# ON AN APPLICATION OF A HIGHER ORDER INVERSION THEOREM TO CERTAIN INTEGRAL EQUATIONS AND BOUNDARY VALUE PROBLEMS 

Piotr FijaŁkowski


#### Abstract

In this paper, we consider examples of the application of a certain higher order inversion theorem to demonstrate the existence of a solution of a certain class of integral equations and a class of boundary value problems equivalent to these equations.


## 1. Introduction

The classical result on the local invertibility of a mapping between Banach spaces is the following well-known Local Inversion Theorem:
Theorem 1. Let $X, Y$ be Banach spaces. Assume $T: X \supset U \rightarrow Y$ is a $C^{1}$-mapping from an open neighbourhood $U$ of a point $x_{0} \in X$ into $Y$.

If the Fréchet derivative $T^{\prime}\left(x_{0}\right): X \rightarrow Y$ of $T$ in $x_{0}$ is an invertible mapping from $X$ onto $Y$, then $T$ is a local diffeomorphism in a neighbourhood of $x_{0}$, this means, there exist neighbourhoods $V \subset U$ of $x_{0}$ and $W$ of $T\left(x_{0}\right)$ such that the mapping $T \mid V$ is a diffeomorphism $V$ onto $W$.

If $T$ is a $C^{k}$-mapping in a certain neighbourhood of $x_{0}$, then the inverse mapping is of the $C^{k}$-class in a certain neighbourhood of $T\left(x_{0}\right)$.

The classical result on the global invertibility of a mapping between Banach spaces is the following Hadamard-Levy's theorem (see [7] and [11] for example):
Theorem 2. Let $X$ and $Y$ be Banach spaces and $T: X \rightarrow Y$ a $C^{1}$-mapping. Suppose that, for any $x \in X, T^{\prime}(x)$ is an isomorphism $X$ onto $Y$ and

$$
\sup _{x \in X}\left\|\left(T^{\prime}(x)\right)^{-1}\right\|<\infty
$$

Then the mapping $T$ is a diffeomorphism $X$ onto $Y$.

[^0]Because of the importance of the topic, the global invertibility was considered by many authors. For example, in [2, there are described sufficient conditions for a locally Lipschitz mapping $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ to be invertible. In [8], there are given some sufficient conditions for a mapping from the Banach space to the Hilbert space to be a diffeomorphism. As well, the homeomorphism between Banach spaces is considered in [3, 9] and [13].

There are many examples of application of local and global inversion theorems to demonstrate solutions of differential equations, including the initial and boundary value problems. Simple examples of this kind are Examples 1.4 and 1.5 in [1], pages 34-35, where Theorem 1 is applied. The same book contains a more complicated example of demonstrating existence of solutions of Dirichlet problems using a specific global inversion theorem (1], pages 61-78).

As well, the papers 10 and [11] deal with an application of global inversion theorem to differential equations.

In [4], [5] and [6], there is described a local and a global inversion theorem for mappings with a singular point in which several derivatives vanish. We shall apply them to integral equations and boundary value problems.

## 2. Local and global inversion theorems for a mapping with a singular point

We use the following local inversion theorem proved in [4, Theorem 1]:
Theorem 3. Let $X, Y$ be Banach spaces, and n a non-negative integer. Suppose $T: X \supset U \rightarrow Y$ is a $C^{2 n+1}$-mapping from an open neighbourhood $U$ of a point $x_{0} \in X$ into $Y$. Let, for $1 \leq k \leq 2 n$, the derivatives of order $k$ satisfy the condition

$$
T^{(k)}\left(x_{0}\right)=0
$$

Let, for $x \in X$,

$$
f(x):=T^{(2 n+1)}\left(x_{0}\right) \cdot x^{(2 n+1)} /(2 n+1)!
$$

and suppose that $f$ maps $X$ onto $Y$ and that, for a certain constant $C>0$,

$$
\begin{equation*}
C\|x\|^{2 n}\|h\| \leq\|f(x+h)-f(x)\| \quad \text { for } \quad x, h \in X \tag{1}
\end{equation*}
$$

Then $T$ is a local homeomorphism in a neighbourhood of $x_{0}$, this means there exist neighbourhoods $V \subset U$ of $x_{0}$ and $W$ of $T\left(x_{0}\right)$ such that $T \mid V$ is a homeomorphism $V$ onto $W$.

The following theorem proved in [5, Theorem 1.6, p. 21] describes conditions equivalent to those from the previous theorem.

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Theorem 4. Let $X, Y$ be Banach spaces. Assume $\tilde{f}: X^{2 n+1} \rightarrow Y$ is continuous $(2 n+1)$-linear mapping. Let us consider the mapping $f: X \rightarrow Y$,

$$
f(x):=\tilde{f}(x, x, \ldots, x)
$$

If $f$ satisfies condition (11) and $f(X)=Y$, then there exists a constant $C_{1}>0$ such that

$$
\begin{equation*}
C_{1}\|x\|^{2 n}\|h\| \leq\left\|f^{\prime}(x) . h\right\| \tag{2}
\end{equation*}
$$

for $x \in X$ and

$$
\begin{equation*}
f^{\prime}(x) X=Y \quad \text { for } \quad x \neq 0 \tag{3}
\end{equation*}
$$

$A$ constant $C_{1}$ satisfying the above condition may be chosen as

$$
\begin{equation*}
C_{1}=C \tag{4}
\end{equation*}
$$

where $C$ is a constant from condition (1).
If the mapping $f$ is injective and satisfies conditions (21) and (3), then $f$ is invertible and satisfies condition (1) with a constant

$$
\begin{equation*}
C=2^{-2 n /(2 n+1)}(2 n+1)^{-1} C_{1} C_{2}^{2 n /(2 n+1)} C_{3}^{-2 n /(2 n+1)}, \tag{5}
\end{equation*}
$$

where $C_{2}, C_{3}$ are positive constants in the estimations

$$
\begin{equation*}
C_{2}\|x\|^{2 n+1} \leq\|f(x)\|, \quad x \in X \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f(x)\| \leq C_{3}\|x\|^{2 n+1}, \quad x \in X \tag{7}
\end{equation*}
$$

In [6, Theorem 4], there was proved the following global inversion theorem for the mapping $T: X \rightarrow Y$ with a singular point of the type described above:

Theorem 5. Let $X, Y$ be Banach spaces and $x_{0} \in X$. Suppose $T: X \rightarrow Y$ is a $C^{2 n+1}$-mapping for a non-negative integer $n$ satisfying all assumptions of Theorem 3. Writing $T(x)$ according to the Taylor's formula as

$$
T(x)=T\left(x_{0}\right)+f\left(x-x_{0}\right)+r\left(x-x_{0}\right)
$$

with

$$
f(x):=T^{(2 n+1)}\left(x_{0}\right) \cdot x^{(2 n+1)} /(2 n+1)!
$$

suppose that

$$
\begin{equation*}
\sup _{x \in X \backslash\{0\}}\left\|r^{\prime}(x)\right\|\|x\|^{-2 n}<C_{1} C_{2}^{2 n /(2 n+1)} C_{3}^{-2 n /(2 n+1)}, \tag{8}
\end{equation*}
$$

where $C_{1}, C_{2} C_{3}$ are the constants from estimations (21), (6) and (7).
Then $T$ is an homeomorphism between $X$ and $Y$.
The proof of the theorem is based on the Theorems 2 and 3 .
Strong assumptions of Theorem 5make it difficult to find a non-trivial example of mapping fulfilling them but we shall make it in the next section.

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## 3. Applications

In the first step, we demonstrate a class of mappings which satisfy the assumptions of Theorem 3 for third order that means for $n=1$ which seems to be difficult in comparison with other works.

Theorem 6. Let $G:[a ; b] \times[a ; b] \rightarrow \mathbb{R}$ be a measurable function with the following properties:

$$
\begin{equation*}
0<K_{1} \leq G(t, s) \leq K_{2}, \quad \text { for } \quad t, s \in[a ; b] \text { (a.e.) } \tag{9}
\end{equation*}
$$

with

$$
\begin{equation*}
K_{1}>2(\sqrt{2}-1) K_{2} \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{a}^{b} x(t)\left(\int_{a}^{b} G(t, s) x(s) d s\right) d t \geq 0 \tag{11}
\end{equation*}
$$

for any $x \in L^{2}([a ; b], \mathbb{R})$.
Then the mapping

$$
L^{2}([a ; b], \mathbb{R}) \ni x \rightarrow x \int_{a}^{b} G(\cdot, s) x^{2}(s) d s+o\left(\|x\|^{3}\right) \in L^{2}([a ; b], \mathbb{R}) \text { (a.e.) }
$$

fulfills assumptions of Theorem 3 for $x_{0}=0$ and

$$
f(x)=x \int_{a}^{b} G(\cdot, s) x^{2}(s) d s(\text { a.e. })
$$

Remark 1. Before proof, we observe that the restrictive assumption (11) is fulfilled by a function of the form

$$
G(t, s)=\sum_{i=1}^{k} a_{i} H(t) H(s)(a . e .)
$$

with a measurable and bounded (a.e.) function $H$ and $a_{i} \geq 0$.
Proof. We have:

$$
f^{\prime}(x) . h=h \int_{a}^{b} G(\cdot, r) x^{2}(r) d r+2 x \int_{a}^{b} G(\cdot, s) x(s) h(s) d s(\text { a.e.) }
$$

and

$$
\begin{align*}
\left\|f^{\prime}(x) . h\right\|^{2}= & \int_{a}^{b} h^{2}(t)\left(\int_{a}^{b} G(t, r) x^{2}(r) d r\right)^{2} d t \\
& +4 \int_{a}^{b}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{a}^{b} G(t, s) x(s) h(s) d s\right) d t \\
& +\int_{a}^{b}\left(2 x(t) \int_{a}^{b} G(t, s) x(s) h(s) d s\right)^{2} d t \tag{12}
\end{align*}
$$

We can estimate the first term in (12) as follows:

$$
\begin{equation*}
\int_{a}^{b} h^{2}(t)\left(\int_{a}^{b} G(t, r) x^{2}(r) d r\right)^{2} d t \geq \int_{a}^{b} h^{2}(t)\left(\int_{a}^{b} K_{1} x^{2}(r) d r\right)^{2} d t=K_{1}\|x\|^{4}\|h\|^{2} \tag{13}
\end{equation*}
$$

and the third one simply as

$$
\begin{equation*}
\int_{a}^{b}\left(2 x(t) \int_{a}^{b} G(t, s) x(s) h(s) d s\right)^{2} d t \geq 0 \tag{14}
\end{equation*}
$$

For the certain estimation of the second component of the sum in (12), we set (with the accuracy to zero measure):

$$
A=\{t \in[a ; b]: x(t) h(t)>0\}
$$

and

$$
B=\{t \in[a ; b]: x(t) h(t) \leq 0\} .
$$

Using (9) and the Schwarz inequality, we obtain

$$
\begin{aligned}
& 4 \int_{a}^{b}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{a}^{b} G(t, s) x(s) h(s) d s\right) d t \\
& =4 \int_{A}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{A} G(t, s) x(s) h(s) d s\right) d t \\
& \quad+4 \int_{B}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{B} G(t, s) x(s) h(s) d s\right) d t
\end{aligned}
$$

$$
\begin{aligned}
& +4 \int_{A}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{B} G(t, s) x(s) h(s) d s\right) d t \\
& +4 \int_{B}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{A} G(t, s) x(s) h(s) d s\right) d t \\
\geq & 4 K_{1}\|x\|^{2} \int_{A}\left(h(t) x(t) \int_{A} G(t, s) x(s) h(s) d s\right) d t \\
& +4 K_{1}\|x\|^{2} \int_{B}(\underbrace{}_{B} h(t) x(t) \int_{B} G(t, s) x(s) h(s) d s) d t \\
& +4 K_{2}\|x\|^{2} \int_{A}\left(h(t) x(t) \int_{B} G(t, s) x(s) h(s) d s\right) d t \\
& +4 K_{2}\|x\|^{2} \int_{B}\left(h(t) x(t) \int_{A} G(t, s) x(s) h(s) d s\right) d t \\
\geq & 4 K_{1}\|x\|^{2} \int_{a}^{b}\left(h(t) x(t) \int_{a}^{b} G(t, s) x(s) h(s) d s\right) d t \\
& +4\left(K_{2}-K_{1}\right)\|x\|^{2}\left(\int_{A}\left(h(t) x(t) \int_{B} G(t, s) x(s) h(s) d s\right) d t\right. \\
& \left.+\int_{B}\left(h(t) x(t) \int_{A} G(t, s) x(s) h(s) d s\right) d t\right) \\
\geq & 4\left(K_{1}-K_{2}\right)\|x\|^{2} \int_{a}^{b}\left(|h(t)||x(t)| \int_{a}^{b} G(t, s)|x(s)||h(s)| d s\right) d t \\
\geq & 4\left(K_{1}-K_{2}\right) K_{2}\|x\|^{4}\|h\|^{2}=\left(4 K_{1} K_{2}-4 K_{2}^{2}\right)\|x\|^{4}\|h\|^{2},
\end{aligned}
$$

hence

$$
\begin{align*}
& 4 \int_{a}^{b}\left(h(t) x(t) \int_{a}^{b} G(t, r) x^{2}(r) d r \int_{a}^{b} G(t, s) x(s) h(s) d s\right) d t \\
& \geq 4\left(K_{1}-K_{2}\right) K_{2}\|x\|^{4}\|h\|^{2} \\
& =\left(4 K_{1} K_{2}-4 K_{2}^{2}\right)\|x\|^{4}\|h\|^{2} . \tag{15}
\end{align*}
$$

From (12), (13), (14) and (15), we obtain

$$
\left\|f^{\prime}(x) . h\right\|^{2} \geq\left(K_{1}^{2}+4 K_{1} K_{2}-4 K_{2}^{2}\right)\|x\|^{4}\|h\|^{2}
$$

Solving the quadratic inequality $K_{1}^{2}+4 K_{1} K_{2}-4 K_{2}^{2}>0$ with respect to the positive $K_{1}$, we obtain (10).

Now, we demonstrate that

$$
f^{\prime}(x) L([a ; b], \mathbb{R})=L([a ; b], \mathbb{R}) \quad \text { for } \quad x \neq 0
$$

The set $f^{\prime}(x) L([a ; b], \mathbb{R})$ is a closed subspace of $L([a ; b], \mathbb{R})$. Supposing that $f^{\prime}(x) L([a ; b], \mathbb{R}) \neq L([a ; b], \mathbb{R})$, we choose a non-zero $y \in L([a ; b], \mathbb{R})$ which is orthogonal to $f^{\prime}(x) L([a ; b], \mathbb{R})$. In particular, $y$ is orthogonal to $f^{\prime}(x) . y$. But, because of (11),

$$
\begin{aligned}
0= & \left\langle y ; f^{\prime}(x) \cdot y\right\rangle \\
= & \int_{a}^{b} y^{2}(t)\left(\int_{a}^{b} G(t, r) x^{2}(r) d r\right) d t \\
& +2 \int_{a}^{b} y(t) x(t)\left(\int_{a}^{b} G(t, s) x(s) y(s) d s\right) d t>0 .
\end{aligned}
$$

The obtained contradiction proves that $f^{\prime}(x) L([a ; b], \mathbb{R})=L([a ; b], \mathbb{R})$.
At the end, we show, that the mapping $f$ is injective. Suppose that $f$ is not injective. Then, for a certain $x, h \in L([a ; b], \mathbb{R}), h \neq 0$, we have $f(x)=f(x+h)$. Let for a real $\tau$

$$
\begin{aligned}
\phi(\tau) & =\langle h ; f(x+\tau h)\rangle \\
& =\int_{a}^{b} h(t)(x(t)+\tau h(t))\left(\int_{a}^{b} G(t, s)(x(s)+\tau h(s))^{2} d s\right) d t .
\end{aligned}
$$

We have

$$
\phi(0)=\langle h ; f(x)\rangle=\langle h ; f(x+h)\rangle=\phi(1),
$$

but, because of (11),

$$
\begin{aligned}
\phi^{\prime}(\tau)= & \int_{a}^{b} h^{2}(t)\left(\int_{a}^{b} G(t, s)(x(s)+\tau h(s))^{2} d s\right) d t \\
& +2 \int_{a}^{b} h(t)(x(t)+\tau h(t))\left(\int_{a}^{b} G(t, s) h(s)(x(s)+\tau h(s)) d s\right) d t \geq 0
\end{aligned}
$$

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and $\phi^{\prime}(\tau)>0$ for a certain subinterval of $[0 ; 1]$, which implies that $\phi(0) \neq \phi(1)$. This contradiction shows that the map $f$ is injective.

The map $f$ fulfills the assumptions of Theorem 5 for $n=1$ with the following constants from estimations (2), (6) and (7):

$$
\begin{aligned}
& C_{1}=\sqrt{K_{1}^{2}+4 K_{1} K_{2}-4 K_{2}^{2}} \\
& C_{2}=K_{1} \\
& C_{3}=K_{2}
\end{aligned}
$$

Applying Theorem 6 for the map described above, we obtain the following.
Example 1. Suppose that a mapping $G$ fulfills assumptions of Theorem 6 and that

$$
r: L^{2}([a ; b], \mathbb{R}) \rightarrow L^{2}([a ; b], \mathbb{R})
$$

is a $C^{3}$-mapping, for which

$$
r(x)=o\left(\|x\|^{3}\right)
$$

and

$$
\sup _{x \in X \backslash\{0\}}\left\|r^{\prime}(x)\right\|\|x\|^{-2}<\sqrt{K_{1}^{2}+K_{1} K_{2}-4 K_{2}^{2}} K_{1}^{2 / 3} K_{2}^{-2 / 3}
$$

(An example of such a mapping may be defined as

$$
r(x)=\|x\|^{4} e^{-\|x\|^{2}} g
$$

for $g \in L^{2}([a ; b], \mathbb{R})$ with sufficiently small norm.)
Then the mapping

$$
L^{2}([a ; b], \mathbb{R}) \ni x \rightarrow x \int_{a}^{b} G(\cdot, s) x^{2}(s) d s+r(x) \in L^{2}([a ; b], \mathbb{R}) \text { (a.e.) }
$$

is a homeomorphism between $L^{2}([a ; b], \mathbb{R})$ and $L^{2}([a ; b], \mathbb{R})$.
Consequently, for any fixed $y \in L^{2}([a ; b], \mathbb{R})$, there is the unique solution $x \in L^{2}([a ; b], \mathbb{R})$ of the equation

$$
x \int_{a}^{b} G(\cdot, s) x^{2}(s) d s+r(x)=y \text { (a.e.). }
$$

Let us consider a specific example. It is going to be interesting even with $r=0$.

Example 2. We shall apply the general Example 1 to particular one with the concrete mapping $G$ defined as the Green function for the following bounded
value problem (see, for example [12], where the construction of the Green function is described):

$$
u^{\prime \prime}-K^{2} u=y, \quad y \in L^{2}([0 ; 1], \mathbb{R}), \quad K u(0)-u^{\prime}(0)=0, \quad K u(1)-u^{\prime}(1)=0
$$

where

$$
K<\frac{1}{2} \ln \frac{\sqrt{2}+1}{2} .
$$

We have

$$
G(t, s)=\left\{\begin{array}{lll}
\frac{e^{K(t-s)}}{2 K} & \text { for } \quad 0 \leq t \leq s \leq 1 \\
\frac{e^{K(s-t)}}{2 K} & \text { for } \quad 0 \leq s<t \leq 1,
\end{array}\right.
$$

and $G$ defines the positive integral operator in the sense of (11). It is clear that, for $t, s \in[0 ; 1]$,

$$
K_{1} \leq G(t, s) \leq K_{2}
$$

with

$$
K_{1}=\frac{e^{-K}}{2 K} \quad \text { and } \quad K_{2}=\frac{e^{K}}{2 K} .
$$

It is obvious that the mapping $G$ fulfills assumptions of Theorem 6, For any fixed $y \in L^{2}([0 ; 1], \mathbb{R})$, let us consider the equation

$$
x \int_{0}^{1} G(\cdot, s) x^{2}(s) d s=y(a . e .)
$$

with $x \in L^{2}([0 ; 1], \mathbb{R})$. Suppose that $y$ is nonnegative, consequently the solution $x$ is nonnegative as well.

Let

$$
\int_{0}^{1} G(\cdot, s) x^{2}(s) d s=u \text { (a.e.). }
$$

Then $u$ is a solution of the boundary value problem

$$
u^{\prime \prime}-K^{2} u=x^{2}, \quad K u(0)-u^{\prime}(0)=0, \quad K u(1)-u^{\prime}(1)=0
$$

so we have constructed a solution of the boundary value problem of the following form:

$$
\sqrt{u^{\prime \prime}-K^{2} u} u=y, \quad K u(0)-u^{\prime}(0)=0, \quad K u(1)-u^{\prime}(1)=0 .
$$

It is possible to describe more examples of the above type considering Green functions of other boundary value problems.

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Received July 20, 2017
Podhalańska Państwowa
Wyższa Szkota Zawodowa
w Nowym Targu
ul. Kokoszków 71
PL-34-400 Nowy Targ
POLAND
E-mail: piotr.fijalkowski@ppwsz.edu.pl


[^0]:    © 2017 Mathematical Institute, Slovak Academy of Sciences.
    2010 Mathematics Subject Classification: 35G30, 45G10.
    Keywords: local inversion, global inversion, integral equation, boundary value problem.

