \alpha_1 \begin{vmatrix} G_{11} & 0 & G_{15} \\ 0 & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}\\
&\quad -\alpha_2 \begin{vmatrix} G_{12} & 0 & 0 \\ 0 & G_{33} & G_{34} \\ 0 & F_{43} & G_{34} \end{vmatrix} - \alpha_2 \begin{vmatrix} G_{12} & 0 & G_{15} \\ 0 & G_{33} & G_{35} \\ 0 & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}\\
&\quad - \begin{vmatrix} G_{12} & 0 & G_{15} \\ 0 & G_{34} & G_{35} \\ 0 & 0 & F_{55}(\beta) \end{vmatrix} - \alpha_2 \begin{vmatrix} G_{33} & G_{34} & G_{35} \\ F_{43} & G_{34} & G_{35} \\ F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix},
\end{aligned}
$$

$$
\begin{aligned}
a_4 &= a_4(\alpha_1,\alpha_2,\beta) = \text{sum of all principal fourth-order minors of matrix } A\\
&= \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & 0 \\ F_{21} & G_{12} & 0 & 0 \\ G_{31} & 0 & G_{33} & G_{34} \\ 0 & 0 & F_{43} & G_{34} \end{vmatrix}\\
&\quad +\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & G_{15} \\ F_{21} & G_{12} & 0 & G_{15} \\ G_{31} & 0 & G_{33} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}
\end{aligned}
$$

$$+\alpha_1 \begin{vmatrix} G_{11} & G_{12} & 0 & G_{15} \\ F_{21} & G_{12} & 0 & G_{15} \\ 0 & 0 & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & 0 & F_{55}(\beta) \end{vmatrix}$$

$$+\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} & 0 & G_{15} \\ G_{31} & G_{33} & G_{34} & G_{35} \\ 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$+\alpha_2 \begin{vmatrix} G_{12} & 0 & 0 & G_{15} \\ 0 & G_{33} & G_{34} & G_{35} \\ 0 & F_{43} & G_{34} & G_{35} \\ 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix},$$

$$\begin{aligned} a_5 = a_5(\alpha_1, \alpha_2, \beta) &= -\det A \\ &= -\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & 0 & G_{15} \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ G_{31} & 0 & G_{33} & G_{34} & G_{35} \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} \\ &= -\alpha_1\alpha_2 G_{12} G_{34} F_{55}(\beta)\big[(G_{11} - F_{21})(G_{33} - F_{43}) - G_{13}G_{31}\big]. \end{aligned}$$

On the base of Assumption 2 we will arrange the coefficients of characteristic equation (6) as polynomials with respect to parameter $\beta$, expressing explicitly only their terms of the highest order. In these relations we use the following notations:

$$\begin{aligned} d_1 &= -\left(\frac{\partial r_1}{\partial M_1} + \frac{1}{E}\frac{\partial r_2}{\partial M_2}\right) > 0, \\ d_2 &= -d_1(G_{12} + G_{34}) > 0, \\ d_3 &= d_1 G_{11} + G_{15}\frac{\partial r_1}{\partial Y_1} > 0, \\ d_4 &= d_1 G_{33} - G_{35}\frac{\partial r_2}{\partial Y_2} > 0, \\ d_5 &= d_1 G_{12} G_{34} > 0, \\ d_6 &= d_3 G_{34} - d_1 G_{12}(F_{21} - G_{11}), \\ d_7 &= d_1 G_{34}(F_{43} - G_{33}) - d_4 G_{12}, \\ d_8 &= d_5(F_{21} - G_{11}) > 0, \\ d_9 &= d_5(F_{43} - G_{33}) > 0, \\ d_{10} &= d_1 G_{12} G_{34}\big[(F_{21} - G_{11})(F_{43} - G_{33}) - G_{13}G_{31}\big] > 0, \\ d_{11} &= d_2 d_6 - d_3 d_5 - d_1 d_8, \end{aligned}$$

$$\begin{aligned}
G &= -\frac{\partial r_1}{\partial Y_1}\big[G_{15}G_{34}(F_{43}-G_{33})+G_{12}G_{13}G_{35}\big] \\
&\quad +\frac{\partial r_2}{\partial Y_2}\big[G_{12}G_{35}(F_{21}-G_{11})+G_{15}G_{31}G_{34}\big] \\
&\quad -d_1\big[G_{12}G_{33}(F_{21}-G_{11})+G_{11}G_{34}(F_{43}-G_{33})+G_{13}G_{31}(G_{12}+G_{34})\big], \\
H &= d_1(G_{11}G_{33}-G_{13}G_{31})-\frac{\partial r_1}{\partial Y_1}(G_{13}G_{35}-G_{15}G_{33}) \\
&\quad +\frac{\partial r_2}{\partial Y_2}(G_{15}G_{31}-G_{11}G_{35}).
\end{aligned}$$

The coefficients $a_j(\alpha_1,\alpha_2,\beta),\ j=1,2,3,4,5$, are given by the relations:

$$\begin{aligned}
a_1(\alpha_1,\alpha_2,\beta) &= d_1\beta+f_1(\alpha_1,\alpha_2), \\
a_2(\alpha_1,\alpha_2,\beta) &= (d_2-d_3\alpha_1-d_4\alpha_2)\beta+f_2(\alpha_1,\alpha_2), \\
a_3(\alpha_1,\alpha_2,\beta) &= (d_5+d_6\alpha_1-d_7\alpha_2+H\alpha_1\alpha_2)\beta+f_3(\alpha_1,\alpha_2), \\
a_4(\alpha_1,\alpha_2,\beta) &= (d_8\alpha_1+d_9\alpha_2-G\alpha_1\alpha_2)\beta+f_4(\alpha_1,\alpha_2), \\
a_5(\alpha_1,\alpha_2,\beta) &= d_{10}\alpha_1\alpha_2\beta,
\end{aligned} \tag{7}$$

where $f_j(\alpha_1,\alpha_2), j=1,2,3,4$, are polynomials with respect to parameters $\alpha_1$, $\alpha_2$ not higher order than two.

The following theorem gives sufficient conditions for the existence of a critical triple of model (4).

**Theorem 1.** *If parameter $\alpha_2$ is sufficiently small and parameter $\beta$ is sufficiently large, then there exists a critical triple $(\alpha_{10},\alpha_{20},\beta_0)$ of model* (4).

Before the proof of Theorem 1 several lemmas will be introduced. First we use the following lemma from [6].

**Lemma 1.** *Characteristic equation* (6) *has a pair of pure imaginary eigenvalues $\lambda_{1,2}=\pm \mathrm{i}\omega$ and three roots $\lambda_j$, $j=3,4,5$, with a negative real part if and only if*

$$\begin{aligned}
\Delta_1 &= a_1>0, \\
\Delta_2 &= a_1a_2-a_3>0, \\
\Delta_3 &= a_3\Delta_2+a_1(a_5-a_1a_4)>0, \\
\Delta_4 &= a_4\Delta_3-a_2a_5\Delta_2+a_5(a_1a_4-a_5)=0, \\
a_5 &> 0,
\end{aligned} \tag{8}$$

*where $a_j=a_j(\alpha_1,\alpha_2,\beta)$, $j=1,2,\dots,5$ are the coefficients of equation* (6).

Using the form of $a_j$ from (7) we can express $\Delta_j$, $j = 1, 2, 3, 4$, as follows:

$$\begin{aligned}
\Delta_1(\alpha_1, \alpha_2, \beta) &= d_1\beta + g_1(\alpha_1, \alpha_2) > 0,\\
\Delta_2(\alpha_1, \alpha_2, \beta) &= d_1(d_2 - d_3\alpha_1 - d_4\alpha_2)\beta^2 + g_2(\alpha_1, \alpha_2, \beta),\\
\Delta_3(\alpha_1, \alpha_2, \beta) &= 7d_1\big(d_2d_5 + d_{11}\alpha_1 - d_3d_6\alpha_1^2 + h_3(\alpha_1, \alpha_2)\big)\beta^3,\\
&\quad + g_3(\alpha_1, \alpha_2, \beta),\\
\Delta_4(\alpha_1, \alpha_2, \beta) &= d_1\left(d_2d_5d_8\alpha_1 + d_8d_{11}\alpha_1^2 - d_3d_6d_8\alpha_1^3 + h_4(\alpha_1, \alpha_2)\right)\beta^4,\\
&\quad + g_4(\alpha_1, \alpha_2, \beta),
\end{aligned} \tag{9}$$

where $h_3$ and $h_4$ are polynomials with respect to $\alpha_1$, $\alpha_2$ not of higher order than five with the property $h_j(\alpha_1, \alpha_2) = \alpha_2\widetilde{h}_j(\alpha_1, \alpha_2)$, $j = 3, 4$, and $g_j$ are polynomials with respect to $\alpha_1, \alpha_2, \beta$ with the property

$$\lim_{\beta\to\infty} \frac{g_j(\alpha_1, \alpha_2, \beta)}{\beta^2} = 0, \qquad j = 1, 2, 3, 4,$$

for arbitrary fixed $\alpha_1$ and $\alpha_2$.

**Lemma 2.** *If parameter $\alpha_2 > 0$ is sufficiently small, and parameter $\beta > 0$ sufficiently large, then there exists $\alpha_1 > 0$ such that*

$$\Delta_4(\alpha_1, \alpha_2, \beta) = 0. \tag{10}$$

P r o o f. Instead of $\beta$ introduce $\gamma = \frac{1}{\beta}$. Put

$$F(\alpha_1, \alpha_2, \gamma) = \gamma^4\Delta_4\left(\alpha_1, \alpha_2, \frac{1}{\gamma}\right).$$

We have

$$F(\alpha_1, 0, 0) = d_1d_8\alpha_1\Big(d_2d_5 + d_{11}\alpha_1 - d_3d_6\alpha_1^2\Big).$$

So, consider equation

$$d_3d_6\alpha_1^2 - d_{11}\alpha_1 - d_2d_5 = 0. \tag{11}$$

The value of $d_6$ may be positive or zero or negative. If $d_6 = 0$, then the root of (11) is

$$\alpha_1^{(1)} = -\frac{d_2d_5}{d_{11}} > 0,$$

because $d_{11} = -d_3d_5 - d_1d_8 < 0$. If $d_6 \neq 0$, then equation (11) has roots

$$\alpha_1^{(1,2)} = \frac{d_{11} \pm \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6}\,. \tag{12}$$

These roots are real, because discriminant $D$ in (12) is positive. If $d_6 > 0$ it this obvious. If $d_6 < 0$, then

$$\begin{aligned} D &= (d_2d_6 - d_3d_5 - d_1d_8)^2 + 4d_2d_3d_5d_6 \\ &= (d_2d_6 + d_3d_5)^2 + 2d_1d_8(d_3d_5 - d_2d_6) + (d_1d_8)^2 \\ &> 0. \end{aligned}$$

If $d_6 > 0$, then the roots have different signs, because

$$\alpha_1^{(1)}\alpha_1^{(2)} = -\frac{d_2d_5}{d_3d_6} < 0.$$

In this case

$$\alpha_1^{(1)} = \frac{d_{11} + \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6} > 0,$$

$$\alpha_1^{(2)} = \frac{d_{11} - \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6} < 0.$$

If $d_6 < 0$ then both roots are positive, because

$$\alpha_1^{(1)}\alpha_1^{(2)} = -\frac{d_2d_5}{d_3d_6} > 0$$

and

$$\alpha_1^{(1)} + \alpha_1^{(2)} = \frac{d_{11}}{d_3d_6} = \frac{d_2d_6 - d_3d_5 - d_1d_8}{d_3d_6} > 0.$$

In this case

$$\alpha_1^{(2)} = \frac{d_{11} - \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6} > \alpha_1^{(1)} = \frac{d_{11} + \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6} > 0.$$

We have

$$F\left(\alpha_1^{(1)}, 0, 0\right) = 0$$

and

$$\frac{\partial F}{\partial \alpha_1}\left(\alpha_1^{(1)}, 0, 0\right) \neq 0.$$

By the implicit function theorem there are $\delta > 0$, $\varepsilon > 0$ such that for any $\alpha_2 \in (0, \delta)$ and $\gamma \in (0, \delta)$ equation

$$F(\alpha_1, \alpha_2, \gamma) = 0$$

has a root $\alpha_1 \in \left(\alpha_1^{(1)} - \varepsilon,\ \alpha_1^{(1)} + \varepsilon\right)$ which depends continuously on $\alpha_2$ and $\gamma$. □

The root $\alpha_1^{(1)}$ of equation (11) from the previous proof plays an important role in the following two Lemmas.

**Lemma 3.** *If $\alpha_2 > 0$ is sufficiently small, $\beta > 0$ sufficiently large and $\alpha_1$ is sufficiently close to $\alpha_1^{(1)}$, then $\Delta_2(\alpha_1, \alpha_2, \beta) > 0$.*

P r o o f. It is sufficient to prove that $d_2 - d_3\alpha_1^{(1)} > 0$. If $d_6 = 0$, then we have to prove

$$d_2 - d_3\alpha_1^{(1)} = d_2 - d_3\frac{-d_2d_5}{d_{11}} > 0,$$

which is equivalent to

$$d_2(-d_3d_5 - d_1d_8) + d_2d_3d_5 < 0.$$

The last inequality is satisfied because $-d_1d_2d_8 < 0$. Let $d_6 > 0$. Then we have to prove

$$d_2 - d_3\alpha_1^{(1)} = d_2 - d_3\frac{d_{11} + \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6} > 0,$$

which is equivalent to

$$\sqrt{d_{11}^2 + 4d_2d_3d_5d_6} < 2d_2d_6 - d_{11}. \tag{13}$$

Note that

$$2d_2d_6 - d_{11} = 2d_2d_6 - (d_2d_6 - d_3d_5 - d_1d_8) = d_2d_6 + d_3d_5 + d_1d_8 > 0.$$

Therefore inequality (13) is equivalent to

$$d_{11}^2 + 4d_2d_3d_5d_6 < 4d_2^2d_6^2 - 4d_2d_6d_{11} + d_{11}^2$$

and

$$d_3d_5 < d_2d_6 - (d_2d_6 - d_3d_5 - d_1d_8),$$

which is satisfied because $d_1d_8 > 0$. Let $d_6 < 0$. Then we have to prove

$$d_2 - d_3\alpha_1^{(1)} = d_2 - d_3\frac{d_{11} + \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}}{2d_3d_6} > 0,$$

which is equivalent to

$$2d_2d_6 - d_{11} < \sqrt{d_{11}^2 + 4d_2d_3d_5d_6}. \tag{14}$$

If $2d_2d_6 - d_{11} \leq 0$, then the last inequality is satisfied. In the opposite case it is equivalent to

$$4d_2^2d_6^2 - d_2d_64d_{11} + d_{11}^2 < d_{11}^2 + 4d_2d_3d_5d_6$$

and

$$d_2d_6 - (d_2d_6 - d_3d_5 - d_1d_8) > d_3d_5,$$

which is satisfied because $d_1d_8 > 0$. □

**Lemma 4.** *If $\alpha_2 > 0$ is sufficiently small, $\beta > 0$ sufficiently large and $\alpha_1$ is sufficiently close to $\alpha_1^{(1)}$, then $\Delta_3(\alpha_1, \alpha_2, \beta) > 0$.*

P r o o f. Consider $\alpha_2 > 0$ sufficiently small and $\beta > 0$ sufficiently large. Arranging $a_4\Delta_3$, $a_2a_5\Delta_2$ and $a_5(a_1a_4 - a_5)$ from $\Delta_4$ in (8) as polynomials with respect to $\beta$, we find out that $a_4\Delta_3$ and $a_2a_5\Delta_2$ have degree 4, while $a_5(a_1a_4 - a_5)$ has degree 3. It means that $a_4\Delta_3$ and $a_2a_5\Delta_2$ have the same sign for sufficiently large $\beta$ whenever $\Delta_4(\alpha_1, \alpha_2, \beta) = 0$. By Lemma 3 $\Delta_2(\alpha_1, \alpha_2, \beta) > 0$. From (7) we see that

$$a_2(\alpha_1, \alpha_2, \beta) > 0, a_4(\alpha_1, \alpha_2, \beta) > 0 \quad \text{and} \quad a_5(\alpha_1, \alpha_2, \beta) > 0.$$

Therefore

$$\Delta_3(\alpha_1, \alpha_2, \beta) > 0$$

for sufficiently small $\alpha_2$ and sufficiently large $\beta$. □

P r o o f o f T h e o r e m 1. Take sufficiently small $\alpha_2 > 0$ and sufficiently large $\beta > 0$. By Lemma 2 there exists $\alpha_1 > 0$ such that $\Delta_4(\alpha_1, \alpha_2, \beta) = 0$. By Lemmas 3 and 4

$$\Delta_2(\alpha_1, \alpha_2, \beta) > 0 \quad \text{and} \quad \Delta_3(\alpha_1, \alpha_2, \beta) > 0.$$

The condition $\Delta_1(\alpha_1, \alpha_2, \beta) > 0$ is satisfied for sufficiently large $\beta$. The coefficient $a_5$ is positive whenever so are $\alpha_1$, $\alpha_2$, $\beta$. Therefore the conditions in Lemma 1 are satisfied. □

Take now an arbitrary critical triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ of model (4), fix $\alpha_{20}$ and $\beta_0$ and investigate model (4) with respect to parameter $\alpha_1 \in (\alpha_{10} - \delta, \alpha_{10} + \delta)$, where $\delta > 0$. Suppose that the first resonant coefficients $\delta_1$ and $\delta_2$ in the partial normal form on invariant surface of model (3) are nonzero (see, e.g., [3]). Then using polar coordinates the bifurcation equation of model (3) is

$$ar^2 + b\varepsilon = 0, \quad a = \operatorname{Re}\delta_2, \quad b = \operatorname{Re}\delta_1, \quad \varepsilon = \alpha_1 - \alpha_{10}.$$

Utilizing the results from the bifurcation theory (see, e.g., [4]) we can formulate the following theorem.

**Theorem 2.** *Let the coefficients $a$, $b$ in the bifurcation equation exist. Then*

(1) *If $a < 0$, then there exists a stable limit cycle for every small enough $\varepsilon > 0$, if $b$ is positive and for every small enough $\varepsilon < 0$, if $b$ is negative.*

(2) *If $a > 0$, then there exists an unstable limit cycle for every small enough $\varepsilon < 0$, if $b$ is positive and for every small enough $\varepsilon > 0$, if $b$ is negative.*

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