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SIMPLIFICATION OF COEFFICIENTS IN DIFFERENTIAL EQUATIONS ASSOCIATED WITH HIGHER ORDER FROBENIUS-EULER NUMBERS

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ABSTRACT. In the paper, the authors apply Faà di Bruno formula, some properties of the Bell polynomials of the second kind, the inversion formulas of binomial numbers and the Stirling numbers of the first and the second kind, to significantly simplify coefficients in two families of ordinary differential equations associated with the higher order Frobenius–Euler numbers.

1. Motivations

In [4, Theorem 2.2], it was inductively and recursively established that the family of differential equations

$$F^{(n)}(t)=\left[\sum_{i=0}^{n}a_i(n)\left(\frac{u}{\mathrm{e}^t-u}\right)^i\right]F(t) \tag{1}$$

for $n\ge 0$, $r\in\mathbb{N}$, and $u\in\mathbb{C}\setminus\{1\}$ has a solution

$$F(t)=F(t;r,u)=\left(\frac{1}{\mathrm{e}^t-u}\right)^r, \tag{2}$$

where $a_0(n)=(-r)^n$,

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$$a_i(n) = (-1)^n (r+i-1)_i \sum_{k_i=0}^{n-i} \sum_{k_{i-1}=0}^{n-i-k_i} \cdots \sum_{k_1=0}^{n-i-k_i-\cdots k_2} r^{n-i-\sum_{\ell=1}^{i} k_\ell} \prod_{\ell=1}^{i} (r+\ell)^{k_\ell} \quad (3)$$

for $1 \le i \le n$, and

$$(x)_n = \prod_{\ell=0}^{n-1} (x+\ell) = \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases}$$

is the rising factorial. Hereafter, the following results were deduced.

(1) For $k, n \ge 0$, we have

$$H_{k+n}^{(r)}(u) = \sum_{i=0}^{n} a_i(n) \Big(\frac{u}{1-u}\Big)^i H_k^{(r+i)}(u),$$

where $H_k^{(r)}$, which can be generated by

$$\Big(\frac{1-u}{\mathrm{e}^t - u}\Big)^r = \sum_{k=0}^{\infty} H_k^{(r)}(u) \frac{t^k}{k!},$$

stand for the Frobenius-Euler numbers of order r. See [4, Theorem 2.3].

(2) For $k, n \ge 0$, we have

$$E_{k+n}^{(r)} = \sum_{i=0}^{n} \Big(-\frac{1}{2}\Big)^i a_i(n) E_k^{(r+i)},$$

where $E_k^{(r)}$, which can be generated by

$$\Big(\frac{2}{e^t + 1}\Big)^r = \sum_{k=0}^{\infty} E_k^{(r)} \frac{t^k}{k!},$$

stand for the Euler numbers of order r. See [4, Corollary 2.5].

(3) When $0 \le k \le r-1$ and $k \ge r+n$,

$$B_k^{(r)} = \frac{1}{(k-r)_n} \sum_{i=\max\{n-k,0\}}^{n} a_i(n) B_{k+i-n}^{(r+i)} (k)_{n-i};$$

when $r \le k \le r-1+n$,

$$\sum_{i=\max\{n-k,0\}}^{n} a_i(n) B_{k+i-n}^{(r+i)} (k)_{n-i} = 0;$$

where $B_k^{(r)}$, which can be generated by

$$\Big(\frac{t}{e^t - 1}\Big)^r = \sum_{k=0}^{\infty} B_k^{(r)} \frac{t^k}{k!},$$

stand for the Bernoulli numbers of order r. See [4, Theorem 2.7].

In [5, Theorem 2.1], it was inductively and recursively proved that the family of differential equations

$$(-1)^{n-1}(r)_n\left(\frac{u}{e^t-u}\right)^n F(t)=\sum_{i=0}^{n} b_i(n)F^{(i)}(t) \tag{4}$$

for $u\in\mathbb{C}$ and $r\in\mathbb{N}$ has a solution $F(t)$ defined in (2), where $b_0(n)=-\langle r+n-1\rangle_n$,

$$\begin{aligned} b_i(n)= &-\sum_{k_i=0}^{n-i}\sum_{k_{i-1}=0}^{n-i-k_i}\cdots\sum_{k_1=0}^{n-i-k_i-\cdots-k_2}\prod_{\ell=1}^{i}\left\langle r+n-i-1-\sum_{j=\ell+1}^{i}k_j+\ell\right\rangle_{k_\ell}\\ &\times\left\langle r+n-i-1-\sum_{j=1}^{i}k_j\right\rangle_{n-i-\sum_{j=1}^{i}k_j}, \qquad 1\le i\le n, \end{aligned} \tag{5}$$

and

$$\langle x\rangle_n=\prod_{\ell=0}^{n-1}(x-\ell)=\begin{cases} x(x-1)(x-2)\dots(x-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

is the falling factorial of $x\in\mathbb{R}$ for $n\in\{0\}\cup\mathbb{N}$. Hereafter, the following conclusions were derived:

(1) For $k,n\ge0$, we have

$$(-1)^{n-1}(r)_n\left(\frac{u}{1-u}\right)^n H_k^{(r+n)}(u)=\sum_{i=0}^{n} b_i(n)H_{k+i}^{(r)}(u). \tag{6}$$

See [5, Theorem 3.1]. In particular, taking $u=-1$ in (6) leads to

$$(-1)^{n-1}(r)_n\left(-\frac{1}{2}\right)^n E_k^{(r+n)}=\sum_{i=0}^{n} b_i(n)E_{k+i}^{(r)}.$$

See [5, Corollary 3.3].

(2) When $0\le k\le n+r-1$, we have

$$B_k^{(r+n)}=(-1)^{n-1}\frac{1}{(r)_n}\sum_{i=\max\{n-k,0\}}^{\min\{r+n-1-k,n\}} b_i(n)B_{k+i-n}^{(r)}\frac{\langle k+i-n-r\rangle_i k!}{(k+i-n)!};$$

when $k\ge n+r$, we have

$$B_k^{(r+n)}=(-1)^{n-1}\frac{1}{(r)_n}\sum_{i=0}^{n} b_i(n)B_{k+i-n}^{(r)}\frac{\langle k+i-n-r\rangle_i k!}{(k+i-n)!}.$$

See [5, Theorem 3.4].

(3) The matrices $\big(a_i(j)\big)_{0\le i,j\le n}$ and $\Big(\frac{b_i(j)}{(-1)^{j-1}(r)_j}\Big)_{0\le i,j\le n}$ are inverse to each other for all n. See [5, Remark 3.2].

It is easy to see that expressions (3) and (5) of the quantities $a_i(n)$ and $b_i(n)$ are too complicated to be computed by hand and computer software. Therefore, can one find simple, meaningful, and significant expressions for the quantities $a_i(n)$ and $b_i(n)$ in (3) and (5)?

2. Lemmas

For answering the above question and proving our main results, we need the following lemmas.

Lemma 1 ([1, p. 134, Theorem A] and [1, p. 139, Theorem C]). *For $n\ge k\ge 0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\Big(\frac{x_i}{i!}\Big)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}\big(h(t)\big)\,\mathrm{B}_{n,k}\Big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\Big). \tag{7}$$

Lemma 2 ([1, p. 135]). *For $n\ge k\ge 0$, we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big) = a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) \tag{8}$$

and

$$\mathrm{B}_{n,k}(1,1,\dots,1) = S(n,k), \tag{9}$$

where a and b are any complex numbers and $S(n,k)$ for $n\ge k\ge 0$, which can be generated by

$$\frac{(\mathrm{e}^x-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!},$$

stand for the Stirling numbers of the second kind.

Lemma 3 ([27, p. 171, Theorem 12.1]). *If b_α and a_k are a collection of constants independent of n, then*

$$a_n = \sum_{\alpha=0}^{n} S(n,\alpha) b_\alpha \qquad \text{if and only if} \qquad b_n = \sum_{k=0}^{n} s(n,k) a_k,$$

where $s(n,k)$ for $n \ge k \ge 0$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \qquad |x|<1,$$

stand for the Stirling numbers of the first kind.

Lemma 4 ([27, p. 83, Eq. (7.12)]). *If a_k and b_k for $k \ge 0$ are a collection of constants independent of n, then*

$$a(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b(k) \qquad \text{if and only if} \qquad b(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a(k).$$

3. Main results and their proofs

Now we are able to answer the above question and to state and prove our main results.

Theorem 1. *For $n \ge 0$, $r \in \mathbb{R}$, and $u \in \mathbb{C}$, the function $F(t)$ defined by (2) satisfies*

$$F^{(n)}(t) = \sum_{\ell=0}^{n}\left[\sum_{k=\ell}^{n}\binom{k}{\ell} S(n,k)\langle -r\rangle_k\right]\left(\frac{u}{e^t-u}\right)^{\ell} F(t) \tag{10}$$

and

$$\sum_{k=0}^{n}\left[\sum_{\ell=k}^{n}(-1)^{\ell}\binom{n}{\ell}\frac{s(\ell,k)}{\langle -r\rangle_\ell}\right] F^{(k)}(t) = (-1)^n\left(\frac{u}{e^t-u}\right)^{n} F(t). \tag{11}$$

Proof. Let

$$F(t) = \frac{1}{w^r} \quad \text{and} \quad w = w(t) = w(t;u) = e^t - u.$$

Then, by the Faà di Bruno formula (7) and the identities (8) and (9) in sequence,

$$\begin{aligned} F^{(n)}(t) &= \sum_{k=0}^{n}\left(\frac{1}{w^r}\right)^{(k)} \mathrm{B}_{n,k}(e^t, e^t, \dots, e^t) \\ &= \sum_{k=0}^{n}\frac{\langle -r\rangle_k}{w^{r+k}} e^{kt}\, \mathrm{B}_{n,k}(1,1,\dots,1) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \frac{\langle -r\rangle_k}{(\mathrm{e}^t-u)^{r+k}} \mathrm{e}^{kt} S(n,k) \\
&= \frac{1}{(\mathrm{e}^t-u)^r} \sum_{k=0}^{n} \frac{\langle -r\rangle_k}{(\mathrm{e}^t-u)^{k}} \mathrm{e}^{kt} S(n,k) \\
&= F(t) \sum_{k=0}^{n} \langle -r\rangle_k S(n,k) \Big(\frac{\mathrm{e}^t}{\mathrm{e}^t-u}\Big)^k \\
&= F(t) \sum_{k=0}^{n} \langle -r\rangle_k S(n,k) \Big(1+\frac{u}{\mathrm{e}^t-u}\Big)^k \\
&= F(t) \sum_{k=0}^{n} \langle -r\rangle_k S(n,k) \sum_{\ell=0}^{k} \binom{k}{\ell} \Big(\frac{u}{\mathrm{e}^t-u}\Big)^\ell \\
&= F(t) \sum_{\ell=0}^{n} \Bigg[\sum_{k=\ell}^{n} \langle -r\rangle_k S(n,k) \binom{k}{\ell}\Bigg] \Big(\frac{u}{\mathrm{e}^t-u}\Big)^\ell.
\end{aligned}
$$

Therefore, the identity (10) follows immediately.

From the above proof of the identity (10), it can be deduced that

$$
F^{(n)}(t) = F(t) \sum_{k=0}^{n} S(n,k) \langle -r\rangle_k \sum_{\ell=0}^{k} \binom{k}{\ell} \Big(\frac{u}{\mathrm{e}^t-u}\Big)^\ell, \qquad n \ge 0.
$$

Utilizing Lemma 3 we obtain

$$
F(t)\langle -r\rangle_n \sum_{\ell=0}^{n} \binom{n}{\ell} \Big(\frac{u}{\mathrm{e}^t-u}\Big)^\ell = \sum_{k=0}^{n} s(n,k) F^{(k)}(t), \qquad n \ge 0
$$

which can be rearranged as

$$
\sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} \Big(-\frac{u}{\mathrm{e}^t-u}\Big)^\ell = \frac{1}{F(t)\langle -r\rangle_n} \sum_{k=0}^{n} s(n,k) F^{(k)}(t), \qquad n \ge 0.
$$

Further use of Lemma 4 derives

$$
\sum_{\ell=0}^{n} (-1)^\ell \binom{n}{\ell} \frac{1}{F(t)\langle -r\rangle_\ell} \sum_{k=0}^{\ell} s(\ell,k) F^{(k)}(t) = \Big(-\frac{u}{\mathrm{e}^t-u}\Big)^n, \qquad n \ge 0
$$

which can be rewritten as (11). The required proof is complete. $\square$

4. Remarks

In this section, we give several remarks and some explanation about our main results.

Remark 1. Theorem 1 extends the range of r from $\mathbb{N}$ to $\mathbb{R}$.

Remark 2. Comparing (1) with (10) one reveals that

$$a_i(n) = \sum_{k=i}^{n} \binom{k}{i} S(n,k)\langle -r\rangle_k, \qquad 0 \le i \le n. \tag{12}$$

This implies that the identity (10) is more meaningful, more significant, more computable than the one (1).

Remark 3. It is not difficult to see that

$$a_0(n) = \sum_{k=0}^{n} S(n,k)\langle -r\rangle_k = (-r)^n, \qquad n \ge 0.$$

Then it is natural to ask a question: Is the finite sum

$$a_i(n) = \sum_{k=i}^{n} \binom{k}{i} S(n,k)\langle -r\rangle_k, \qquad 1 \le i \le n$$

summarizable?

Remark 4. Comparing (4) with (11) one obtains

$$b_i(n) = (r)_n \sum_{\ell=i}^{n} (-1)^{\ell+1} \binom{n}{\ell} \frac{s(\ell,i)}{\langle -r\rangle_\ell}, \qquad n \ge i \ge 0. \tag{13}$$

This means that the identity (11) is more meaningful, more significant, more computable than (4).

Remark 5. By virtue of the expressions (12) and (13), all the above mentioned results in the papers [4, 5] can be reformulated simpler, more meaningfully, and more significantly. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in detail here.

Remark 6. Till now we can see that the method used in this paper is simpler, shorter, nicer, more meaningful, and more significant than the inductive and recursive method used in [4, 5] and closely related references therein.

Remark 7. In the papers and preprints [2, 3, 6–20, 22–26, 28], there are similar ideas, methods, techniques, and purposes to this paper.

Remark 8. This paper is a slightly revised version of the preprint [21].

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