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Mathematical Publications

ON THE OSCILLATION OF THE SOLUTIONS TO DELAY AND DIFFERENCE EQUATIONS

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ABSTRACT. Consider the first-order linear delay differential equation

$$x'(t) + p(t)x\big(\tau(t)\big) = 0, \qquad t \geq t_0, \tag{1}$$

where $p, \tau \in C\big([t_0, \infty), \mathbb{R}^+\big)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, and the (discrete analogue) difference equation

$$\Delta x(n) + p(n)x\big(\tau(n)\big) = 0, \qquad n = 0, 1, 2, \dots, \tag{1$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n - 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$. Optimal conditions for the oscillation of all solutions to the above equations are presented.

1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x\big(\tau(t)\big) = 0, \qquad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C\big([t_0, \infty), \mathbb{R}^+\big)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [11], [15], [17], [21]–[26], [28], [29]–[32], [33]–[42], [44], [47]–[52], [54], [55], [59], [60], [66], [73]–[80], [82]–[84], [90] and the references cited therein.

By a solution of the equation (1) we understand a continuously differentiable function defined on $\big[\tau(T_0), \infty\big)$ for some $T_0 \geq t_0$ and such that the equation (1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

The oscillation theory of the (discrete analogue) delay difference equation

$$\Delta x(n) + p(n)x\big(\tau(n)\big) = 0, \qquad n = 0, 1, 2, \dots, \tag{1$'$}$$

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where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \le n-1$ for all $n \ge 0$ and $\lim_{n\to\infty} \tau(n) = \infty$, has also attracted growing attention in the last decades, especially in the case where the delay $n - \tau(n)$ is a constant, that is, in the special case of the difference equation,

$$\Delta x(n) + p(n)x(n-k) = 0, \qquad n = 0,1,2,\dots, \tag*{(1)$''$}$$

where k is a positive integer. The reader is referred to [5]–[10], [12], [13], [16], [18]–[20], [43], [46], [53], [56], [57], [61]–[65], [67]–[72], [81], [85]–[89] and the references cited therein.

By a solution of the equation $(1)'$ we mean a sequence $x(n)$ which is defined for $n \ge -k$ and which satisfies $(1)'$ for $n \ge 0$. A solution $x(n)$ of the equation $(1)'$ is said to be *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is said to be *nonoscillatory*. (Analogously for the equation $(1)''$.)

In this paper our main purpose is to present the state of the art on the oscillation of all solutions to the equation (1), especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s \le \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,\mathrm{d}s < 1,$$

and to (the discrete analogues) the equation $(1)''$ when

$$0 < \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) < 1$$

and the equation $(1)'$ when

$$0 < \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) \le \frac{1}{e} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) < 1.$$

2. Oscillation criteria for equation (1)

In this section we study the delay equation

$$x'(t) + p(t)x\big(\tau(t)\big) = 0, \qquad t \ge t_0, \tag{1}$$

where the functions $p, \tau \in C\big([t_0,\infty), \mathbb{R}^+\big)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \ge t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

The first systematic study for the oscillation of all solutions to the equation (1) was made by Myshkis. In 1950 [58] he proved that every solution of the equation (1) oscillates if

$$\limsup_{t\to\infty}\bigl[t-\tau(t)\bigr]<\infty \quad \text{and} \quad \liminf_{t\to\infty}\bigl[t-\tau(t)\bigr]\liminf_{t\to\infty}p(t)>\frac{1}{e}. \tag{C_1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [44] proved that the same conclusion holds if

$$A:=\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)\,\mathrm{d}s>1. \tag{C_2}$$

In 1979, Ladas [42] established integral conditions for the oscillation of the equation (1) with constant delay. Tomaras [77]–[79] extended this result to the equation (1) with variable delay. Related results see Ladde [49]–[51]. The following most general result is due to Koplatadze and Canturija [37].

If

$$\mathfrak{a}:=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)\,\mathrm{d}s>\frac{1}{e}, \tag{C_3}$$

then all solutions of the equation (1) oscillate; if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)\,\mathrm{d}s<\frac{1}{e}, \tag{N_1}$$

then the equation (1) has a nonoscillatory solution.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit $\lim_{t\to\infty}\int_{\tau(t)}^{t}p(s)\,\mathrm{d}s$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [26] developed new oscillation criteria by employing the upper bound of the ratio $x\bigl(\tau(t)\bigr)/x(t)$ for possible nonoscillatory solutions $x(t)$ of the equation (1). Their result says that all the solutions of the equation (1) are oscillatory, if $0<\mathfrak{a}\le\frac{1}{e}$ and

$$A>1-\frac{\mathfrak{a}^2}{4}. \tag{C_4}$$

Since then several authors tried to obtain better results by improving the upper bound for $x\bigl(\tau(t)\bigr)/x(t)$.

In 1991, Jian [35] derived the condition

$$A>1-\frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}, \tag{C_5}$$

while in 1992, Yu and Wang [83] and Yu, Wang, Zhang and Qian [84] obtained the condition

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}. \tag{C$_6$}$$

In 1990, Elbert and Stavroulakis [23] and in 1991 Kwong [41], using different techniques, improved (C_4), in the case where $0 < \mathfrak{a} \leq \frac{1}{e}$, to the conditions

$$A > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \tag{C$_7$}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C$_8$}$$

respectively, where λ_1 is the smaller real root of the equation $\lambda = e^{\mathfrak{a}\lambda}$.

In 1994, Koplatadze and Kvinikadze [38] improved (C_6), while in 1998, Philos and Sficas [59] and in 1999, Zhou and Yu [90] and Jaroš and Stavroulakis [34] derived the conditions

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})} - \frac{\mathfrak{a}^2}{2}\lambda_1, \tag{C$_9$}$$

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \tag{C$_{10}$}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}, \tag{C$_{11}$}$$

respectively.

Consider the equation (1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p\big(\tau(t)\big)\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [36] and in 2003, Sficas and Stavroulakis [60] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{(1-\mathfrak{a})^2 - 4\Theta}}{2} \tag{2.1}$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta\lambda_1 M}}{\theta\lambda_1}, \tag{2.2}$$

respectively, where

$$\Theta = \frac{e^{\lambda_1\theta\mathfrak{a}} - \lambda_1\theta\mathfrak{a} - 1}{(\lambda_1\theta)^2} \quad \text{and} \quad M = \frac{1 - \mathfrak{a} - \sqrt{(1-\mathfrak{a})^2 - 4\Theta}}{2}.$$

Remark 2.1 ([36], [60]). Observe that when $\theta = 1$, then $\Theta = \frac{\lambda_1 - \lambda_1 \mathfrak{a} - 1}{\lambda_1{}^2}$, and (2.1) reduces to

$$A > 2\mathfrak{a} + \frac{2}{\lambda_1} - 1, \tag{C_{12}}$$

while in this case it follows that $M = 1 - \mathfrak{a} - \frac{1}{\lambda_1}$ and (2.2) reduces to

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\mathfrak{a}\lambda_1}}{\lambda_1}. \tag{C_{13}}$$

In the case where $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{13}) leads to

$$A > \frac{\sqrt{7 - 2e}}{e} \approx 0.459987065 .$$

Note that as $\mathfrak{a} \to 0$, then all the previous conditions (C_4)–(C_{12}) reduce to the condition (C_2), i.e., $A > 1$. However, the condition (C_{13}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (C_{13}) improves all the above conditions when $0 < \mathfrak{a} \leq \frac{1}{e}$ as well. Note that the value of the lower bound on A can not be less than $\frac{1}{e} \approx 0.367879441$. Thus the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\mathfrak{a} = \frac{1}{e}$.

(C_4):	0.966166179,
(C_5):	0.892951367,
(C_6):	0.863457014,
(C_7):	0.845181878,
(C_8):	0.735758882,
(C_9):	0.709011646,
(C_{10}):	0.708638892,
(C_{11}):	0.599215896,
(C_{12}):	0.471517764,
(C_{13}):	0.459987065.

We see that the condition (C_{13}) essentially improves all the known results in the literature.

EXAMPLE 2.1 ([60]). Consider the delay differential equation

$$x'(t) + px\left(t - q\sin^2\sqrt{t} - \frac{1}{pe}\right) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p\,\mathrm{d}s = \liminf_{t\to\infty} p\left(q\sin^2\sqrt{t} + \frac{1}{pe}\right) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p\,\mathrm{d}s = \limsup_{t\to\infty} p\left(q\sin^2\sqrt{t} + \frac{1}{pe}\right) = pq + \frac{1}{e} = 0.46\,.$$

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions (C_4)–(C_{12}) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s = \frac{1}{e}$$

this problem was studied in 1995, by Elbert and Stavroulakis [24], by Kozakiewicz [39], by Li [54], [55] and in 1996, by Domshlak and Stavroulakis [22].

3. Oscillation criteria for equation (1)″

In this section we study the difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \qquad n = 0, 1, 2, \dots, \tag{1$''$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and k is a positive integer.

In 1981, Domshlak [12] was the first who studied this problem in the case where $k = 1$. Then, in 1989, Erbe and Zhang [27] established that all solutions of equation (1)″ are oscillatory if

$$\liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{3.1}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1. \tag{$C_2$$''$}$$

In the same year, 1989, L a d a s, P h i l o s and S f i c a s [46] proved that a sufficient condition for all solutions of the equation $(1)''$ to be oscillatory is that

$$\liminf_{n\to\infty}\sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1}. \tag{C_3}''$$

Therefore they improved the condition (3.1) by replacing the $p(n)$ of (3.1) by the arithmetic mean of $p(n-k),\dots,p(n-1)$ in $(\mathrm{C}_3)''$.

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (3.1) it should be emphasized that, as it is shown in [27], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}},$$

then the equation $(1)''$ has a nonoscillatory solution.

In 1990, L a d a s [43] conjectured that the equation $(1)''$ has a nonoscillatory solution if

$$\sum_{i=n-k}^{n-1} p(i) < \left(\frac{k}{k+1}\right)^{k+1}$$

holds eventually. However, a counterexample to this conjecture was given in 1994, by Y u, Z h a n g and W a n g [86].

It is interesting to establish sufficient oscillation conditions for the equation $(1)''$ in the case where neither $(\mathrm{C}_2)''$ nor $(\mathrm{C}_3)''$ is satisfied.

In 1995, the following oscillation criterion was established by S t a v r o u l a k i s [63].

Theorem 3.1 ([63]). *Assume that*

$$\boldsymbol{\alpha}_0 := \liminf_{n\to\infty}\sum_{i=n-k}^{n-1} p(i) \le \left(\frac{k}{k+1}\right)^{k+1}$$

and

$$\limsup_{n\to\infty} p(n) > 1 - \frac{\boldsymbol{\alpha}_0^2}{4} \tag{3.2}$$

then all solutions of the equation $(1)''$ *oscillate.*

In 2004, the same author [64] improved the condition (3.2) as follows:

Theorem 3.2 ([64]). *If* $0 < \boldsymbol{\alpha}_0 \le \left(\frac{k}{k+1}\right)^{k+1}$, *then either one of the conditions*

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\boldsymbol{\alpha}_0^2}{4} \tag{C_4}''$$

or

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n-1} p(i) > 1 - \boldsymbol{\alpha}_0^k \tag{3.3}$$

implies that all solutions of the equation $(1)''$ *oscillate.*

In 2006 Chatzarakis and Stavroulakis [5] established the following.

Theorem 3.3 ([5]). *If* $0 < \boldsymbol{\alpha}_0 \le \left(\frac{k}{k+1}\right)^{k+1}$ *and*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\boldsymbol{\alpha}_0^2}{2(2-\boldsymbol{\alpha}_0)}, \tag{3.4}$$

then all solutions of the equation $(1)''$ *oscillate.*

Also, Chen and Yu in [6] obtained the following oscillation condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p\,(i) > 1 - \frac{1-\boldsymbol{\alpha}_0-\sqrt{1-2\boldsymbol{\alpha}_0-\boldsymbol{\alpha}_0^2}}{2}. \tag{C_6}''$$

Remark 3.1. Observe that the conditions $(C_2)''$, $(C_3)''$, $(C_4)''$ and $(C_6)''$ are the discrete analogues of the conditions (C_2), (C_3), (C_4) and (C_6), respectively, for the equation $(1)''$.

4. Oscillation criteria for equation (1)′

In this section we study the difference equation

$$\Delta x(n) + p(n)x\big(\tau(n)\big) = 0, \qquad n = 0,1,2,\dots, \tag{1'}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \le n-1$ for all $n \ge 0$ and $\lim_{n\to\infty} \tau(n) = \infty$.

In the case of the equation $(1)'$ with a general delay argument $\tau(n)$, from Chatzarakis, Koplatadze and Stavroulakis [1], it follows the following

Theorem 4.1 ([1]). *If*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) > 1, \tag{C_2}'$$

then all solutions of the equation $(1)'$ *oscillate.*

This result generalizes the oscillation criterion $(C_2)''$. Also Chatzarakis, Koplatadze and Stavroulakis [2] extended the oscillation criterion $(C_3)''$ to the general case of the equation $(1)'$. More precisely, the following theorem has been established in [2].

Theorem 4.2 ([2]). *Assume that*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) < +\infty \tag{3.5}$$

and

$$\boldsymbol{\alpha} := \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) > \frac{1}{e}\,. \tag{$(\mathrm{C}_3)'$}$$

Then all solutions of the equation $(1)'$ *oscillate.*

Remark 4.1. It is to be pointed out that the conditions $(\mathrm{C}_2)'$ and $(\mathrm{C}_3)'$ are the discrete analogues of the conditions (C_2) and (C_3) and also the analogues of the conditions $(\mathrm{C}_2)''$ and $(\mathrm{C}_3)''$ for the equation $(1)'$ in the case of a general delay argument $\tau(n)$.

Remark 4.2 ([2]). The condition $(\mathrm{C}_3)'$ is optimal for the equation $(1)'$ under the assumption that $\lim_{n\to+\infty}\big(n-\tau(n)\big)=\infty$, since in this case the set of natural numbers increases infinitely in the interval $\big[\tau(n), n-1\big]$ for $n\to\infty$.

Now, we are going to present an example to show that the condition $(\mathrm{C}_3)'$ is optimal, in the sense that it cannot be replaced by the non-strong inequality.

Example 4.1 ([2]). Consider the equation $(1)'$, where

$$\tau(n) = [\beta n],\ p(n) = \Big(n^{-\lambda} - (n+1)^{-\lambda}\Big)\big([\beta n]\big)^{\lambda}, \quad \beta\in(0,1),\ \lambda = -\ln^{-1}\beta \tag{3.6}$$

and $[\beta n]$ denotes the integer part of βn.

It is obvious that

$$n^{1+\lambda}\Big(n^{-\lambda} - (n+1)^{-\lambda}\Big) \to \lambda \qquad \text{for} \quad n\to\infty\,.$$

Therefore,

$$n\Big(n^{-\lambda} - (n+1)^{-\lambda}\Big)\big([\beta n]\big)^{\lambda} \to \frac{\lambda}{e} \qquad \text{for} \quad n\to\infty. \tag{3.7}$$

Hence, in view of (3.6) and (3.7), we have

$$\begin{aligned}\liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) &= \frac{\lambda}{e} \liminf_{n\to\infty} \sum_{i=[\beta n]}^{n-1} \frac{e}{\lambda} i\Big(i^{-\lambda} - (i+1)^{-\lambda}\Big)\big([\beta i]\big)^{\lambda}\cdot\frac{1}{i} \\ &= \frac{\lambda}{e} \liminf_{n\to\infty} \sum_{i=[\beta n]}^{n-1} \frac{1}{i} = \frac{\lambda}{e}\ln\frac{1}{\beta} = \frac{1}{e}\end{aligned}$$

or

$$\liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) = \frac{1}{e}\,. \tag{3.8}$$

Observe that all the conditions of Theorem 3.2 are satisfied except the condition $(C_3)'$. In this case it is not guaranteed that all solutions of the equation $(1)'$ oscillate. Indeed, it is easy to see that the function $u = n^{-\lambda}$ is a positive solution of the equation $(1)'$.

As it has been mentioned above, it is an interesting problem to find new sufficient conditions for the oscillation of all solutions of the delay difference the equation $(1)'$, in the case where neither $(C_2)'$ nor $(C_3)'$ is satisfied.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [1] investigated for the first time this question for the equation $(1)'$ in the case of a general delay argument $\tau(n)$ and derived the following theorem.

Theorem 4.3 ([1]). *Assume that* $0 < \boldsymbol{\alpha} \leq \frac{1}{e}$. *Then we have:*

(I) *If*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \left(1 - \sqrt{1-\boldsymbol{\alpha}}\right)^2, \tag{3.9}$$

then all solutions of the equation $(1)'$ *oscillate.*

(II) *If in addition,*

$$p(n) \geq 1 - \sqrt{1-\boldsymbol{\alpha}} \qquad \textit{for all large } n, \tag{3.10}$$

and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \boldsymbol{\alpha}\frac{1-\sqrt{1-\boldsymbol{\alpha}}}{\sqrt{1-\boldsymbol{\alpha}}}, \tag{3.11}$$

then all solutions of the equation $(1)'$ *oscillate.*

Recently, the above result was improved in [3] and [4] as follows:

Theorem 4.4 ([3]).

(I) *If* $0 < \boldsymbol{\alpha} \leq \frac{1}{e}$ *and*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \boldsymbol{\alpha} - \sqrt{1-2\boldsymbol{\alpha}}\right), \tag{3.12}$$

then all solutions of the equation $(1)'$ *oscillate.*

(II) *If* $0 < \boldsymbol{\alpha} \leq 6 - 4\sqrt{2}$ *and in addition,*

$$p(n) \geq \frac{\boldsymbol{\alpha}}{2} \qquad \textit{for all large } n, \tag{3.13}$$

and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3\boldsymbol{\alpha} - \sqrt{4 - 12\boldsymbol{\alpha} + \boldsymbol{\alpha}^2}\right), \tag{3.14}$$

then all solutions of the equation $(1)'$ *are oscillatory.*

Theorem 4.5 ([4]). *Assume that* $0 < \alpha \le -1 + \sqrt{2}$, *and*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right), \qquad (C_6)'$$

then all solutions of the equation $(1)'$ *oscillate.*

Remark 4.3. Observe the following:

(i) When $0 < \alpha \le \frac{1}{e}$, it is easy to verify that

$$\frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} > \alpha \frac{1 - \sqrt{1-\alpha}}{\sqrt{1-\alpha}} > \frac{1 - \alpha - \sqrt{1 - 2\alpha}}{2} > (1 - \sqrt{1-\alpha})^2$$

and therefore the condition $(C_6)'$ is weaker than the conditions (3.11), (3.12) and (3.9).

(ii) When $0 < \alpha \le 6 - 4\sqrt{2}$, it is easy to show that

$$\frac{1}{4}\left(2 - 3\boldsymbol{\alpha} - \sqrt{4 - 12\boldsymbol{\alpha} + \boldsymbol{\alpha}^2}\right) > \frac{1}{2}\left(1 - \boldsymbol{\alpha} - \sqrt{1 - 2\boldsymbol{\alpha} - \boldsymbol{\alpha}^2}\right),$$

and therefore in this case and when (3.13) holds, inequality (3.14) improves the inequality $(C_6)'$ and especially, when $\boldsymbol{\alpha} = 6 - 4\sqrt{2} \simeq 0.3431457$, the lower bound in $(C_6)'$ is 0.8929094 while in (3.14) it is 0.7573593.

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