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ASYMPTOTIC PROPERTIES OF TRINOMIAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. New criteria for asymptotic properties of the solutions of the third order delay differential equation

$$\left(\frac{1}{r(t)}y''(t)\right)' - p(t)y'(t) + g(t)y\big(\tau(t)\big) = 0,$$

by transforming this equation to its binomial canonical form are presented.

We consider the third order delay differential equation

$$\left(\frac{1}{r(t)}y''(t)\right)' - p(t)y'(t) + g(t)y\big(\tau(t)\big) = 0 \tag{1}$$

and the corresponding second order differential equation

$$\left(\frac{v'(t)}{r(t)}\right)' = p(t)v(t). \tag{2}$$

We always assume that

(H1) $r(t)$, $p(t)$ and $g(t) \in C\big([t_0, \infty)\big)$, $p(t) \geq 0$, $r(t) > 0$, $g(t) > 0$;

(H2) $\tau(t) \in C\big([t_0, \infty)\big)$, $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$;

(H3) $\int_{t_0}^{\infty} r(s)\, \mathrm{d}s = \infty$.

The particular case of (1), namely differential equation without delay

$$y'''(t) - p(t)y'(t) + g(t)y(t) = 0 \tag{3}$$

has been investigated in earlier papers [1, 3, 4, 7, 10, 12, 17] and [20]. In these papers the following structure of nonoscillatory solutions is presented:

Lemma 1. *Let $y(t)$ be a nonoscillatory solution of (3). Then there exists $t_1 \geq t_0$ such that either*

$$y(t)y'(t) < 0 \tag{4}$$

or

$$y(t)y'(t) \geq 0 \tag{5}$$

for $t \geq t_1$ and moreover if $y(t)$ satisfies (4), then also

$$(-1)^i y(t)y^{(i)}(t) > 0, \quad 0 \leq i \leq 3, \quad t \geq t_1. \tag{6}$$

It is known that equation (3) always has a solution satisfying (6). For the next considerations we say that (3) has property (P_0) if every nonoscillatory solution of (3) satisfies (6).

L a z e r presented a sufficient condition for property (P_0) of (3) (see [17]). This result has been improved by several authors (see, e.g., [6, 10, 12, 19] and [20]). We denote

$$D_0 y = y, \quad D_1 y = (D_0 y)', \quad D_2 y = \frac{1}{r}(D_1 y)', \quad D_3 y = (D_2 y)'.$$

We say that (1) has property (P_0) if every nonoscillatory solution $y(t)$ of (1) satisfies

$$(-1)^i y(t) D_i y(t) \geq 0, \qquad 0 \leq i \leq 3. \tag{7}$$

Lemma 2. *The operator $Ly \equiv \left(\frac{1}{r(t)}y''(t)\right)' - p(t)y'(t)$ can be written as*

$$Ly \equiv \frac{1}{v}\left(\frac{v^2}{r}\left(\frac{1}{v}y'\right)'\right)',$$

where $v(t)$ is a positive solution of (2).

P r o o f. Straightforward computation shows that

$$Ly = \frac{1}{v}\left(\frac{v^2}{r}\left[\frac{y''}{r}\frac{1}{v} - \frac{v'}{rv^2}y'\right]\right)' = \left(\frac{y''}{r}\right)' - \left(\frac{v'}{r}\right)'\frac{y'}{v} = \left(\frac{1}{r(t)}y''(t)\right)' - p(t)y'(t).$$

$\square$

Corollary 1. *If $v(t)$ is a positive solution of (2), then equation (1) can be rewritten as*

$$\left(\frac{v^2}{r}\left(\frac{1}{v}y'\right)'\right)' + v(t)g(t)y\big(\tau(t)\big) = 0. \tag{8}$$

We shall study properties of trinomial equation (1) with help of its binomial form (8). The properties of canonical equations are well known. For this reason it is useful for (8) to be in the canonical form. The equation (8) is in the canonical form if $\int_{t_0}^{\infty} \frac{r(t)}{v^2(t)}\,\mathrm{d}t = \infty$ and $\int_{t_0}^{\infty} v(t)\,\mathrm{d}t = \infty$.

LEMMA 3. *Equation*(2) *possesses the following couple of solutions*

$$v(t) > 0, \quad v'(t) \le 0 \quad and \quad \left(\frac{v'(t)}{r(t)}\right)' \ge 0 \tag{9}$$

and

$$v(t) > 0, \quad v'(t) \ge 0 \quad and \quad \left(\frac{v'(t)}{r(t)}\right)' \ge 0, \tag{10}$$

for all t large enough.

We say that a solution $v(t)$ of (2) is of degree 0 if it satisfies (9), on the other hand $v(t)$ is said to be of degree 2 if it satisfies (10).

LEMMA 4. *Let* (H3) *hold. If $v(t)$ is a solution of degree 0 of*(2), *then*

$$\int\limits_{t_0}^{\infty} \frac{r(t)}{v^2(t)}\,\mathrm{d}t = \infty.$$

P r o o f. It is easy to see that $v(t)$ satisfies $c > v(t)$, eventually which implies assertion of the lemma. $\square$

A solution $v(t)$ of degree 0 is the key solution of (2). If it satisfies the condition

$$\int\limits_{t_0}^{\infty} v(t)\,\mathrm{d}t = \infty, \tag{11}$$

then equation (1) can be represented in the canonical form (8).

We present sufficient condition for every solution of degree 0 of (2) to satisfy (11). Let us denote $\widetilde{P}_r(t) = r(t)\int_t^{\infty} p(s)\,\mathrm{d}s$ (we suppose that $\int_{t_0}^{\infty} p(s)\,\mathrm{d}s < \infty$).

LEMMA 5. *Assume that*

$$\int\limits_{t_0}^{\infty} \mathrm{e}^{-\int_{t_0}^{t} \widetilde{P}_r(s)\,\mathrm{d}s}\,\mathrm{d}t = \infty. \tag{12}$$

Then every solution of degree 0 of(2) *satisfies* (11).

P r o o f. Let $v(t)$ satisfy (9). Integrating (2) from t to ∞, one gets

$$\ell - \frac{v'(t)}{r(t)} = \int\limits_{t}^{\infty} p(s)v(s)\,\mathrm{d}s,$$

where $\ell = \lim\limits_{t\to\infty} \frac{v'(t)}{r(t)}$. We claim that $\lim\limits_{t\to\infty} \frac{v'(t)}{r(t)} = 0$. If not, then $\lim\limits_{t\to\infty} \frac{v'(t)}{r(t)} = \ell$, $\ell < 0$. Then $v'(t) \leq \ell\, r(t)$. Integrating from t_1 to t, we have $v(t) \leq v(t_1) + \ell \int_{t_1}^t r(t)\,\mathrm{d}s \to -\infty$ as $t \to \infty$. This is a contradiction. Thus we conclude that

$$-v'(t) = r(t)\int\limits_t^\infty p(s)v(s)\,\mathrm{d}s \leq v(t)r(t)\int\limits_t^\infty p(s)\,\mathrm{d}s = v(t)\widetilde{P}_r(t).$$

Then integrating from t_1 to t, we have

$$v(t) \geq v(t_1)\,\mathrm{e}^{-\int_{t_1}^t \widetilde{P}_r(s)\,\mathrm{d}s}.$$

Integrating from t to ∞ we get the desired property. $\qquad\square$

To simplify our notation, we set

$$L_0 y = y, \quad L_1 y = \frac{1}{v}(L_0 y)', \quad L_2 y = \frac{v^2}{r}(L_1 y)', \quad L_3 y = (L_2 y)'.$$

We present structure of nonoscillatory solutions $y(t)$ of (8), where (8) is in canonical form (see, e.g., [9] or [14]).

Lemma 6. *Let* (8) *be in canonical form. Every positive solution of* (8) *satisfies either*

$$L_0 y(t) > 0, \quad L_1 y(t) < 0, \quad L_2 y(t) > 0, \quad L_3 y(t) < 0 \tag{13}$$

or

$$L_0 y(t) > 0, \quad L_1 y(t) > 0, \quad L_2 y(t) > 0, \quad L_3 y(t) < 0 \tag{14}$$

for large t.

Remark 1. It follows from Lemma (6) that if (8) is in canonical form, then the set of nonoscillatory solutions of (1) has the same structure as those of (2). That is

$$y(t)y'(t) < 0$$

or

$$y(t)y'(t) > 0,$$

where $y(t)$ is a nonoscillatory solutions of (1). Note that the corresponding inequality (5) is sharp for (1).

Lemma 7. *If* $v(t)$ *is a positive solution of degree 0 of* (2) *satisfying* (11), *then a positive solution of* (8) *satisfies* (13) *if and only if it satisfies* (7).

P r o o f. We have $L_0 y = y > 0$, $L_1 y = \frac{1}{v}y' < 0$, eventually. From this we know that $D_0 y > 0$ and $D_1 y = y' < 0$. From equation (1) we see that $D_3 y = \left(\frac{1}{r}y''\right)' < 0$.

So D_2y is of constant sign. If we admit $D_2y < 0$, then $D_1y = y'$ is decreasing so $y'(t) \leq y'(t_1) = \ell < 0$. Integrating from t_1 to t we get

$$y(t) \leq y(t_1) + \ell(t_1 - t) \to -\infty \qquad \text{as} \quad t \to \infty.$$

A contradiction with the positivity of $y(t)$. So we conclude $D_2y > 0$ and y satisfies (7).

Now, $D_0y > 0$ and $D_1y < 0$ implies $L_0y > 0$ and $L_1y < 0$. It follows from (8) that $L_3y < 0$. Then $L_2y(t)$ is decreasing. If we admit $L_2y(t) < 0$ for $t \geq t_1$, then $L_1y(t) \leq \ell < 0$ and integrating from t_1 to t one gets

$$y(t) < y(t_1) + \ell \int_{t_1}^{t} v(s)\,\mathrm{d}s \to -\infty \qquad \text{as} \quad t \to \infty.$$

$\square$

Following K i g u r a d z e [13] we say that (8) has property (A) if every its nonoscillatory solution is of degree 0. Lemma 7 can now be formulated in the form:

THEOREM 1. *If $v(t)$ is a positive solution of degree 0 of (2) satisfying (11), then (1) has property (P_0) if and only if ((8) has property (A).*

THEOREM 2. *Let $v(t)$ be a positive solution of degree 0 of (2) satisfying (11). Let*

$$\tau'(t) > 0. \tag{15}$$

If the differential inequality

$$\left(\frac{v^2(t)}{r(t)}z'(t)\right)' + \left[v\big(\tau(t)\big)\tau'(t)\int_{t}^{\infty} v(s)g(s)\,\mathrm{d}s\right]z\big(\tau(t)\big) \leq 0 \tag{16}$$

has no positive solution, then (1) has property (P_0).

P r o o f. Assume that (1) has not property (P_0). It implies from Theorem 1 that (8) has not property (A). Consequently a solution $y(t)$ exists that satisfies (14).

Integrating (8) from t to ∞, we have

$$L_2y(t) = c + \int_{t}^{\infty} v(s)g(s)y\big(\tau(s)\big)\,\mathrm{d}s, \tag{17}$$

where $c = \lim_{t\to\infty} L_2y(t)$. Since

$$y(t) = y(t_1) + \int_{t_1}^{t} v(x)L_1y(x)\,\mathrm{d}x,$$

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then it implies from (17) that

$$L_2 y(t) \geq \int_t^\infty v(s)g(s)\left(\int_{t_1}^{\tau(s)} v(x)L_1 y(x)\,dx\right)ds.$$

We may assume that $\tau(t) > t_1$, so

$$L_2 y(t) \geq \int_t^\infty v(s)g(s)\int_{\tau(t)}^{\tau(s)} v(x)L_1 y(x)\,dx\,ds.$$

After changing the order of integration we get

$$L_2 y(t) \geq \int_{\tau(t)}^\infty v(x)L_1 y(x)\int_{\tau^{-1}(x)}^\infty v(s)g(s)\,ds\,dx = \int_{\tau(t)}^\infty L_1 y(x)G(x)\,dx,$$

where

$$G(x) = v(x)\int_{\tau^{-1}(x)}^\infty v(s)g(s)\,ds,$$

and $\tau^{-1}(t)$ is the inverse function to $\tau(t)$. Integration from t_1 to t leads to

$$L_1 y(t) \geq L_1 y(t_1) + \int_{t_1}^t \frac{r(s)}{v^2(s)}\int_{\tau(s)}^\infty L_1 y(x)G(x)\,dx. \tag{18}$$

Let us denote the right hand side of (18) by $z(t)$. Then $z(t) > 0$ and

$$0 = \left(\frac{v^2(t)}{r(t)}z'(t)\right)' + L_1 y\big(\tau(t)\big)G\big(\tau(t)\big)\tau'(t) \geq \left(\frac{v^2(t)}{r(t)}z'(t)\right)' + G\big(\tau(t)\big)\tau'(t)z\big(\tau(t)\big)$$

so $z(t)$ is a positive solution of (16). A contradiction. $\qquad\square$

COROLLARY 2. *Let $v(t)$ be a positive solution of degree 0 of (2) satisfying (11). Let (15) hold. If the second order differential equation*

$$\left(\frac{v^2(t)}{r(t)}z'(t)\right)' + \left[v\big(\tau(t)\big)\tau'(t)\int_t^\infty v(s)g(s)\,ds\right]z\big(\tau(t)\big) = 0 \tag{19}$$

is oscillatory, then (1) has property (P_0).

It is known that (19) is oscillatory if and only if (16) has no positive solution (see [9], [14]).

THEOREM 3. *Let $v(t)$ be a positive solution of degree 0 of* (2) *satisfying* (11). *Let* (15) *hold. If*

$$\liminf_{t \to \infty} \left(\int_{t_1}^{\tau(t)} \frac{r(s)}{v^2(s)} \, ds \right) \left(\int_t^\infty v\big(\tau(x)\big)\tau'(x) \int_x^\infty v(s)g(s) \, ds \, dx \right) > \frac{1}{4} \qquad (20)$$

then equation (1) *has property* (P_0).

P r o o f. Condition (20) guaranties oscillation of (19) (see, e.g., [8, 9] or [14]). $\square$

EXAMPLE 1. Let us consider delay differential equation

$$\left(\frac{1}{3t} \, y''(t) \right)' - \frac{1}{t^3} \, y'(t) + g(t)y(\lambda t) = 0, \qquad 0 < \lambda < 1. \qquad (21)$$

Now $v(t) = \frac{1}{t}$ is a solution of degree 0 of equation $\big(\frac{v'(t)}{3t}\big)' = \frac{1}{t^3} \, v(t)$. Moreover, it satisfies (11), then follows from Theorem 3 that (21) satisfies (P_0) if

$$\liminf_{t \to \infty} \lambda^4 t^4 \int_t^\infty \frac{g(s)}{s} \, \ln\left| \frac{s}{t} \right| ds > \frac{1}{3}. \qquad (22)$$

THEOREM 4. *Let all assumptions of Theorem* 3 *hold. Moreover, if*

$$\int_{t_0}^\infty v(s_2) \int_{s_2}^\infty \frac{r(s_1)}{v^2(s_1)} \int_{s_1}^\infty v(s_0)g(s_0) \, ds_0 \, ds_1 \, ds_2 \; = \infty, \qquad (23)$$

then every nonoscillatory solution of (1) *tends to zero as* $t \to \infty$.

P r o o f. Let $y(t) > 0$ be a solution of (1). From Theorem 3 it follows that $y(t)$ satisfies (7), so $y'(t) < 0$. There exists $\lim_{t \to \infty} y(t) = \ell \geq 0$. If $\ell > 0$ then $y\big(\tau(t)\big) > \ell$. Integrating (8) twice from t to ∞ and then from t_1 to t, we have

$$y(t) = y(t_1) - \int_{t_1}^t v(s_2) \int_{s_2}^\infty \frac{r(s_1)}{v^2(s_1)} \int_{s_1}^\infty v(s_0)g(s_0)y\big(\tau(s_0)\big) \, ds_0 \, ds_1 \, ds_2$$

$$\leq y(t_1) - \ell \int_{t_0}^\infty v(s_2) \int_{s_2}^\infty \frac{r(s_1)}{v^2(s_1)} \int_{s_1}^\infty v(s_0)g(s_0) \, ds_0 \, ds_1 \, ds_2 \to -\infty \qquad (24)$$

as $t \to \infty$. Therefore, $\lim_{t \to \infty} y(t) = 0$. $\square$

EXAMPLE 2. We consider a differential equation

$$\left(\frac{1}{3t} \, y''(t) \right)' - \frac{1}{t^3} \, y'(t) + \frac{a}{t^4} y(\lambda t) = 0, \qquad 0 < \lambda < 1, \quad a > 0. \qquad (25)$$

It follows from Theorem 3 that (E_3) has a property (P_0) and from Theorem 4 that every nonoscillatory solution of (25) tends to zero as $t \to \infty$.

Now we recall the following result (see, e.g., [9] or [14]), which enables us to deduce property (A) of delay equation from that of equation without delay:

THEOREM 5. *Assume that $v(t)$ is a solution of degree 0 of (2) satisfying (11). Let (15) hold. If*

$$\left(\frac{v^2(t)}{r(t)} \left(\frac{1}{v(t)}\, y'(t) \right)' \right)' + \frac{v\big(\tau^{-1}(t)\big) g\big(\tau^{-1}(t)\big)}{\tau'\big(\tau^{-1}(t)\big)}\, y(t) = 0 \qquad (26)$$

has property (A), then so does (8).

When we combine Theorem 1 and Theorem 5, we get:

THEOREM 6. *Assume that $v(t)$ is a solution of degree 0 of (2) satisfying (11). Let (15) hold. If (26) has property (A) then (1) has property (P_0).*

Applying Theorem 1 to (26), we get

THEOREM 7. *Assume that $v(t)$ is a solution of degree 0 of (2) satisfying (11). Let (15) hold. Then (??) has property (A) if and only if*

$$\left(\frac{1}{r(t)} y''(t) \right)' - p(t) y'(t) + \frac{v\big(\tau^{-1}(t)\big) g\big(\tau^{-1}(t)\big)}{v(t)\tau'\big(\tau^{-1}(t)\big)}\, y(t) = 0 \qquad (27)$$

has property (P_0).

Finally Theorems 6 and 7 provide:

THEOREM 8. *Assume that $v(t)$ is a solution of degree 0 of (2) satisfying (11). Let (15) hold. If (27) has property (P_0), then so does (1).*

Remark 2. Our results here present easily verifiable criteria for studied properties of (1). We have reduced properties of general delay trinomial equation to oscillation of the corresponding second order differential equation. Moreover, using M a h f o u d [18] types of comparison theorems, we can examine properties of delay trinomial equation with help of that without deviating argument.

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