Organon F 32 (1) 2025: 92–106 ISSN 2585-7150 (online)
https://doi.org/10.31577/orgf.2025.32105

DISCUSSION NOTE

Rectification Note to "Riemann's Philosophy of Geometry and Kant's Pure Intuition"

(Organon F, 31(2), 114–140)

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I would like to extend my gratitude to those who have given me the opportunity to present my perspective on this matter.

The philosophy of geometry is a highly specialized field of study. More specifically, research focusing on Riemann's philosophy of geometry is exceptionally limited. Of course, this cannot justify inappropriate use of others' work, it will be observed in the details provided, many of the analyses (if not all) rely on a shared set of interpretations regarding Riemann's *Habilitationsvortrag.* As a result, it is exceedingly challenging to introduce entirely novel claims or terminology. In this regard, based on my observations in the literature, similarities are often unavoidable within the scope of what is considered "common knowledge." What is common knowledge? According to *The Harvard Guide to Using Sources*, it is defined as follows: "The only source material that you can use in an essay without attribution is material that is considered common knowledge and is therefore not attributable to one source. Common knowledge is information generally known to

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an educated reader, such as widely known facts and dates, and, more rarely, ideas or language."[1]

One should be careful to distinguish between specific details, claims, and views, and what is considered common knowledge—characterized by its cumulative, general, and widely shared nature. This pertains to the content. Additionally, one should be cautious about similarities in form, such as structural and grammatical patterns. As the simple yet powerful motto states, "Correlation does not imply causation."

Therefore, apart from content-related issues—which largely rely on common knowledge, as I will demonstrate—the other examples provided by the anonymous scholar pertain to grammatical structures.

Particulars

Arguably, Riemann and Gauss are two of the most important figures in the field of geometry; their ideas were groundbreaking. It is not uncommon for discussions about revolutionary concepts, claims, and approaches within a specific area to lack significant expansion or diversification.

I have been in ongoing communication with prominent researchers in the field, including those with the scholars I am accused of not properly citing. I have been in communication with them since the period of my master's degree, including during the preparation process of this paper, and I am grateful to them. On the other hand, I only heard of names like Windham—whom the anonymous researcher claims I cited improperly—for the first time. I was not even aware that this scholar worked in this field.

Different versions of the paper were reviewed by the prominent researchers. Before being submitted to *Organon-F*, the paper was also sent to two other well-known international journals in the fields of philosophy and history of science.

In fact, during the revision process of the paper, I again contacted one of the researchers in question regarding a reference for a piece of information

[1] https://usingsources.fas.harvard.edu/exception-commonknowledge#:~:text=Common%20knowledge%20is%20information%20generally,and%20must%20always%20be%20cited. (Accessed 16.10.2024).

I used in the paper. All these details, along with evidence, have already been presented in the letter previously sent to the editorial board of *Organon-F*.

My detailed explanations, along with examples from the literature, are provided in the continuation of the letter.

DÇ (PAGE 125): In the mathematical part of his *Habilitationsvortrag*, Riemann follows Gauss' most fundamental steps, by extending Gaussian concepts and results for surfaces to n -dimensional manifolds, such as the measure of curvature and some properties of geodesic lines. Like Gauss, Riemann's approach is metric; the concept of distance plays a fundamental role both in the theory of curved surfaces and in Riemannian manifolds; in addition, the essential properties of manifolds are expressed by means of the linear element.

Original Rossana Tazzioli: "In his memoir Riemann follows Gauss's fundamental steps, by extending Gaussian concepts and results for surfaces to n-dimensional manifolds, such as the measure of curvature and some properties of geodesic lines. Riemann's approach, as that of Gauss, is metric; in fact, the concept of distance plays the fundamental role whether in the theory of curved surfaces or in Riemannian theory of manifolds; moreover, the essential properties of manifolds are expressed by means of the linear element."

Explanation: It is well-known that Gauss developed a method to describe the geometry of surfaces using what we now call the metric tensor. This tensor encodes information about distances and angles on a surface. Riemann later introduced the concept of a Riemannian manifold, a generalization of surfaces to spaces of any dimension. He extended the notion of a metric from surfaces to arbitrary spaces, introducing what is now known as the Riemannian metric. This innovation allows for the measurement of distances and angles in spaces of any dimension, whether they are curved or flat. To the best of my knowledge, there is no alternative approach to the metric other than the one I have described, which directly traces Riemann's ideas back to Gauss.

It is common to analyze the relationship between Riemann and Gauss using the conceptual resources mentioned above: the role of distance, the line element, curved surfaces, geodesic lines, and similar concepts. These ideas, among others, were inherited by Riemann from Gauss. There are numerous sources that illustrate these connections. For instance, Olivier Darrigol's 2015 paper frequently addresses the underlying relationships between the methodologies and conceptual frameworks of Gauss and Riemann. I will quote a few key points from the paper to highlight these connections.

In the second section, I use this Gaussian background to analyze Riemann's results regarding the curvature of a manifold in his habilitation lecture on the one hand, and his results regarding the transformation properties of quadratic differential forms in the *Commentatio* on the other hand (a *quadratic differential form* is an expression of the type…) (2015, 48-49).[2] See another example where V. F. Kagan presents and explains the line element (mentioned seven times in the article) and other conceptual resources used to define how Riemann generalizes Gauss's metric approach[3]. Kagan also elaborates on the generalization process:

> All their originality notwithstanding, Riemann's ideas are an extension of the methods of investigation of surfaces presented by Gauss in his *Disquisitiones generales* [4] of 1827. Gauss's key idea is that a point on a surface (to be sure, in ordinary Euclidean space) is determined by two coordinates x^1 and x^2 (this is modernized symbolism), and a line element is expressed in terms of a given positive definite quadratic form in the differentials of these coordinates. Specifically (again using modem symbolism) (2005, p.80).

You will notice similar points. It is well-known that both Riemann's and Gauss's approaches are metric-based. It is also widely recognized that both make use of curvature, geodesic lines, and similar concepts. Additionally, the concept of the line element is well-established. Therefore, the conclusion is that Riemann recognized the path Gauss had opened and followed the "fundamental steps" Gauss had already taken. It is generally agreed that

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2 Darrigol, O. (2015). The mystery of Riemann's curvature. *Historia Mathematica*, 42 (1): 47-83.

3 V. F. Kagan, (2005). Riemann's Geometrical Ideas. *The American Mathematical Monthly*, Vol. 112, No. 1, pp. 79-86).

Riemann followed the path laid out by Gauss, relied on the conceptual framework he inherited from him, and built his own methodology upon it.

DÇ: Developing this approach enabled Riemann to investigate the links between different laws of nature-knowledge of which is based on the exactness of our description of phenomena in infinitesimal regions. Gaining knowledge of the external world from the behaviour of infinitesimal parts constitutes the backbone of Riemann's research program.

Rossana Tazzioli: This approach is typical of all Riemann's work and leads him to investigate links between different laws of nature, knowledge of which is based on the exactness of our description of phenomena in infinitesimal regions. As Hermann Weyl wrote in *Space, time, matter*, "the principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry."

Explanation: Riemann's philosophy of geometry is also closely related to physics. One of the key starting points for analyzing this relationship is the use of infinitesimals and the laws of nature. For example, W. Ehm emphasizes points similar to those I focus on in my paper. Riemann too, wants to circumvent mysterious actions at a distance and postulates the existence of a space-filling stuff of which he assumes that it behaves like anincompressible homogeneous fluid without inertia. He largely avoids using the term aether and rather speaks of 'stuff' (Stoff), but he clearly adopts the aether hypothesis. Often he speaks of Stofftheilchen if he wishes to indicate that *infinitesimally small portions* of the aether are being considered (emphasis added, 2010, p. 148).

See another example from Papadopoulos, who discusses the importance of the infinitely small and large in Riemann's research program: "Likewise, Riemann's speculations on the infinitely small and the infinitely large go beyond the mathematical and physical setting, and they had a non-negligible impact on philosophy"[4] (Papadopoulos 2017, p.1).

See Papadopoulos quoting Klein:

[4] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

> I must mention, first of all, that Riemann devoted much time and thought to physical considerations. Grown up under the tradition which is represented by the combinations of the names of Gauss and Wilhelm Weber, influenced on the other hand by Herbart's philosophy, he endeavored again and again to find a general mathematical formulation for *the laws underlying all natural phenomena.* [...] The point to which I wish to call your attention is that these physical views are the mainspring of Riemann's purely mathematical investigations (emphasis added, 2017, p.8).[5]

Elsewhere, Papadopoulos directly quotes Riemann himself in the context of the laws of nature:

> Right after the submission of my HabilitationsschriftindexRiemann! habilitation text I resumed my investigations into the coherence of the *laws of Nature* and got so involved in it that I could not tear myself loose. The continuing preoccupation with it has become bad for my health, in fact, right after New Year's my usual affliction set in which such persistence, that I could only obtain relief through the strongest remedies. As a result I felt very ill, felt unable to work, and sought to again put my health in order through long walks (emphasis added, 2017, p.37).[6]

In fact, it is easy to provide numerous examples showing that one of the fundamental issues in Riemann's research program was to delve into the details of nature and the laws governing it. In this regard, W. Ehm quotes Riemann: "[The] purpose [of the paper] is to penetrate the inner of nature, beyond the foundations of astronomy and physics layed by Galilei and Newton." (Riemann quoted in Ehm, 2010, p.147).[7]

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[5] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

[6] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

[7] Ehm, W. (2010). Broad views of the philosophy of nature: Riemann, Herbart, and the "matter of the mind." *Philosophical Pschhology*, Volume 23, Issue 2, pp. 141-162.

In another quotation that illustrates the significance of discovering the laws of nature within the context of Riemann's research program, L. Ji quotes Klein:

> He [Riemann] endeavored again and again to find a general mathematical formulation for the laws underlying all natural phenomena… these physical views are the mainspring of Riemann's purely mathematical investigations.[8](emphasis added, Ji, 2017, p. 167).[9]

DÇ: Accordingly, Riemann set himself two tasks: The first (a philosophical task) was to define a manifold extension. The second (an empirical task) was to give definitions of intrinsic curvature and measure determined from within extension.

Original Banks: The first task is to define a manifold extension. The second task is to give definitions of intrinsic curvature and measure determined from within this extension, say by introducing rigid meter sticks or light beams.

Explanation: Indeed, in the abstract of the paper, I already provide hints about the philosophical and empirical tasks: "The aim of this paper is twofold: first to explicate how Riemann's philosophy of geometry is organized around the concept of manifold. Second, to argue that Riemann's philosophy of geometry does not dismiss Kant's spatial intuition." What I want to emphasize by saying this is that in the article, I rely on Riemann's methodology, which is built upon philosophy, mathematics, and physics through the concept of the manifold. Papadopoulos puts this idea by saying that "Roughly speaking, the first part is philosophical, the second one is mathematical, and the third one deals with applications to physics. But to some extent philosophy and physics are present in the three parts" (2017, p. 27).[10]

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[8] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

[9] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

[10] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

Anyone familiar with Riemann's interest in philosophy would explain the methodology of his 1854 paper as relating the concept of a manifold to its philosophical and mathematical context, before transitioning to its physical applications. For instance, see the following: "For B. Riemann it was *an intricate task to formulate his general concept of manifolds (discrete or continuous) in 1854*. At that time, it was technically impossible to give a formal definition of topological spaces and to specify manifolds among them. Not even a general concept of set was around; Riemann's step rather contributed to bring it about (Ferreiros [Ferreiros 1999]. Around the turn from the 19th to the 20th century the situation started to change [Scholz 1999]"[11](emphasis added). E. Scholz also explains that "On the other hand Herbart's epistemology and his ideas on the relationship between philosophy and sciences do seem have influenced Riemann and thus Riemann's perception of *the task of mathematics*" (p.427).[12] He goes on to say that "In fact, it is precisely for this reason that Riemann's approach to mathematics is sometimes referred to as 'conceptual mathematics'" (Ibid.). [13]

In another paper Scholz underlines the following:

> As is well known, Riemann organized his approach to geometry around the new concept of manifold (Mannigfaltigkeit) which for obvious reasons he could not define in a mathematical technical sense. He therefore did it in a semi-philosophical way, drawing consciously and cautiously upon hints by C. F. Gauss who had spoken geometrically about complex numbers (Gauss 1831) and J. F. Herbart who had argued for the use of geometrical imagery in all kind of concept formation, his so-called serial forms (Reihenformen) (emphasis added, Scholz, 2005, p.22).[14]

[11] http://www.map.mpim-bonn.mpg.de/Axiomatization_of_the_manifold_concept (Accessed: 16.10.2024).

[12] Scholz, E. (1982). Herbart's Influence on Bernhard Riemann. *Historia Mathematica*, Volume 9, Issue 4, pp. 413-440.

[13] Scholz, E. (1982). Herbart's Influence on Bernhard Riemann. *Historia Mathematica*, Volume 9, Issue 4, pp. 413-440.

[14] Scholz, E. (2005). Riemann's Vision of a New Approach to Geometry. In *1830-1930: A Century of Geometry,* pp. 22-34.

The second task (i.e., the empirical task) is directly related to the first. This is already evident in Riemann's own article:

> ... Now it seems that the empirical concepts on which *the metrical determinations of space* are founded, namely, the concept of a rigid body and that of a light ray, are not applicable in the infinitely small; it is therefore quite conceivable that *the metrical relations of space* in the infinitely small do not agree with the assumptions of geometry; and indeed we ought to hold that this is so if phenomena can thereby be explained in a simpler fashion (emphasis added, Riemann, quoted in Boi, p.199).

Hence, it is evident that Riemann's program is related to a) the concept of the manifold (philosophical task), and b) the determination of metrical relations within extension.

DÇ: Riemann discusses the problem of what he calls 'multiply extended magnitude' in his famous lecture "On the Hypotheses Which Lie at the Foundation of Geometry." Riemann's introduction clearly shows that he saw himself involved in a philosophical as well as mathematical enterprise.

Original Banks: "Bernhard Riemann discusses the problem of what he calls multiply extended magnitude in his famous lecture "On the Hypotheses that Lie at the Foundation of Geometry.'"

Nowak: "Riemann's introduction made it clear that he saw himself involved in a philosophical as well as a mathematical enterprise."

Explanation: The concept of a manifold is also referred to as a multiply extended magnitude. Riemann articulates this fundamental concept in his "On the Hypotheses Which Lie at the Foundation of Geometry." Erhard Scholz's 1982 article has already put the idea that the Riemann's concept of manifold is a "semi-philosophical" in character. Also, Laugwitz (1999)[15], Ferreirós and several other scholars have pointed out that Riemann's approach in his lecture is philosophical, mathematical, and physical. That is

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[15] Laugwitz, D. (2008). *Bernhard Riemann, 1826-1866: Turning Points in the Conception of Mathematics*. Birkhäuser.

why Ferreirós refers to Riemann's *Habilitationsvortrag* as the "Magic Triangle" (2006, p.67).[16]

See Freudenthal regarding the philosophical dimension of Riemann's geometry:

> one of the most profound and imaginative mathematicians of all time, he had a strong inclination to philosophy, indeed, was a great philosopher. Had he lived and worked longer, philosophers would acknowledge him as one of them[17] (Freudenthal, 1975, p. 448).

DÇ: Riemann's main concern was construction of space, rather than construction in space.

Nowak: "First, Herbart's constructive approach to space, already cited, mirrored the content of Riemann's reference to Gauss in that both discussed construction of spaces rather than construction in space."

This section relates to a conclusion I reached as a result of discussions conducted via email with another prominent philosopher in the field (the details of which, along with supporting evidence, have been submitted to the journal's editorial board).

I chose to mention Nowak in a controlled manner because, in discussions with a prominent philosopher of geometry in the field, it was agreed that his views are no longer considered relevant, and his paper was rather inconclusive. On the other hand, anyone who has worked to some extent on Riemann's 1846 paper will see that his primary concern was the construction of space. The concept of a manifold is primarily related to the construction of space, not to specific geometric and topological objects within a given space.

DÇ: In Herbart's view, experience shows us properties and bundles of properties, while the underlying reality must be searched for within the things to which properties are ascribed. This distinction between the phenomena and a more stable underlying reality, and an investigation of the

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[16] Ferreiros, J and Gray, J. (2006). *The Architecture of Modern Mathematics: Essays in History and Philosophy.* Oxford University Press.

[17] Freudenthal, H. (1975). Riemann, Georg Friedrich Bernhard. In: *Dictionary of Scientific Biography* vol. 11. New York, 447-456.

relationship between them, is essential in Riemann's own reflections about the epistemology of science.

Original Scholz: In the first place, according to Herbart, experience shows us properties and bundles [Complexionen] of properties, theunderlying reality of which must first be sought in things to which the properties are ascribed....... The distinction however between the phenomena and a more stable underlying reality, with an intense relationship between both became an essential point in Riemann's own reflections about the epistemology of science.

Explanation: Herbart himself explains that:

> Are the forms of experience given? Yes indeed they are given, although only as determinations of the manner in which sensations are bound up together. Were they not given, we could not only sunder them from sensation in such a way that the sensed could occur completely isolated, without any connection; rather we could also, at pleasure, see different shapes, hear other time intervals; similarly we could put things together arbitrarily out of properties and change them (Herbart quoted in Banks, 2005, p. 209).[18]

Someone who recognizes the empiricist aspects of Herbart's philosophy would make similar observations. Herbart's initial adherence to Kant also enables analyses concerning the relationship between empiricism and the unchanging reality found here. With this understanding, Riemann's effort to grasp the laws of nature and nature itself from within, through infinitesimals, can also be linked. Similar interpretations have also been made by other researchers in the field:

> Another important element that Riemann inherited from Herbart was a developmental, genetic understanding of science. Far from the usual idea that there exists (in some Platonic realm) a ready-made theory of everything, in his view all concepts of natural science, and of mathematics in particular, have evolved gradually

[18] Banks, E. (2005). Kant, Herbart and Riemann. *Kant-Studien*, 96(2), pp. 208-234.

> from older explanatory systems. Scientific theories are for Riemann the outcomes of a process of gradual transformation of concepts, starting from the basic ideas of object, causality, and continuity. Development takes place under the pressure of contradictions or else implausibilities [*Unwahrscheinlichkeiten*] revealed by unexpected observations – unexpected in light of the hypotheses proposed by reflection at some particular stage (Ferreiros, 2006, p. 77).[19]

> Riemann and Herbart tended to prefer a Leibnizian view on space, seeing space not as a basic entity, but as an expression of the relations between physical entities (see end of Section 3.1). It is particularly interesting that this viewpoint led Herb- art to think geometrically about all kinds of subjects, which again was noticed and further developed by Riemann. This is a feature of Herbart's work that makes it vaguely reminiscent of modern mathematics, and perhaps it has not been sufficiently appreciated (Ferreiros, 2006, p.75).[20]

DÇ: In *Habilitationsvortrag*, Riemann generalizes the Gaussian theory of curved spaces to n -dimensions. Such manifolds are characterized by the fact that each point within them can be uniquely specified by n real numbers. The introduction of the concept of distance into a manifold follows the Gaussian model. Analogously to the two-dimensional case, infinitesimal distances are expressed by processing differentials given in terms of some internal coordinate system, u, with the help of the metric tensor g ij. Thus, Riemann arrives at a formula that is identical to the Gaussian expression for the surfaces.

Original Carrier: In 1854, Bernhard Riemann generalized the Gaussian theory of curved surfaces to n-dimensional manifolds. Such manifolds are characterized by the fact that each point in them can be uniquely specified by n real numbers. The introduction of the concept of length or dis-

[19] Ferreiros, J and Gray, J. (2006). The Architecture of Modern Mathematics: Essays in History and Philosophy. Oxford University Press.

[20] Ferreiros, J and Gray, J. (2006). The Architecture of Modern Mathematics: Essays in History and Philosophy. Oxford University Press.

tance into such a manifold closely follows the Gaussian model. Quite analogous to the two-dimensional case we express infinitesimal lengths by processing coordinate differences (or differentials) as given in terms of some internal coordinate system u with the help of the metric tensor gik. One arrives at a formula that is identical, mutatis mutandis, with the corresponding Gaussian expression for surfaces.

Explanation: This part is technical and common. One of the most important results of the Riemann's 1846 lecture is his generalization of Gaussian view of curved spaces to n-dimensions. This is a *technical fact.*

> [...] he [Riemann] expresses the square of a line element by means of a positive definite quadratic form in the differentials dx^i of the coordinates whose co- efficients are functions of the coordinates x^i .This *relation is not just an extension of Gauss's formula to an n-dimensional manifold.* Rather, it introduces the completely new idea of determining the metric on a manifold by specifying it in an infinitely small portion of that manifold... (Kagan, 2005, p. 81).[21]

Indeed, it can be said that one of the most important developments Riemann brought about was this generalization. What I mean is that this is a paradigmatic example of common knowledge that illustrates how Riemann reaches n-dimensions by following Gauss's footsteps. This technical aspect can be seen in any discussion of Riemannian spaces:

> The theory of Riemannian spaces. A Riemannian space is an n-dimensional connected differentiable manifold M^n on which a differentiable tensor field g of rank 2 is given which is covariant, symmetric and positive definite. The tensor g is called metric-tensor. Riemannian geometry is a multi-dimensional generalization of the intrinsic geometry (cf. İnterior geometry) of two-dimensional surfaces in the Euclidean space E^3. The metric of a Riemannian space coincides with the Euclidean metric of the domain under consideration up to the first order of smallness. The

[21] V. F. Kagan, (2005). Riemann's Geometrical Ideas. *The American Mathematical Monthly*, Vol. 112, No. 1, pp. 79-86).

> difference between these metrics is (locally) estimated by Riemannian curvature- a multi-dimensional generalization of the concept of the Gaussian curvature of a surface in E^3.[22]

See also the following which explains the generalization procedure in a less technical sense:

> Gauss's theorem suggests that one could see the surface as an independent curved manifold and then to generalize this concept to higher dimensions via "the concept of a multiply extended magnitude," … which is what Riemann did. These concepts also helped him to generalize to higher dimensions Gauss's concept of curvature. To do so yer another new concept, another great invention of Riemann, the tensor of curvature, and a new form of differential calculus, tensor calculus on manifolds, a generalization of differential calculus[23] (Plotnitsky, p.344-345, 2017).

Conclusion

Of course, points of differentiation and approaches should be conveyed according to the relevant references. On the other hand, in studies on the philosophy of geometry and Riemann, some themes, fundamental concepts, Riemann's innovations, and observations, claims, and concepts about how he achieved them have now become mostly common knowledge.

One could establish connections between various relations and believe that the relations they themselves connect are factual. However, there is a significant difference between believing something and justifying it. As I have shown above, the allegations put forward by the anonymous scholar are unjustifiable. One may notice some similarities between words, names, claims, or views. However, the fact that there are *similarities* in how the

[22] https://encyclopediaofmath.org/wiki/Riemannian_geometry (Accessed: 16. 10. 2024).

[23] Ji, Lishen et. al. (2017). *From Riemann to Differential Geometry and Relativity.* Springer.

issues are presented does not mean they are the *same*; fundamentally, *similarity* does not equal *equality*, especially given that there is limited literature regarding Riemann that is common to all scholars.

Sincerely,
Dinçer Çevik