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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2018-0023
Tatra Mt. Math. Publ. **72** (2018), 77–105

RÉNYI ENTROPY AND RÉNYI DIVERGENCE IN THE INTUITIONISTIC FUZZY CASE

Beloslav Riečan [1,2,†] — Dagmar Markechová[3]

[1]Institute of Mathematics, Slovak Academy of Sciences, Bratislava, SLOVAKIA

[2]Department of Mathematics, Faculty of Natural Sciences, MBU, Banská Bystrica, SLOVAKIA

[3]Institute of Mathematics, Faculty of Natural Sciences CPU, Nitra, SLOVAKIA

ABSTRACT. Our objective in this paper is to define and study the Rényi entropy and the Rényi divergence in the intuitionistic fuzzy case. We define the Rényi entropy of order of intuitionistic fuzzy experiments (which are modeled by IF-partitions) and its conditional version and we examine their properties. It is shown that the suggested concepts are consistent, in the case of the limit of q going to 1, with the Shannon entropy of IF-partitions. In addition, we introduce and study the concept of Rényi divergence in the intuitionistic fuzzy case. Specifically, relationships between the Rényi divergence and Kullback-Leibler divergence and between the Rényi divergence and the Rényi entropy in the intuitionistic fuzzy case are studied. The results are illustrated with several numerical examples.

1. Introduction

The Shannon entropy [31] and Kullback-Leibler divergence [22] are among the most important quantities in information theory [17] and its applications. Because of their success, many attempts have been made to generalize these concepts. As is known, the Rényi entropy and Rényi divergence [26] represent their significant generalizations. These quantities have important applications in ecology, in statistics, and they are also important in quantum information.

If we consider a probability space (X, S, P) and a finite measurable partition $\mathfrak{A} = \{E_1, \dots, E_n\}$ of (X, S, P) with probabilities $p_i = P(E_i)$, $i = 1, 2, \dots, n$, then the Shannon entropy of $\mathfrak{A}$ is defined as the number $H(\mathfrak{A}) = -\sum_{i=1}^{n} p_i \cdot \log p_i$.

2010 Mathematics Subject Classification: 28AXX; 37A35; 54C70.

Keywords: intuitionistic fuzzy event; IF-partition; Rényi entropy; Rényi conditional entropy; Rényi divergence.

The first author was supported by the grant VEGA 2/0034/15.

If $\mathfrak{A} = \{E_1, \dots, E_n\}$ and $\mathfrak{B} = \{F_1, \dots, F_m\}$ are two finite measurable partitions of (X, S, P), then the conditional Shannon entropy of $\mathfrak{A}$ assuming a realization of $\mathfrak{B}$ is defined as the number

$$H(\mathfrak{A}|\mathfrak{B}) = -\sum_{i=1}^{n}\sum_{j=1}^{m} P(E_i \cap F_j) \cdot \log \frac{P(E_i \cap F_j)}{P(F_j)}.$$

It is assumed in the definitions above that $0 \cdot \log \frac{0}{x} = 0$ if $x \geq 0$. The Rényi entropy of order q, where $q > 0$ and $q \neq 1$, of $\mathfrak{A}$ is defined as the number $H_q(\mathfrak{A}) = \frac{1}{1-q} \log \sum_{i=1}^{n} p_i^q$. It can be shown that $\lim_{q \to 1} H_q(\mathfrak{A}) = \sum_{i=1}^{n} p_i \cdot \log \frac{1}{p_i}$, thus the Shannon entropy is a limiting case of the Rényi entropy for $q \to 1$. It is known that there is no generally accepted definition of conditional Rényi entropy. In [33], three definitions of conditional Rényi entropy are described, which can be found in the literature. In [16], an overview of various approaches to defining the conditional Rényi entropy can be found, and moreover, a new definition of conditional Rényi entropy has been suggested. In [20], the authors suggested a general type of conditional Rényi entropy that includes previously defined conditional Rényi entropies as special cases. The suggested concepts have been successfully used in time series analysis [21], in information theory [1] and in cryptographic applications [25]. None of the proposed generalizations, however, fulfills all the basic properties of Shannon conditional entropy, so the choice of the definition depends on the purpose of use.

The present article is devoted to the study of Rényi entropy and Rényi divergence in the intuitionistic fuzzy case. The intuitionistic fuzzy sets theory [2,3,5] was introduced by A t a n a s s o v as an extension of the Zadeh fuzzy set theory [34]. Recall that while a fuzzy set is a mapping $f_A : X \to [0,1]$ (where the considered fuzzy set is identified with its membership function f_A), the Atanassov intuitionistic fuzzy set (shortly IF-set) is a pair $A = (f_A, g_A) : X \to [0,1] \times [0,1]$ of fuzzy sets for which the condition $f_A(x) + g_A(x) \leq 1$, for every $x \in X$, is satisfied. The functions f_A and g_A define the degree of membership and the degree of non-membership of the element $x \in X$ to the considered IF-set A, respectively. It is obvious that every fuzzy set $f_A : X \to [0,1]$ can be considered as an IF-set $A = (f_A, 1 - f_A)$. All results that apply to IF-sets are also valid in the fuzzy case. Naturally, the opposite implication is not true; the intuitionistic fuzzy sets theory is a nontrivial generalization of the fuzzy set theory. This means that the IF-sets provide opportunities for modeling a larger class of real situations. In the last three decades, many authors have dealt with the theory of IF-sets, which has been successfully applied in various mathematical disciplines and has also important practical applications. We refer the interested reader to the article [19], which contains an overview of highly cited intuitionistic fuzzy publications and provides their characteristics. Of course, many papers (see, e.g., [7, 9, 10, 14, 15, 23, 24, 32, 35]) are devoted also to study of entropy in the intuitionistic fuzzy case.

In [24], we introduced the concepts of Shannon entropy and Kullback-Leibler divergence in the intuitionistic fuzzy case. Instead of measurable partitions, we considered so-called intuitionistic fuzzy partitions (IF-partitions, for brevity) that can be useful for modeling experiments with inaccurate IF-information. The aim of the present paper is to extend our study concerning the Shannon entropy and Kullback-Leibler divergence in the intuitionistic fuzzy case to the case of Rényi entropy and Rényi divergence.

The rest of the paper is structured as follows. In the following section we provide basic definitions, notations and facts used in the article. Our main results are discussed in Sections 3–5. In Section 3, we define the Rényi entropy of an IF-partition and examine its properties. It is shown that for $q \to 1$ the Rényi entropy of order q converges to the Shannon entropy of an IF-partition. In Section 4, we introduce the concept of Rényi conditional entropy of IF-partitions and study its properties. It is shown that the suggested definition of the conditional Rényi entropy is consistent, in the case of the limit of q going to 1, with the conditional Shannon entropy of IF-partitions, and it satisfies the property of monotonicity and a weak chain rule. Section 5 is devoted to the study of Rényi divergence in the intuitionistic fuzzy case. The results are explained with several examples to illustrate the theory developed in the article. The final section provides a brief summary.

2. Preliminaries

In this section, we provide basic definitions, notations and facts used in the paper.

Definition 1. Let X be a non–empty set. By an intuitionistic fuzzy set (IF-set for short) we will understand a pair $A = (f_A, g_A)$ of functions $f_A, g_A : X \to [0, 1]$ such that $f_A + g_A \leq 1$ for every $x \in X$.

Analogously as in the fuzzy set theory, there are many possibilities to define operations over intuitionistic fuzzy sets (cf. [4, 6, 13, 29, 30]). We will use the partial binary operation $\oplus$ and the binary operation $\cdot$ defined as follows. If $A = (f_A, g_A)$, and $B = (f_B, g_B)$ are two IF-sets, then we define

$$A \oplus B = (f_A + f_B, g_A + g_B - 1_X) \quad \text{and} \quad A \cdot B = (f_A \cdot f_B, g_A + g_B - g_A \cdot g_B).$$

Here, 1_X denotes the constant function with the value 1; similarly, 0_X denotes the constant function with the value 0. It is evident that if A, B are two IF-sets, then $A \oplus B$ is an IF-set if and only if $f_A + f_B \leq 1_X$ and $g_A + g_B \geq 1_X$. In the case that $A \oplus B$ is an IF set, we will say that $A \oplus B$ exists. Put

$$1 = (1_X, 0_X), \quad 0 = (0_X, 1_X).$$

It can be verified that for any IF-sets A, B, C, the following conditions are satisfied:

(F1) $A \cdot 1 = A$, $A \oplus 0 = A$;

(F2) $A \cdot B = B \cdot A$; if $A \oplus B$ exists, then $B \oplus A$ exists, and $A \oplus B = B \oplus A$ (commutativity);

(F3) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$; if $(A \oplus B) \oplus C$ exists, then $A \oplus (B \oplus C)$ exists and $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (associativity);

(F4) if $A \oplus B$ exists then $C \cdot A \oplus C \cdot B$ exists and $C \cdot (A \oplus B) = C \cdot A \oplus C \cdot B$ (distributivity).

In the class of all IF-sets we define the partial order relation $\leq$ in the following way: if $A = (f_A, g_A)$ and $B = (f_B, g_B)$ are two IF-sets, then $A \leq B$ if and only if $f_A \leq f_B$ and $g_A \geq g_B$. It is easy to see that $0 \leq A \leq 1$ for any IF-set A. In [18] Gutierrez Garcia and Rodabaugh have proved that intuitionistic fuzzy sets ordering and topology are reduced to the ordering and topology of fuzzy sets. Another situation is in measure theory where the intuitionistic fuzzy case cannot be reduced to the fuzzy one (cf. [8]). We note that a probability theory for the intuitionistic fuzzy case was developed in [28], see also [12].

EXAMPLE 1. As already mentioned in Introduction, any fuzzy set $f_A : X \to [0, 1]$ can be regarded as an IF-set, if we put $A = (f_A, 1_X - f_A)$. If $f_A = I_A$ where I_A is the indicator function of a set $A \subset X$, then the corresponding IF-set has the form $A = (I_A, 1_X - I_A) = (I_A, I_{A^c})$. Here, A^c denotes the complement of a set $A \subset X$. In this case, $A \oplus B$ corresponds to the union of sets $A, B \subset X$ with empty intersection, $A \cdot B$ to the intersection of sets $A, B \subset X$ and the relation $\leq$ to the inclusion of sets $A, B \subset X$.

In what follows we shall denote by the symbol $\mathcal{F}$ any family of IF-sets satisfying the following two conditions:

(F5) $0 \in \mathcal{F}$, $1 \in \mathcal{F}$;

(F6) if $A, B \in \mathcal{F}$, then $A \cdot B \in \mathcal{F}$.

Any IF-set from the family $\mathcal{F}$ is interpreted as an intuitionistic fuzzy event. The IF-set $0 = (0_X, 1_X)$ is considered as an impossible event; the IF-set $1 = (1_X, 0_X)$ as a certain event. In an analogous way as in [23], we define the state on the family $\mathcal{F}$. It plays the role of a probability measure on the family $\mathcal{F}$ of IF-events.

DEFINITION 2. A mapping $s : \mathcal{F} \to [0, 1]$ is called a state if the following two conditions are satisfied:

(i) $s(1) = 1$;

(ii) if $A, B \in \mathcal{F}$ such that $A \oplus B \in \mathcal{F}$ then $s(A \oplus B) = s(A) + s(B)$.

Notice that the disjointness of IF-sets $A = (f_A, g_A)$ and $B = (f_B, g_B)$ of $\mathcal{F}$ is expressed in Definition 2 by the condition that $f_A + f_B \leq 1_X$ and $g_A + g_B \geq 1_X$ (or equivalently by the condition that $f_A \leq 1_X - f_B$ and $g_A \geq 1_X - g_B$).

Definition 3 ([23])**.** By a measurable IF-partition of $(1_X, 0_X)$ with respect to a state $s : \mathcal{F} \to [0,1]$, we will mean an n-tuple $\alpha = (A_1, \dots, A_n)$ of (not necessarily different) members of $\mathcal{F}$ such that $\oplus_{i=1}^n A_i \in \mathcal{F}$ and $s\left(\oplus_{i=1}^n A_i\right) = 1$.

Example 2. Let us consider a classical probability space (X, S, P) and put $\mathcal{F} = \{(I_E, 1_X - I_E); E \in S\}$. Then the mapping $s : \mathcal{F} \to [0,1]$ defined by $s\big((I_E, 1_X - I_E)\big) = P(E)$, $e \in S$, is a state. A measurable partition $(E_1, \dots, E_n)$ of (X, S, P) can be regarded as an IF-partition, if we consider $(I_{E_i}, 1_X - I_{E_i})$ instead of E_i.

For two finite tuples $\alpha = (A_1, \dots, A_k)$ and $\beta = (B_1, \dots, B_\ell)$ of elements of $\mathcal{F}$ we define $\alpha \vee \beta$ as an r-tuple (where $r = k \cdot \ell$) consisting of the elements $(A_i \cdot B_j)$, $i = 1, 2, \dots, k,\ j = 1, 2, \dots, \ell$. If $\alpha = (A_1, \dots, A_k)$ and $\beta = (B_1, \dots, B_\ell)$ are two measurable IF-partitions of $(1_X, 0_X)$, then it can be shown that the r-tuple $\alpha \vee \beta$ is also a measurable IF-partition of $(1_X, 0_X)$. The proof can be found in [23]. Let $s : \mathcal{F} \to [0,1]$ be a state. Two measurable IF-partitions $\alpha = (A_1, \dots, A_k)$, $\beta = (B_1, \dots, B_\ell)$ of $(1_X, 0_X)$ are called statistically independent with respect to s if $s(A_i \cdot B_j) = s(A_i) \cdot s(B_j)$ for $i = 1, \dots, k,\ j = 1, \dots, \ell$. We will say that the IF-partition β is a refinement of α and write $\alpha \prec \beta$ if for each $A_i \in \alpha$ there exists a subset $I(i) \subset \{1, \dots, \ell\}$ such that $A_i = \oplus_{j \in I(i)} B_j$, where $I(i) \cap I(j) = \emptyset$ whenever $i \neq j$, and $\bigcap_{i=1}^k I(i) = \{1, \dots, \ell\}$. It can be proved that for arbitrary measurable IF-partitions α, β of $(1_X, 0_X)$, it holds $\alpha \prec \alpha \vee \beta$ and $\beta \prec \alpha \vee \beta$.

Definition 4 ([24])**.** We define the entropy of a measurable IF-partition $\alpha = (A_1, A_2, \dots, A_n)$ of $(1_X, 0_X)$ with respect to s by the formula

$$H_s(\alpha) = -\sum_{i=1}^n s(A_i) \log s(A_i). \tag{1}$$

Definition 5 ([24])**.** If $\alpha = (A_1, A_2, \dots, A_k)$ and $\beta = (B_1, b_2, \dots, B_\ell)$ are two measurable IF-partitions of $(1_X, 0_X)$ with respect to s, then we define the conditional Shannon entropy of α given $B_j \in \beta$ by

$$H_s(\alpha | B_j) = -\sum_{i=1}^k s(A_i | B_j) \log s(A_i | B_j),$$

where

$$s(A_i | B_j) = \begin{cases} \frac{s(A_i \cdot B_j)}{s(B_j)} & \text{if } s(B_j) > 0; \\ 0 & \text{if } s(B_j) = 0. \end{cases}$$

The conditional entropy of α assuming a realization of experiment β is defined by the formula

$$H_s(\alpha|\beta) = \sum_{j=1}^{\ell} s(B_j) H_s(\alpha|B_j). \tag{2}$$

Evidently, formula (2) may be written in the following equivalent form:

$$H_s(\alpha|\beta) = -\sum_{i=1}^{k}\sum_{j=1}^{\ell} s(A_i \cdot B_j) \log \frac{s(A_i \cdot B_j)}{s(B_j)}. \tag{3}$$

It is assumed (based on continuity arguments) that $0 \cdot \log \frac{0}{x} = 0$ if $x \geq 0$. The base of the logarithm can be any positive number, but as a rule one takes logarithms to the base 2. The entropy is then expressed in bits. It can be shown that the Shannon entropy of IF-partitions satisfies the properties analogous to properties of Shannon entropy of classical measurable partitions.

Example 3. Let (X, S, P) be a probability space. Consider the family $\mathcal{F}$ of all S-measurable IF-sets, i.e., the family

$$\mathcal{F} = \{A = (f_A, g_A); f_A, g_A : X \to [0,1] \text{ are } S\text{-measurable with } f_A + g_A \leq 1_X\}.$$

Let $c \in [0,1]$. Then it can be verified that the mapping $s : \mathcal{F} \to [0,1]$ defined, for any element $A = (f_A, g_A)$ of $\mathcal{F}$ by the formula

$$s(A) = \int_X f_A \, \mathrm{d}P + c\left(1 - \int_X (f_A + g_A)\mathrm{d}P\right), \tag{4}$$

is a state.

Remark 1. We note that any continuous state s defined on the family $\mathcal{F}$ of all S-measurable IF-events (i.e., a state $s : \mathcal{F} \to [0,1]$ satisfying the condition $A_n \nearrow A \Rightarrow s(A_n) \nearrow s(A)$) has the form (4); for more details, see [11, 27]. The entropy theory of Shannon type for the case of the class $\mathcal{F}$ of all S-measurable IF-events was constructed by Ďurica in [14]. We remark that while an IF-partition considered by Ďurica is a set $\alpha = \{A_1, \dots, A_n\}$ of S--measurable IF-events such that $\oplus_{i=1}^{n} A_i = 1$, the model studied in [23, 24] as well as the model studied in this paper is more general.

In [24], the concept of Kullback-Leibler divergence for the intuitionistic fuzzy case was introduced as follows.

Definition 6. Let s, t be two states on $\mathcal{F}$ and $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and t. Then we define the Kullback-Leibler divergence $d_\alpha(s\|t)$ as the number

$$d_\alpha(s\|t) = \sum_{i=1}^{n} s(A_i) \cdot \log \frac{s(A_i)}{t(A_i)}. \tag{5}$$

In the succeeding sections we will use the known Jensen inequality: for a real convex function F real numbers $x_1, x_2, \dots, x_n$ in its domain and nonnegative real numbers $a_1, a_2, \dots, a_n$ such that $\sum_{i=1}^n a_i = 1$ it holds

$$f\left(\sum_{i=1}^n a_i x_i\right) \leq \sum_{i=1}^n a_i F(s_i)$$

and the inequality is reversed if F is a real concave function. The equality applies if and only if $x_1 = \dots = x_n$ or F is a linear function.

Further, we recall the following notions.

Definition 7. Let D be an arbitrary non-empty set and $f : D \to \mathbb{R}$ be a real function defined on it. Then the support of f is defined by

$$\mathrm{supp}(f) = \{x \in D; f(x) \neq 0\}.$$

Definition 8. Let $f : D \to \mathbb{R}$ be a real function on a non-empty set D. We define the q-norm, for $a \leq q < \infty$, or q-quasinorm, for $0 < q < 1$, of f as

$$\|f\|_q = \left(\sum_{x\in D} |f(x)|^q\right)^{\frac{1}{q}}.$$

3. The Rényi entropy of IF-partitions

In this section we define the Rényi entropy of IF-partitions and examine its properties. In the following, we assume that $s : F \to [0, 1]$ is a state.

Definition 9. Let $\alpha = (A_1, A_2, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s. Then we define the Rényi entropy of order q, where $q > 0$, $q \neq 1$, of α by the formula

$$H_q^s(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^n s(A_i)^q. \tag{6}$$

Remark 2. In accordance with the classical theory the log is to the base 2 and the Rényi entropy is expressed in bits. For simplicity, we write $s(A_i)^q$ instead of $\big(s(A_i)\big)^q$ and $\log \sum_{i=1}^n s(A_i)^q$ instead of $\log\left(\sum_{i=1}^n s(A_i)^q\right)$.

Let $\alpha = (A_1, A_2, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s. If we consider the function $s_\alpha : \alpha \to \mathbb{R}$, defined by $s_\alpha(A_i) = s(A_i)$ for every $A_i \in \alpha$ then we have

$$\|s_\alpha\|_q = \left(\sum_{i=1}^n s(A_i)^q\right)^{\frac{1}{q}},$$

and formula (6) can be expressed in the following equivalent form

$$H_q^s(\alpha) = \frac{q}{1-q} \log(\|s_\alpha\|_q). \tag{7}$$

Remark 3. Let $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to a state s. Let us assume that the state s is uniform over α, i.e., $s(A_i) = \frac{1}{n}$ for $i = 1, 2, \dots, n$. Then

$$H_q^2(\alpha) = \frac{1}{1-q} \log n^{1-q} = \log n.$$

Example 4. Let us consider the IF-partition $\alpha_0 = \{1\}$ representing an experiment resulting in a certain event. It is easy to see that $H_q^s(\alpha_0) = 0$.

Remark 4. It can be verified that the Rényi entropy $H_q^s(\alpha)$ is always non-negative. Namely, for $0 < q < 1$ and $i = 1, 2, \dots, n$, it holds $s(A_i)^q \geq s(A_i)$, hence

$$\sum_{i=1}^{n} s(A_i)^q \geq \sum_{i=1}^{n} s(A_i) = s\left(\oplus_{i=1}^{n} A_i\right) = 1.$$

It follows that $H_s^q(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \geq 0$. On the other hand, for $q > 1$ and $i = 1, 2, \dots, n$, it holds $s(A_i)^q \leq s(A_i)$, hence $\sum_{i=1}^{n} s(A_i)^q \leq \sum_{i=1}^{n} s(A_i) = 1$. In this case we have $\frac{1}{1-q} < 0$, therefore, $H_q^s(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \geq 0$.

Example 5. Consider a family $\mathcal{F}$ of IF-events and a state s defined on $\mathcal{F}$. Let $\alpha = (A_1, A_2)$ be a measurable IF-partition of $(1_X, 0_X)$ with $s(A_i) = p$, where $p \in (0, 1)$. Then $s(A_2) = 1 - p$, and the Rényi entropy $H_q^s(\alpha)$ of order $q = \frac{1}{2}$ is $H_q^s(\alpha) = 2 \log\left(\sqrt{p} + \sqrt{1-p}\right)$. If we put $p = \frac{1}{3}$, then we have $H_q^s(\alpha) \doteq 0.958$ bit.

In the following theorem it is proved that $H_q^s(\alpha)$ is monotonically decreasing in q.

Theorem 1. *Let $\alpha = (A_1, A_2, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and q_1, q_2 be positive real numbers, $q_1 \neq 1$, $q_2 \neq 1$. Then $q_1 \geq q_2$ implies $H_{q_1}^s(\alpha) \leq H_{q_2}^s(\alpha)$.*

Proof. Suppose that $q_1, q_2 \in (1, \infty)$. Then the claim is evident to the inequality

$$\left(\sum_{i=1}^{n} s(A_i)^{q_1}\right)^{\frac{1}{q_1-1}} \geq \left(\sum_{i=1}^{n} s(A_i)^{q_2}\right)^{\frac{1}{q_2-1}}.$$

This inequality follows by applying the Jensen inequality to the function F defined by $F(x) = x^{\frac{q_2-1}{q_1-1}}$, for every $x \in [0, \infty)$, and putting $\alpha_i = s(A_i)$, $x_i = s(A_i)^{q_1-1}$, $i = 1, 2, \dots, n$. The assumption $q_1 \geq q_2$ implies $\frac{q_2-1}{q_1-1} \leq 1$, hence, the function F is concave.

Therefore, we get

$$\begin{aligned}\left(\sum_{i=1}^{n} s(A_i)_i^q\right)^{\frac{1}{q_1-1}} &= \left(\sum_{i=1}^{n} s(A_i)s(A_i)^{q_1-1}\right)^{\frac{q_2-1}{(q_1-1)(q_2-1)}} \\ &= \left(\left(\sum_{i=1}^{n} s(A_i)s(A_i)^{q_1-1}\right)^{\frac{q_2-1}{q_1-1}}\right)^{\frac{1}{q_2-1}} \\ &\geq \left(\sum_{i=1}^{n} s(A_i)s(A_i)^{q_2-1}\right)^{\frac{1}{q_2-1}} = \left(\sum_{i=1}^{n} s(A_i)^{q_2}\right)^{\frac{1}{q_2-1}}.\end{aligned}$$

The case of $q_1, q_2 \in (0,1)$ is obtained by similar arguments. Finally, the case $q_1 \in (1,\infty)$ and $q_2 \in (0,1)$ follows by transitivity. $\square$

Example 6. Consider the following family $\mathcal{F}$ of Borel measurable IF-events: $\mathcal{F} = \{A = (f_A, g_A); f_A, g_A : [0,1] \to [0,1]$ are Borel measurable with the property $f_A + g_A \leq 1_X\}$. Further, define a state $s : \mathcal{F} \to [0,1]$ by the formula

$$s(A) = \int_0^1 f_A \,\mathrm{d}x + 1 - \int_0^1 (f_A + g_A)\,\mathrm{d}x$$

for any element $A = (f_A, g_A)$ of $\mathcal{F}$. We put $A_1 = \left(I_{[0,\frac{1}{3}]}, I_{(\frac{1}{3},1]}\right)$, and $A_2 = \left(I_{(\frac{1}{3}]}, I_{[0,\frac{1}{3}]}\right)$. Since $A_1 \oplus A_2 = \left(I_{[0,\frac{2}{3}]}, 0\right)$ (and hence $A_1 \oplus A_2 \in \mathcal{F}$), and $s(A_1 \oplus A_2) = 1$, the pair $\alpha = (A_1, A_2)$ is a measurable IF-partition of $(1_X, 0_X)$. Simple calculation will show that $s(A_1) = \frac{1}{3}$ and $s(A_2) = \frac{2}{3}$. The Rényi entropy $H_{q_1}^s(\alpha)$ of order $q_1 = \frac{1}{2}$ is $H_{q_1}^s(\alpha) = 2\log\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}}\right) \doteq 0.9758144$ bit, and the Rényi entropy $H_{q_2}^s(\alpha)$ of order $q_2 = \frac{1}{3}$ is $H_{q_2}^s(\alpha) \doteq 0.971927$ bit. So, it holds $H_{\frac{1}{2}}^s(\alpha) < H_{\frac{1}{3}}^s(\alpha)$, which is consistent with the claim of Theorem 1.

At $q = 1$ the value of the quantity $H_q^s(\alpha)$ is undefined as it generates the form $\frac{0}{0}$. In the following theorem it is shown that for $q \to 1$ the Rényi entropy $H_q^s(\alpha)$ converges to the Shannon entropy of an IF-partition defined by formula (1).

Theorem 2. *Let $\alpha = (A_1, A_2, \ldots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s. Then*

$$\lim_{q\to 1} H_q^s(\alpha) = -\sum_{i=1}^{n} s(A_i)\log s(A_i).$$

P r o o f. In the proof we use L'Hôpital's rule $\lim_{q\to 1} \frac{f(q)}{g(q)} = \lim_{q\to 1} \frac{f'(q)}{g'(q)}$. We put $f(q) = \log\sum_{i=1}^{n} s(A_i)^q$, and $g(q) = 1 - q$, for every $q \in (0,\infty)$.

Then, for every $q \in (0,1) \cup (1,\infty)$, we have $H_q^s(\alpha) = \frac{f(q)}{g(q)}$, and the functions f, g are differentiable. Evidently, $\lim_{q\to 1} g(q) = 0$, and we have also

$$\lim_{q\to 1} f(q) = \log \sum_{i=1}^{n} s(A_i) = \log s\left(\oplus_{i=1}^{n} A_i\right) = \log 1 = 0.$$

It holds $\frac{d}{dq} g(q) = -1$, and

$$\frac{d}{dq} f(q) = \frac{1}{\sum\limits_{i=1}^{n} s(A_i)^q \cdot \ln 2} \sum_{i=1}^{n} \frac{d}{dq} s(A_i)^q = \frac{1}{\sum\limits_{i=1}^{n} s(A_i)^q} \sum_{i=1}^{n} s(A_i)^q \log s(A_i).$$

Using L'Hôpital's rule, this yields $\lim\limits_{q\to 1} H_q^s(\alpha) = \lim\limits_{q\to 1} \frac{f'(q)}{g'(q)}$, under the assumption that the right-hand side exists. Therefore, we have

$$\begin{aligned}\lim_{q\to 1} H_q^s(\alpha) &= \lim_{q\to 1} \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \\ &= \lim_{q\to 1} \frac{-1}{\sum\limits_{i=1}^{n} s(A_i)^q} \sum_{i=1}^{n} s(A_i)^q \log s(A_i) = -\sum_{i=1}^{n} s(A_i) \log s(A_i),\end{aligned}$$

which is the Shannon entropy of α defined by formula (1). □

Theorem 3. *Let α and β be measurable IF-partitions of $(1_X, 0_X)$ with respect to s such that $\alpha \prec \beta$. Then $H_q^s(\alpha) \le H_q^s(\beta)$.*

Proof. Assume that $\alpha = (A_1, \dots, A_n)$, $\beta = (B_1, \dots, B_m)$, $\alpha_0 \prec \beta$. Then there exists a partition $\{I(1), \dots, I(n)\}$ of the set $\{1, 2, \dots, m\}$ such that it holds $A_i = \oplus_{j \in I(i)} B_j$, for $i = 1, 2, \dots, n$. Hence,

$$s(A_i) = s\left(\oplus_{j\in I(i)} B_j\right) = \sum_{j\in I(i)} s(B_j), \quad \text{for } i = 1, 2, \dots, n.$$

Consider the case of $q > 1$. Then $s(A_i)^q = \left(\sum_{j\in I(i)} s(B_j)\right)^q \ge \sum_{j\in I(i)} s(B_j)^q$, for $i = 1, 2, \dots, n$, and consequently,

$$\sum_{i=1}^{n} s(A_i)^q \ge \sum_{i=1}^{n} \sum_{j\in I(i)} s(B_j)^q = \sum_{j=1}^{m} s(B_j)^q.$$

Therefore, we get

$$\log \sum_{i=1}^{n} s(A_i)^q \geq \log \sum_{j=1}^{m} s(B_j)^q.$$

In this case $\frac{1}{1-q} < 0$, hence, we obtain

$$H_q^s(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \leq \frac{1}{1-q} \log \sum_{j=1}^{m} s(B_j)^q = H_q^s(\beta).$$

Consider the case when $0 < q < 1$. Then

$$s(A_i)^q = \left(\sum_{j \in I(i)} s(B_j) \right)^q \leq \sum_{j \in I(i)} s(B_j)^q, \quad \text{for } i = 1, 2, \dots, n,$$

and consequently,

$$\sum_{i=1}^{n} s(A_i)^q \leq \sum_{i=1}^{n} \sum_{j \in I(i)} s(B_j)^q = \sum_{j=1}^{m} s(B_j)^q.$$

Therefore, we get

$$\log \sum_{i=1}^{n} s(A_i)^q \leq \log \sum_{j=1}^{m} s(B_j)^q.$$

In this case $\frac{1}{1-q} > 0$, hence, we obtain

$$H_q^s(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \leq \frac{1}{1-q} \log \sum_{j=1}^{m} s(B_j)^q = H_q^s(\beta).$$

□

Corollary 1. *Let α and β be measurable IF-partitions of $(1_X, 0_X)$ with respect to s. Then*

$$H_q^s(\alpha \vee \beta) \geq \max\bigl(H_q^s(\alpha), H_q^s(\beta)\bigr).$$

Theorem 4. *If measurable IF-partitions α and β are statistically independent with the respect to s, then*

$$H_q^s(\alpha \vee \beta) = H_q^s(\alpha) + H_q^s(\beta).$$

Proof. Assume that $\alpha = (A_1, \dots, A_n)$ and $\beta = (B_1, \dots, B_m)$. Let us calculate

$$\begin{aligned} H_q^s(\alpha \vee \beta) =& \frac{1}{1-q} \log \sum_{i=1}^{n} \sum_{j=1}^{m} s(A_i \cdot B_j)^q = \frac{1}{1-q} \log \left(\sum_{i=1}^{n} s(A_i)^q \cdot \sum_{j=1}^{m} s(B_j)^q \right) \\ =& \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q + \frac{1}{1-q} \log \sum_{j=1}^{m} s(B_j)^q = H_q^s(\alpha) + H_q^s(\beta). \end{aligned}$$

□

THEOREM 5. *Let $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s. If we denote*

$$a = \max\left\{\frac{1}{s(A_i)}; a_i \in \operatorname{supp}(s), i = 1, \dots, n\right\}, \quad \textit{then} \quad H_q^s(\alpha) \leq \log a.$$

P r o o f. Put $\delta = \{i; s(A_i) > 0\}$.

Let $0 < q < 1$. Then

$$\sum_{i=1}^{n} s(A_i)^q = \sum_{i\in\delta} \left(\frac{1}{s(A_i)}\right)^{1-q} \cdot s(A_i)$$
$$\leq \sum_{i\in\delta} a^{1-q} \cdot s(A_i) = a^{1-q} \sum_{i=1}^{n} s(A_i) = a^{1-q},$$

and consequently,

$$\log \sum_{i=1}^{n} s(A_i)^q \leq \log a^{1-q} = (1-q)\log a.$$

Therefore, we have

$$H_q(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \leq \frac{1}{1-q}(1-q)\log a = \log a.$$

Suppose that $q > 1$. Then

$$\sum_{i=1}^{n} s(A_i)^q = \sum_{i\in\delta} \left(\frac{1}{s(A_i)}\right)^{1-q} \cdot s(A_i)$$
$$\geq \sum_{i\in\delta} a^{1-q} \cdot s(A_i) = a^{1-q},$$

and consequently

$$\log \sum_{i=1}^{n} s(A_i)^q \geq \log a^{1-q} = (1-q)\log a.$$

Hence, we get

$$H_q(\alpha) = \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \leq \frac{1}{1-q}(1-q)\log a = \log a.$$

$\square$

4. The conditional Rényi entropy of IF-partitions

In this section, we introduce the concept of conditional Rényi entropy of IF--partitions and examine its properties. Let

$$\alpha = (A_1, \ldots, A_n), \quad \text{and} \quad \beta = (B_1, \ldots, B_m)$$

be IF-partitions of F. If we consider the function $s_{\alpha|B_j} : \alpha \to \mathbb{R}$, defined by $s_{\alpha|B_j}(A_i) = s(A_i|B_j)$, for every $A_i \in \alpha$, then we have

$$\|s_{\alpha|B_j}\|_q = \left(\sum_{i=1}^{n} s(A_i|B_j)^q\right)^{\frac{1}{q}}.$$

Definition 10. Let $\alpha = (A_1, \ldots, A_n)$ and $\beta = (B_1, \ldots, B_m)$ be measurable IF-partitions of $(1_X, 0_X)$ with respect to s. We define the conditional Rényi entropy of order q, where $q > 0$, $q \neq 1$, of α given β by the formula

$$H_q^s(\alpha|\beta) = \frac{q}{1-q} \log\left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_q\right).$$

Remark 5. In the same way as in the unconditional case of Rényi entropy $H_q^s(\alpha)$, it can be verified that the conditional Rényi entropy $H_q^s(\alpha|\beta)$ is always nonnegative. Let $\alpha = (A_1, \ldots, A_n)$ be any IF-partition with respect to s and let $\alpha_0 = \{1\}$ be the IF-partition representing the experiment resulting in a certain event. Since $s(A_i|1) = s(A_i)$, for $i = 1, 2, \ldots, n$, it holds $\|s_{\alpha|1}\|_q = \|s_\alpha\|_q$, and consequently,

$$H_q^s(\alpha|\alpha_0) = \frac{q}{1-q} \log\left(s(1)\|s_{\alpha|1}\|_q\right) = \frac{q}{1-q} \log\left(\|s_\alpha\|_q\right) = H_q^s(\alpha).$$

Proposition 1. *Let $\alpha = (A_1, \ldots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s. Then*

(i): $\sum_{i=1}^{n} s(A_i \cdot B) = s(B)$, *for any $B \in \mathcal{F}$;*

(ii): $\sum_{i=1}^{n} s(A_i|B) = 1$, *for any $B \in \mathcal{F}$ such that $s(B) > 0$.*

Proof. The claim (i) is proved in [23]. If $B \in \mathcal{F}$ such that $s(B) > 0$, then using the previous equality, we get

$$\sum_{i=1}^{n} s(A_i|B) = \sum_{i=1}^{n} \frac{s(A_i \cdot B)}{S(B)} = \frac{s(B)}{s(B)} = 1. \qquad \square$$

Theorem 6. *Let $\alpha = (A_1, \dots, A_n)$ and $\beta = (B_1, \dots, B_m)$ be measurable IF--partitions of $(1_X, 0_X)$ with respect to s. Then*

$$\lim_{q\to 1} H_q^s(\alpha|\beta) = -\sum_{i=1}^{n}\sum_{j=1}^{m} s(A_i \cdot B_j) \log \frac{s(A_i \cdot B_j)}{s(B_j)}.$$

Proof. We can write

$$\begin{aligned} H_q^s(\alpha|\beta) = & \frac{q}{1-q} \log \left(\sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j} \|_q \right) \\ = & -\frac{1}{1-\frac{1}{q}} \log \left(\sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j} \|_q \right) = -\frac{f(q)}{g(q)}, \end{aligned}$$

where f a g are continuous functions defined, for every $q \in (0, \infty)$, in the following way: $f(q) = \log\bigl(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_q\bigr), g(q) = 1 - \frac{1}{q}$. The functions f and g are differentiable and $\lim_{q\to 1} g(q) = 0$. If we put $\delta = \{j; s(B_j) > 0\}$ then, using Proposition 1, we get

$$\begin{aligned} \lim_{q\to 1} f(q) = & \log \left(\sum_{j=1}^{m} s(B_j) \sum_{i=1}^{n} s(A_i|B_j) \right) \\ = & \log \left(\sum_{j\in\delta}^{m} s(B_j) \sum_{i=1}^{n} \frac{s(A_i \cdot B_j)}{s(B_j)} \right) = \log \left(\sum_{i\in\delta}\sum_{i=1}^{n} s(A_i \cdot B_j) \right) \\ = & \log \left(\sum_{i\in\delta} s(B_j) \right) = \log \left(\sum_{j=1}^{m} s(B_j) \right) = \log 1 = 0. \end{aligned}$$

We have $\frac{d}{dq} q(q) = \frac{1}{q^2}$ and $\frac{d}{dq} f(q) = \frac{h'(q)}{h(q)\ln 2}$, where h is the continuous function defined, for every $q \in (0, \infty)$, by the formula

$$h(q) = \sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j} \|_q,$$

with continuous derivative h' for which we have

$$h'(q) = \sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j} \|_q \cdot \left(-\frac{\ln \sum\limits_{i=1}^{n} s(A_i|B_j)^q}{q^2} + \frac{\ln \sum\limits_{i=1}^{n} s(A_i|B_j)^q \ln s(A_i|B_j)}{q \sum\limits_{i=1}^{n} s(A_i|B_j)^q} \right).$$

From this we see that

$$\lim_{q\to 1} f'(q) = -\sum_{j=1}^{m} s(B_j) \sum_{i=1}^{n} s(A_i|B_j) \log s(A_i|B_j).$$

Using L'Hopital's rule, we get that $\lim_{q\to1} H_q^s(\alpha|\beta) = -\lim_{q\to1} \frac{f'(q)}{g'(q)}$, under the assumption that the right-hand side exists. It follows

$$\begin{aligned}\lim_{q\to1} H_q^s(\alpha|\beta) &= \lim_{q\to1} f'(q)\\ &= -\sum_{j=1}^{m} s(B_j) \sum_{i=1}^{n} s(A_i|B_j) \log s(A_i|B_j)\\ &= -\sum_{j=1}^{m} \sum s(A_i \cdot B_j) \log \frac{s(A_i \cdot B_j)}{s(B_j)}\end{aligned}$$

which is the conditional Shannon entropy of α given β defined by Equation (3). □

Theorem 7 (monotonicity)**.** *Let α and β be measurable IF-partitions of $(1_X, 0_X)$ with respect to s. Then $H_q^s(\alpha|\beta) \le H_q^s(\alpha)$.*

Proof. Let $\alpha = (A_1, \dots, A_n)$, $\beta = (B_1, \dots, B_m)$. Then by Proposition 1, we have $s(A_i) = \sum_{j=1}^{m} s(A_i \cdot B_j)$, for $i = 1, 2, \dots, n$. Suppose that $q > 1$. Then using the triangle inequality of q-norm, we get

$$\begin{aligned}\sum_{i=1}^{n} s(A_i)^q &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} s(A_i \cdot B_j)\right)^q\\ &= \left(\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{m} s(A_i \cdot B_j)\right)^q\right)^{\frac{1}{q}}\right)^q = \left(\left\|\sum_{j=1}^{m} s_{\alpha\vee\{B_j\}}\right\|_q\right)^q\\ &\le \left(\sum_{j=1}^{m} \left\|s_{\alpha\vee\{B_j\}}\right\|_q\right)^q = \left(\sum_{j=1}^{m} s(B_j) \left\|s_{\alpha\vee\{B_j\}}\right\|_q\right)^q.\end{aligned}$$

It follows that

$$\log \sum_{i=1}^{n} s(A_i)^q \le \log \left(\sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j}\right)^q,$$

and consequently,

$$\begin{aligned} H_q^s(\alpha) &= \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q \\ &\geq \frac{q}{1-q} \log \left(\sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j} \|_q \right) \\ &= H_q^s(\alpha|\beta). \end{aligned}$$

For the case where $0 < q < 1$, we put $r = \frac{1}{q}$. By writing the Rényi entropy in terms of the $\frac{1}{q}$-norm and using the triangle inequality for the $\frac{1}{q}$-norm, we get

$$\begin{aligned} H_q^s(\alpha) &= \frac{1}{1-q} \log \sum_{i=1}^{n} s(A_i)^q = \frac{r}{r-1} \log \sum_{i=1}^{n} s(A_i)^{\frac{1}{r}} \\ &= \frac{r}{r-1} \log \sum_{i=1}^{n} \left(\sum_{j=1}^{m} s(A_i \cdot B_j) \right)^{\frac{1}{r}} \\ &= \frac{r}{r-1} \log \sum_{i=1}^{n} \left\| s_{\{A_i\} \vee \beta}^{\frac{1}{r}} \right\|_r \geq \frac{r}{r-1} \log \left\| \sum_{i=1}^{n} s_{\{A_i\} \vee \beta}^{\frac{1}{r}} \right\|_r \\ &= \frac{r}{r-1} \log \left(\sum_{j=1}^{m} \left(\sum_{i=1}^{n} s(A_i \cdot B_j)^{\frac{1}{r}} \right)^r \right)^{\frac{1}{r}} \\ &= \frac{r}{r-1} \log \left(\sum_{j=1}^{m} s(B_j) \left(\sum_{i=1}^{n} s(A_i|B_j)^{\frac{1}{r}} \right)^r \right)^{\frac{1}{r}} \\ &= \frac{1}{r-1} \log \left(\sum_{j=1}^{m} s(B_j) \left(\sum_{i=1}^{n} s(A_i|B_j)^{\frac{1}{r}} \right)^r \right) \\ &= \frac{q}{1-q} \log \left(\sum_{j=1}^{m} s(B_j) \| s_{\alpha|B_j} \|_q \right) \\ &= H_q^s(\alpha|\beta). \end{aligned}$$

□

Theorem 8. *If IF-partitions α and β are statistically independent with respect to s, then*

$$H_q^s(\alpha|\beta) = H_q^s(\alpha).$$

P r o o f. Let $\alpha = (A_1, \dots, A_n)$, $\beta = (B_1, \dots, B_m)$. Put $\delta = \{j; s(B_j) > 0\}$. Since

$$\sum_{j\in\delta} s(B_j) = \sum_{j=1}^{m} s(B_j) = 1,$$

we have

$$\begin{aligned}
H_q^s(\alpha|\beta) &= \frac{q}{1-q}\log\left(\sum_{j=1}^{m} s(B_j)\left(\sum_{i=1}^{n} s(A_i|B_j)^q\right)^{\frac{1}{q}}\right)\\
&= \frac{q}{1-q}\log\left(\sum_{j\in\delta} s(B_j)\left(\sum_{i=1}^{n} \frac{s(A_i)^q s(B_j)^q}{s(B_j)^q}\right)^{\frac{1}{q}}\right)\\
&= \frac{q}{1-q}\log\left(\sum_{j\in\delta} s(B_j)\left(\sum_{i=1}^{n} s(A_i)^q\right)^{\frac{1}{q}}\right)\\
&= \frac{q}{1-q}\log\sum_{i=1}^{n} s(A_i)^q\\
&= H_q^s(\alpha). \qquad \square
\end{aligned}$$

Theorem 9. *Let α and β be measurable IF-partitions of $(1_X, 0_X)$ with respect to s and q_1, q_2 be two positive real numbers, $q_1 \neq 1$, $q_2 \neq 1$. Then $q_1 \geq q_2$ implies*

$$H_{q_1}^s(\alpha|\beta) \leq H_{q_2}^s(\alpha|\beta).$$

P r o o f. Assume that $\alpha = (A_1, \dots, A_n)$, $\beta = (B_1, \dots, B_m)$. Let $q_1, q_2 \in (1, \infty)$. Then the claim is equivalent to the inequality

$$\left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_{q_1}\right)^{\frac{q_1}{q_1-1}} \geq \left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_{q_2}\right)^{\frac{q_2}{q_2-1}}.$$

We prove this inequality by applying twice the Jensen inequality for concave functions. First, we apply the Jensen inequality for the function F_1 defined by

$$F_1(x) = x^{\frac{q_1(q_2-1)}{q_2(q_1-1)}},$$

for every $x \in [0, \infty)$, where $a_j = s(B_j)$, $x_j = \|s_{\alpha|B_j}\|_{q_1}$, $j = 1, 2, \dots, m$. The assumption $q_1 \geq q_2$ implies that $\frac{q_1(q_2-1)}{q_2(q_1-1)} \leq 1$, hence, the function F_1 is concave.

We obtain

$$
\begin{aligned}
&\left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_{q_1}\right)^{\frac{q_1}{q_1-1}} \\
&= \left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_{q_1}\right)^{\frac{q_2(q_2-1)q_1}{(q_1-1)q_2(q_2-1)}} \\
&= \left(\left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_{q_1}\right)^{\frac{q_1(q_2-1)}{q_2(q_1-1)}}\right)^{\frac{q_2}{q_2-1}} \\
&\geq \left(\sum_{j=1}^{m} s(B_j)\left(\|s_{\alpha|B_j}\|_{q_1}\right)^{\frac{q_1(q_2-1)}{q_2(q_1-1)}}\right)^{\frac{q_2}{q_2-1}} \\
&= \left(\sum_{j=1} s(B_j)\left(\sum_{i=1}^{n} s(A_i|B_j)s(A_i|B_j)^{q_1-1}\right)^{\frac{q_2-1}{q_2(q_1-1)}}\right)^{\frac{q_2}{q_2-1}}.
\end{aligned}
$$

Now, we apply the Jensen inequality for the function F_2 defined by

$$F_2(x) = x^{\frac{q_2-1}{q_1-1}},$$

for $x \in [0, \infty)$, where $a_i = s(A_i|B_j)$, $x_i = s(A_i|B_j)^{q_1-1}$, $i = 1, 2, \ldots, n$. Note that $\sum_{i=1}^{n} a_i = 1$ according to Proposition 1. The assumption $q_1 \geq q_2$ implies $\frac{q_2-1}{q_1-1} \leq 1$, hence, the function F_2 is concave. We get

$$
\begin{aligned}
&\left(\sum_{j=1}^{m} s(B_j)\left(\sum_{i=1}^{n} s(A_i|B_j)s(A_i|B_j)^{q_1-1}\right)^{\frac{q_2-1}{q_2(q_1-1)}}\right)^{\frac{q_2}{q_2-1}} \\
&\geq \left(\sum_{j=1}^{m} s(B_j)\left(\sum_{i=1}^{n} s(A_i|B_j)^{q_2}\right)^{\frac{1}{q_2}}\right)^{\frac{q_2}{q_2-1}} \\
&= \left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_{q_2}\right)^{\frac{q_2}{q_2-1}}.
\end{aligned}
$$

By combining the previous results, we obtain the required inequality. Analogously, we can prove the case where and q_1, $q_2 \in (0, 1)$. Finally, the case where $q_1 \in (1, \infty)$ and $q_2 \in (0, 1)$ follows by transitivity. $\square$

In the following theorem, a weak chain rule for Rényi entropy of IF-partitions is given.

Theorem 10. *Let $\alpha = (A_1, \dots, A_n)$, $\beta = (B_1, \dots, B_m)$ be measurable IF--partitions of $(1_X, 0_X)$ with respect to s. If we denote*

$$b = \max\left\{ \frac{1}{s(B_j)}; B_j \in \operatorname{supp}(s),\ j = 1, \dots, m \right\},$$

then

$$H_q^s(\alpha \vee \beta) \leq H_q^s(\alpha|\beta) + \log b.$$

P r o o f. The assertion follows by applying the Jensen inequality to the function F defined by $F(x) = x^q$, $x \in [0, \infty)$, and putting $a_i = s(B_j)$, $x_j = \|s_{\alpha|B_j}\|_q$, for $j = 1, 2, \dots, m$.

Let $0 < q < 1$. Then the function F is concave, and therefore, we get

$$\begin{aligned}
&\left(\sum_{j=1}^{m} s(Bj) \|a_{\alpha|B_j}\|_q \right)^q \\
&\geq \sum_{j=1}^{m} s(B_j) \sum_{i=1}^{n} s(A_i|B_j)^q \\
&= \sum_{j \in \delta} \left(\frac{1}{s(B_j)} \right)^{q_1 - 1} \sum_{i=1}^{n} s(A_i \cdot B_j)^q \\
&\geq \sum_{j \in \delta} b^{q-1} \sum_{i=1}^{n} s(A_i \cdot B_j)^q.
\end{aligned}$$

It follows

$$\begin{aligned}
H_q^s(\alpha|\beta) &= \frac{q}{1-q} \log \left(\sum_{j=1}^{m} s(B_j) \|s_{\alpha|B_j}\|_q \right) \\
&\geq \frac{1}{1-q} \log \left(b^{q-1} \sum_{j \in \delta} \sum_{i=1}^{n} s(A_i \cdot B_j)^q \right) \\
&= -\log b + \frac{1}{1-q} \log \sum_{j=1}^{m} \sum_{i=1}^{n} s(A_i \cdot B_j)^q \\
&= -\log b + H_q^s(\alpha \vee \beta).
\end{aligned}$$

Consider now the case where $q > 1$. The function F is in this case convex, and therefore, we have

$$\begin{aligned}\left(\sum_{j=1}^{m} s(B_j)\|a_{\alpha|B_j}\|_q\right)^q &\le \sum_{j=1}^{m} s(B_j)\left(\|s_{\alpha|B_j}\|_q\right)^q\\ &= \sum_{j=1}^{m} s(B_j)\sum_{i=1}^{n} s(A_i|B_j)^q\\ &= \sum_{j\in\delta}\left(\frac{1}{s(B_j)}\right)^{q-1}\sum_{i=1}^{n} s(A_i|B_j)^q\\ &\le \sum_{j\in\delta} b^{q-1}\sum_{i=1}^{n} s(A_i|B_j)^q.\end{aligned}$$

Thus

$$q\log\left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j}\|_q\right) \le (q-1)\log b + \log\sum_{j=1}^{m}\sum_{i=1}^{n} s(A_i\cdot B_j)^q.$$

Since $1-q<0$, we get

$$\begin{aligned}H_q^s(\alpha|\beta) &= \frac{q}{1-q}\log\left(\sum_{j=1}^{m} s(B_j)\|s_{\alpha|B_j\|_q}\right)\\ &\ge \frac{q-1}{1-q}\log b + \frac{1}{1-q}\log\sum_{j=1}^{m}\sum_{i=1}^{n} s(A_i|B_j)^q\\ &= -\log b + H_q^s(\alpha\vee\beta).\end{aligned}$$

□

Remark 6. Let $\alpha = (A_1,\dots,A_n)$, $\beta = (B_1,\dots,B_m)$ be measurable IF-partitions of $(1_X, 0_X)$ with respect to s. Since $\alpha\vee\beta = \beta\vee\alpha$, it holds also the inequality

$$H_q^s(\alpha\vee\beta) \le H_q^s(\beta|\alpha) + \log a,$$

where

$$a = \max\left\{\frac{1}{s(A_i)};\, A_i\in\operatorname{supp}(s),\ i=1,2,\dots,n\right\}.$$

Theorem 11. *Let $\alpha = (A_1,\dots,A_n)$, $\beta = (B_1,\dots,B_m)$ be measurable IF-partitions of $(1_X, 0_X)$ with respect to s. If we denote*

$$b = \max\left\{\frac{1}{s(B_j)};\, B_j\in\operatorname{supp}(s),\ j=1,\dots,m\right\},$$

then

$$H_q^s(\alpha\vee\beta) \le H_q(\alpha) + \log b.$$

P r o o f. The claim is a direct consequence of Theorems 6 and 8. □

5. The Rényi divergence in the intuitionistic fuzzy case

In this section, we define the concept of the Rényi divergence in the intuitionistic fuzzy case. We will prove basic properties of this quantity, and for illustration, we provide some numerical examples.

Definition 11. Let s, t be two states on $\mathcal{F}$ and let $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and t such that $t(A_i) > 0$, for $i = 1, \dots, n$. Then we define the Rényi divergence of order q where $q > 0$, $q \neq 1$, of the state s from the state t with respect to α as the number

$$D_q^{\alpha}(s\|t) = \frac{1}{q-1} \log \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q}. \tag{8}$$

Remark 7. It is easy to see that, for any measurable IF partition α of $(1_X, 0_X)$, we have $D_q^{\alpha}(s\|s) = 0$.

The following theorem states that the Rényi entropy $H_q(\alpha)$ can be expressed in terms of the Rényi divergence $D_q^{\alpha}(s|t)$ of a state s from a state t that is uniform over $\alpha = (A_1, \dots, A_n)$.

Theorem 12. *Let s, t be two states on $\mathcal{F}$ and $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and t. If the state t is uniform over α, i.e., $t(A_i) = \frac{1}{n}$, for $i = 1, 2, \dots, n$, then*

$$H_q^s(\alpha) = H_q^t(\alpha) - D_q^{\alpha}(s\|t).$$

Proof. Let us calculate:

$$\begin{aligned}
D_q^{\alpha}(s\|t) &= \frac{1}{q-1} \log \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q} \\
&= \frac{1}{q-1} \log \sum_{i=1}^{n} s(A_i)^q \left(\frac{1}{n}\right)^{1-q} \\
&= \frac{1}{q-1} \log \left(\frac{1}{n}\right)^{1-q} + \frac{1}{q-1} \log \sum_{i=1}^{n} s(A_i)^q \\
&= H_q^t(\alpha) - \frac{1}{q-1} \log \sum_{i=1}^{n} s(A_i)^q = \log n - H_q^s(\alpha).
\end{aligned}$$

From this follows the claim. □

Example 7. Consider a family $\mathcal{F}$ of IF-events and a state s defined on $\mathcal{F}$. In Example 5 we dealt with the IF-partition $\alpha = (A_1, A_2)$ of $(1_X, 0_X)$ with $s(A_1) = \frac{1}{3}$, $s(A_2) = \frac{2}{3}$, and we calculated that the Rényi entropy

$$H_q^s(\alpha) \quad \text{of order} \quad q = \frac{1}{2} \quad \text{is} \quad H_q^s(\alpha) \doteq 0.9581 \text{ bit}.$$

Let t be a state defined on $\mathcal{F}$ that is uniform over α, i.e., $t(A_1) = t(A_2) = \frac{1}{2}$. Then the Rényi divergence of order $q = \frac{1}{2}$ is $D^{\alpha}_{\frac{1}{2}}(s\|t) = -2\log\left(\sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\right) \doteq$ 0.04186 bit, and we get bit

$$H^{t}_{\frac{1}{2}}(\alpha) - D^{\alpha}_{\frac{1}{2}}(s\|t) = \log 2 - D^{\alpha}_{\frac{1}{2}}(s\|t) \doteq 1 - 0.04186 \doteq 0.9581.$$

It can be verified that the equality $H^{s}_{\frac{1}{2}}(\alpha) = \log 2 - D^{\alpha}_{\frac{1}{2}}(s\|t)$ holds.

Theorem 13. *Let s, t be two states on $\mathcal{F}$ and $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and t such that $s(A_i) > 0$ and $t(A_i) > 0$, for $i = 1, 2, \dots, n$. Then $D^{\alpha}_{q}(s\|t) \geq 0$ with the equality if and only if*

$$s(A_i) = t(A_i) \quad \textit{for} \quad i = 1, 2, \dots, n.$$

Proof. The inequality follows by applying the Jensen inequality for the functions F defined by $F(x) = x^{1-q}$, $x \in [0, \infty)$, and putting

$$a_i = s(A_i), \quad x_i = \left(\frac{t(A_i)}{s(A_i)}\right)^{1-q}, \quad i = 1, 2, \dots, n.$$

Let us consider the case $q > 1$. Then $1 - q < 0$, therefore, the function F is convex. By the Jensen inequality we obtain

$$\begin{aligned} 1 = \left(\sum_{i=1}^{n} t(A_i)\right)^{1-q} &= \left(\sum_{i=1}^{n} s(A_i)\frac{t(A_i)}{s(A-i)}\right)^{1-q} \\ &\leq \sum_{i=1}^{n} s(A_i)\left(\frac{t(A_i)}{s(A_i)}\right)^{1-q} = \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q}, \end{aligned} \tag{9}$$

and consequently,

$$\log \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q} \geq \log 1 = 0.$$

Since $\frac{1}{q-1} > 0$ for $q > 1$, it follows that

$$D^{\alpha}_{q}(a\|t) = \frac{1}{q-1}\log \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q} \geq 0.$$

Let $0 < q < 1$. Then the function F is concave, therefore, we get

$$\sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q} \leq 1,$$

and consequently,

$$\log \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q} \leq \log 1 = 0.$$

Since $\frac{1}{q-1} < 0$ for $0 < q < 1$, it follows that

$$D_q^\alpha(a\|t) = \frac{1}{q-1} \log \sum_{i=1}^{n} s(A_i)^q t(A_i)^{1-q} \geq 0.$$

The equality in (9) holds if and only if $\frac{t(A_i)}{s(A_i)}$ is constant, for $i = 1, 2, \dots, n$, i.e., if and only if $t(A_i) = c \cdot s(A_i)$, for $i = 1, 2, \dots, n$. By summing over $i = 1, 2, \dots, n$, we get $\sum_{i=1}^{n} t(A_i) = c \cdot \sum_{i=1}^{n} s(A_i)$, which implies that $c = 1$. Hence $s(A_i) = t(A_i)$, for $i = 1, 2, \dots, n$. Therefore, we conclude that $D_q^\alpha(a\|t) = 0$ if and only if $s(A_i) = t(A_i)$, for $i = 1, 2, \dots, n$. □

Corollary 2. *Let $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to a state s defined on $\mathcal{F}$ such that $s(A_i) > 0$, for $i = 1, 2, \dots, n$. Then*

$$H_q^s(\alpha) \leq \log n$$

with the equality if and only if the state s is uniform over α.

Proof. Let $t : \mathcal{F} \to [0, 1]$ be a state uniform over α, i.e., $t(A_i) = \frac{1}{n}$ for $i = 1, 2, \dots, n$. Then according to Theorems 12 and 13, it holds

$$0 \leq D_q^\alpha(s\|t) = \log n - H_q^s(\alpha),$$

which implies that $H_q^s(\alpha) \leq \log n$. Since the equality $D_q^\alpha(s\|t) = 0$ applies if and only if $s(A_i) = t(A_i)$, for $i = 1, 2, \dots, n$, the equality $H_q^s(\alpha) = \log n$ holds if and only if s is a state uniform over α. □

Example 8. Consider an arbitrary family $\mathcal{F}$ of IF-events and states s_1, s_2, s_3 defined on it. Let $\alpha = (A_1, A_2)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to states s_1, s_2, s_3 with $s_1(A_1) = p_1$, $s_2(A_1) = p_2$, $s_3(A_1) = p_3$, where $p_1, p_2, p_3 \in (0, 1)$. Then $s_1(A_2) = 1 - p_1$, $s_2(A_2) = 1 - p_2$, $s_3(A_2) = 1 - p_3$. Putting $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$, $p_3 = \frac{1}{4}$ and $q = 2$, we obtain:

$$D_2^\alpha(s_1\|s_2) = \log \sum_{i=1}^{2} s_1(A_i)^2 s_2(A_i)^{-1} \doteq 0.169925 \text{ bit};$$

$$D_2^\alpha(s_1\|s_3) = \log \sum_{i=1}^{2} s_1(A_i)^2 s_3(A_i)^{-1} \doteq 0.415037 \text{ bit};$$

$$D_2^\alpha(s_2\|s_3) = \log \sum_{i=1}^{2} s_2(A_i)^2 s_3(A_i)^{-1} \doteq 0.052467 \text{ bit}.$$

Evidently, $D_2^\alpha(s_1\|s_3) > D_2^\alpha(s_1\|s_2) + D_2^\alpha(s_2\|s_3)$. If we put $q = \frac{1}{2}$, then by simple calculations we get: $D_q^\alpha(s_1\|s_2) \doteq 0.04186$ bit, $D_q^\alpha(s_1\|s_3) \doteq 0.1$ bit, $D_q^\alpha(s_2\|s_3) \doteq 0.0122$ bit. In this case also applies that $D_{\frac{1}{2}}^\alpha(s_1\|s_3) > D_{\frac{1}{2}}^\alpha(s_1\|s_2) + D_{\frac{1}{2}}^\alpha(s_2\|s_3)$.

This means that the triangle inequality for the Rényi divergence $D_q^\alpha(s\|t)$ generally does not apply. In the same way it can be shown that the equality $D_q^\alpha(s\|t) = D_q^\alpha(t\|s)$ is not necessarily true, thus the Rényi divergence $D_q^\alpha(s\|t)$ is not symmetric. The result means that it is not a metric in a true sense.

The following theorem states that for $q \to 1$ the Rényi divergence $D_q^\alpha(s\|t)$ converges to the Kullback-Leibler divergence $d_\alpha(s\|t)$ defined by formula (5).

Theorem 14. *Let s, t be two states on $\mathcal{F}$ and $\alpha = (A_1, \dots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and t such that $t(A_i) > 0$, for $i = 1, \dots, n$. Then*

$$\lim_{q\to 1} D_q^\alpha(s\|t) = \sum_{i=1}^n s(A_i) \cdot \log \frac{s(A_i)}{t(A_i)}.$$

P r o o f. For every $q \in (0,1) \cup (1,\infty)$, we can write:

$$D_q^\alpha(s\|t) = \frac{1}{q-1} \log \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q} = \frac{f(q)}{g(q)},$$

where f, g are continuous functions defined, for every $q \in (0,\infty)$, in the following way

$$f(q) = \log \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q}, \quad g(q) = q - 1.$$

By continuity of the functions f, g, we have $\lim_{q\to 1} g(q) = g(1) = 0$, and

$$\lim_{q\to 1} f(q) = f(1) = \log \sum_{i=1}^n s(A_i) t(A_i)^0 = \log 1 = 0.$$

Using L'Hôspital's rule, we get that $\lim_{q\to 1} D_q^\alpha(s\|t) = \lim_{q\to 1} \frac{f'(q)}{g'(q)}$, under the assumption that the right-hand side exists. Since

$$g'(q) = 1 \quad \text{and} \quad f'(q) = \frac{h'(q)}{h(q)\ln 2},$$

where

$$h(q) = \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q}, \quad \text{and} \quad h'(q) = \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q} \big(\ln s(A_i) - \ln t(A_i)\big),$$

we obtain

$$\begin{aligned} \lim_{q\to 1} D_q^\alpha(s\|t) = \lim_{q\to 1} f'(q) &= \frac{1}{\ln 2} \sum_{i=1}^n s(A_i) \big(\ln s(A_i) - \ln t(A_i)\big) \\ &= \sum_{i=1}^n s(A_i) \cdot \log \frac{s(A_i)}{t(A_i)}. \end{aligned}$$

□

Theorem 15. *Let s, t be two states on $\mathcal{F}$ and $\alpha = (A_1, \ldots, A_n)$ be a measurable IF-partition of $(1_X, 0_X)$ with respect to s and t such that*

$$s(A_i) > 0, \quad \text{and} \quad t(A_i) > 0, \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Then

(i): $0 < q < 1$ *implies* $D_q^\alpha(a\|t) \leq d_\alpha(s\|t)$;

(ii): $q > 1$ *implies* $D_q^\alpha(s\|t) \geq d_\alpha(s\|t)$,

where

$$d_\alpha(s\|t) = \sum_{i=1} s(A_i) \cdot \log \frac{s(A_i)}{t(A_i)}.$$

Proof. By using the Jensen inequality for the concave function F defined by $F(x) = \log x$, for $x \in (0, \infty)$, and putting $a_i = s(A_i)$, $x_i = \left(\frac{s(A_i)}{t(A_i)}\right)^{q-1}$, for $i = 1, 2, \ldots, n$, we get

$$\begin{aligned} \log \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q} &= \log \sum_{i=1}^n s(A_i) \left(\frac{s(A_i)}{t(A_i)}\right)^{q-1} \\ &\geq \sum_{i=1}^n s(A_i) \log \left(\frac{s(A_i)}{t(A_i)}\right)^{q-1} \\ &= (q-1) \sum_{i=1}^n s(A_i) \log \frac{s(A_i)}{t(A_i)}. \end{aligned}$$

Let $0 < q < 1$. Then $\frac{1}{q-1} < 0$, and therefore, we have

$$\begin{aligned} D_q^\alpha(s\|t) =& \frac{1}{q-1} \log \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q} \\ \leq& \sum_{i=1}^n s(A_i) \log \frac{s(A_i)}{t(A_i)} = d_\alpha(s\|t). \end{aligned}$$

Let us consider the case of $q > 1$. Since $\frac{1}{1-q} > 0$ for $q > 1$, we get

$$\begin{aligned} D_q^\alpha(s\|t) =& \frac{1}{q-1} \log \sum_{i=1}^n s(A_i)^q t(A_i)^{1-q} \\ \geq& \sum_{i=1}^n s(A_i) \log \frac{s(A_i)}{t(A_i)} = d_\alpha(s\|t). \end{aligned}$$

$\square$

EXAMPLE 9. Consider the states s_1, s_2, s_3 and the IF-partition $\alpha = (A_1, A_2)$ from Example 8. It can be calculated that the Kullback-Leibler divergences

$$d_\alpha(s_1 \| s_2) \doteq 0.084963 \text{ bit},$$
$$d_\alpha(s_1 \| s_3) \doteq 0.207518 \text{ bit},$$

and

$$d_\alpha(s_2 \| s_3) \doteq 0.025062 \text{ bit}.$$

Based on the previous results, we have:

$$D_2^\alpha(s_1 \| s_2) \doteq 0.169925 \text{ bit},$$
$$D_2^\alpha(s_1 \| s_3) \doteq 0.415037 \text{ bit},$$
$$D_2^\alpha(s_2 \| s_3) \doteq 0.052467 \text{ bit}.$$

Evidently, the obtained results correspond to the claim (ii) of Theorem 15. For $q = \frac{1}{2}$ we have:

$$D_q^\alpha(s_1 \| s_2) \doteq 0.04186 \text{ bit},$$
$$D_q^\alpha(s_1 \| s_3) \doteq 0.1 \text{ bit},$$
$$D_q^\alpha(s_2 \| s_3) \doteq 0.0122 \text{ bit}.$$

which corresponds to the claim (i) of Theorem 15.

6. Conclusion

The aim of this paper was to extend the study concerning the Shannon entropy and Kullback-Leibler divergence in the intuitionistic fuzzy case to the case of Rényi entropy and Rényi divergence. The results are contained in Sections 3–5. In Section 3, we have introduced the concept of Rényi entropy of IF-partitions and we examined properties of this entropy measure. Specifically, it was shown that the Rényi entropy $H_q^s(\alpha)$ is monotonically decreasing in q. In Section 4, we have defined the Rényi conditional entropy of IF-partitions. It was shown that the suggested concepts are consistent with the Shannon entropy of IF-partitions defined and studied by the authors in [24]. Section 5 was devoted to the study of Rényi divergence in the intuitionistic fuzzy case. We have proved that the Kullback-Leibler divergence of states on a family of IF-events can be derived from their Rényi divergence as the limiting case for q going to 1. Theorem 13 enables the interpretation of Rényi divergence as a measure of the distance between two states over the same IF-partition. In addition, we have investigated the relationship between the Rényi divergence and the Rényi entropy of IF-partitions (Theorem 12) as well as the relationship between the Rényi divergence and Kullback-Leibler divergence in the intuitionistic fuzzy case (Theorem 15).

In the proofs we used L'Hôpital's rule, the triangle inequality of q-norm and the Jensen inequality. To illustrate the results, we have provided several numerical examples.

Acknowledgement. The second author thanks the editor and the referees for their valuable comments and suggestions. It is my sad duty to announce that the first author Professor Beloslav Riečan, passed away, unexpectedly, on August 13, 2018, before this article was published. Rest in peace, our beloved friend, co-worker, and teacher.

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Received October 9, 2018

Beloslav Riečan †
Institute of Mathematics
Slovak Academy of Sciences
Štefánikova 49
SK-814–73 Bratislava

Department of Mathematics
Faculty of Natural Sciences
Matej Bel University
Tajovského 40
SK-974–01 Banská Bystrica
SLOVAKIA

Dagmar Markechová
Department of Mathematics
Faculty of Natural Sciences
Constantine the Philosopher University
A. Hlinku 1
SK-949–01 Nitra
SLOVAKIA
E-mail: dmarkechova@ukf.sk