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ON THE DIOPHANTINE EQUATION $11 + 2^{x+2} + (7)3^y = z^2$

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ABSTRACT. In this note, we investigate solutions of the Diophantine equation
$$11 + 2^{x+2} + (7)3^y = z^2 \qquad (x, y, z) \in \mathbb{N}^3.$$

1. Introduction

In 1844, Catalan in [2] conjectured that $(a, b, x, y) = (3, 2, 2, 3)$ is a unique solution for the exponential Diophantine equation
$$a^x - b^y = 1 \qquad (a, b, x, y) \in \mathbb{N}^4 \quad \text{and} \quad \min(a, b, x, y) > 1.$$
The proof of this conjecture is due to Mihailescu [3, 2004]. In 2011, Suvarnamani [1] found all non-negative integer solutions of diophantine equation
$$2^x + p^y = z^2.$$

In 2012, Chotchaisthit [4], using this theorem, found all solutions for the Diophantine equation $4^x + p^y = z^2$. In 2014, Yahui Yu and Xiaoxue Li [5] investigated solutions of equations of type
$$2^x + b^y = c^z, \qquad x, y, z \in \mathbb{N},$$
where b and c are fixed coprime odd positive integers with $\min(b, c) > 1$.

In this note, we give a proof of the following result.

2. Main theorem

Theorem 2.1. *The Diophantine equation* $11 + 2^{x+2} + (7)3^y = z^2$ *has a unique solution given by* $(x, y, z) = (3, 1, 8)$.

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P r o o f. Let x, y be non-negative integers and z be an even non-negative integer such that

$$11 + 2^{x+2} + (7)3^y = z^2.$$

We need to consider several cases:

Case 1. If $y = 0$. Then, $2^{x+2} + 18 = z^2$. If we take $z = 2k$, we obtain $2^{x+1} = 2k^2 - 9$. But $2^{x+1} = 0 \pmod 2$. This is a contradiction.

Case 2. If $y = 1$. Then, the equation becomes $2^{x+2} + 2^5 = z^2$. The solution of this equation is $(x, z) = (3, 8)$ (see [1]). Thus, the triplet $(3, 1, 8)$ is a solution of our equation.

Case 3. If $y > 1$. We have $11 + 2^{x+2} + (7)3^y = 4(1 + 2^x) + 7(1 + 3^y) = z^2$. Since z is even, then, y is odd because $7(1 + 3^{2l+1}) = 0 \pmod 4$. Next, we can write $4(1 + 2^x) + 7(1 + 3^y)$ as $2^j c + 2^{j'} c'$. Then, we deduce that $j = 2$ and $1 + 3^y = 2^{j'}$. The unique solution of the last equation is $(y, j') = (1, 2)$. Hence, $(3, 1, 8)$ is the unique solution.

Corollary 2.1. $(3, 1, 2)$ *is a unique non-negative integer solution* (x, y, t) *for the Diophantine equation* $11+2^{x+2}+(7)3^y = t^6$, *where* x,y *and* t *are non-negative integers.*

P r o o f. Let x,y and t be non-negative integers such that $11 + 2^{x+2} + (7)3^y = t^6$. Let $z = t^3$. Then $11 + 2^{x+2} + (7)3^y = z^2$. By Theorem 2.1, we obtain that $(x, y, z) = (1, 3, 8)$. Then $t^3 = z = 8$. Hence, $t = 2$. □

3. Open problem

Let a, b and c be positive odd prime numbers such that $(a, b, c) = 1$. We may ask for the set of all solutions (x, y, z) for the Diophantine equation

$$a + b2^x + c3^y = z^2,$$

where x, y and z are non-negative integers.

□

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