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COMPUTATION OF DEFINITE INTEGRAL OVER REPEATED INTEGRAL

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ABSTRACT. The tasks involving repeated integral occur from time to time in technical practice. This paper introduces the research of authors in the field of repeated integrals within the required class of functions. Authors focus on the definite integral over repeated integral and they develop a tool for its computation. It involves two principal steps, analytical and numerical step.

In the analytical step, the definite integral over a repeated integral is decomposed into n integrals and then the Cauchy form is used for further rearrangement. Numerical step involves Gauss type integration slightly modified by the authors. Several examples illustrating the operation of both analytical and numerical steps of the method are provided in the paper.

Introduction

Nowadays, with a huge development of computers, the technical analysis tends towards the numerical computation of its problems including nonlinear differential equations of higher order. These problems often involve repeated integrals. Repeated integral is an integral taken multiple times over a single variable:

$$f(x)^{(-n)} = \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)\mathrm{d}x_n \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1. \tag{1}$$

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Further, in the paper, we will use Cauchy form as well, which rearranges the repeated integral as follows [4]:

$$f(x)^{(-n)} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t. \tag{2}$$

Motivation—technical example

The deflection of a loaded beam is governed by the differential equation of the 4th order [3, 6]:

$$\frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = \frac{q(x)}{EI}, \qquad x \in \langle 0, L\rangle, \tag{3}$$

where $w(x)[m]$ is the deflection at the point x, $q(x)[N/m]$ distributed load, a piecewise continuous, in general not continuous function on $\langle 0, L\rangle$, $I[m^4]$ is the second moment of inertia, $E[Pa]$ is the modulus of elasticity. Taking an interval $\langle a, b\rangle \subset \langle 0, L\rangle$, where $q(x)$ is continuous, we can express general solution of (3) on this interval $\langle a, b\rangle$ as follows:

$$\begin{aligned} w(x) &= w_a + \varphi_a (x-a) + \frac{M_a}{EI}\frac{(x-a)^2}{2} + \frac{V_a}{EI}\frac{(x-a)^3}{6} \\ &\quad + \frac{1}{EI}\int_a^x \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} q(x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1, \qquad x \in \langle a, b\rangle. \end{aligned} \tag{4}$$

Function $w(x)$ represents the deflection curve of the beam within the interval $\langle a, b\rangle \subset \langle 0, L\rangle$ and the deflection at an arbitrary point $p \in \langle a, b\rangle$ is then

$$\begin{aligned} w(p) &= w_a + \varphi_a \tilde{p} + \frac{M_a}{EI}\frac{\tilde{p}^2}{2} + \frac{V_a}{EI}\frac{\tilde{p}^3}{6} \\ &\quad + \frac{1}{EI}\int_a^p \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} q(x_4)\mathrm{d}x_4\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \quad \text{with } \tilde{p} = p - a. \end{aligned} \tag{5}$$

Remark 1. In (4) and (5) the integration constants have their physical meaning: $w_a = w(a)$ is the deflection, $\varphi_a = \varphi(a) \approx \tan\varphi_a = \frac{\mathrm{d}w(a)}{\mathrm{d}x}$, the slope of the beam, φ_a being “sufficiently small”, $-V_a = -V(a)$ the shear force and $-M_a = -M(a)$ the bending moment at the point $x = a$. By exploiting the lower order integral, we keep direct connection with these physical phenomena. It can be seen in that the deflection curve, the equation (5) involves a definite integral over a repeated integral. So far no numerical method that resolves this type of problems is known. We have developed one that is introduced in this paper. The crucial point of the paper is in the Theorem 1. The formula (7) enables the decomposition of a definite integral on the interval $\langle a, b\rangle$ over an $n-1$ times repeated integral into two members. The first member is the sum of n–1 definite integrals on the interval $\langle 0, a\rangle$ over a repeated integral, where the repetition goes from 1 to $n-1$ and the lower bound is 0. The second member is the definite integral on the interval $\langle a,b\rangle$ over an $n-1$ times repeated integral, each integral within the repetition has the lower bound equal to a. All addends can be further regarded and treated as repeated integrals.

1. Formulation of the problem, the analytical step

Definition 1. Let us have a functional space $\mathscr{P}_{(\langle a,b\rangle,0)}$ of continuous n times integrable functions of one variable over the interval $\langle \min\{0,a\}, \max\{0,b\}\rangle$, where $a \leq b$. Functional $D^{(-n)}_{\langle a,b\rangle}(f) : \mathscr{P}_{(\langle a,b\rangle,0)} \to \mathbf{R}$ will stand for a definite integral over $n-1$ times repeated integral, n is an arbitrary integer number, $n \geq 2$:

$$D^{(-n)}_{\langle a,b\rangle}(f) = \int\limits_a^b \int\limits_0^{x_1} \int\limits_0^{x_2} \cdots \int\limits_0^{x_{n-1}} f(x_n)\mathrm{d}x_n \ldots \mathrm{d}x_2 \mathrm{d}x_1. \tag{6}$$

Theorem 2. *If $f(x) \in \mathscr{P}_{(\langle a,b\rangle,0)}$, then functional $D^{(-n)}_{\langle a,b\rangle}(f)$ with integer $n \geq 2$ can be expressed in the form*

$$D^{(-n)}_{\langle a,b\rangle}(f) = \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} \int\limits_0^a \int\limits_0^{x_1} \cdots \int\limits_0^{x_{n-i-1}} f(x_{n-i})\mathrm{d}x_{n-i} \ldots \mathrm{d}x_1 + \int\limits_a^b \int\limits_a^{x_1} \cdots \int\limits_a^{x_{n-1}} f(x_n)\mathrm{d}x_n \ldots \mathrm{d}x_1. \tag{7}$$

P r o o f.
The proof for integer $n \geq 2$ can be carried out by using mathematical induction:
Base case: $n = 2$.
When employing function $g(x) = \int\limits_0^x f(t)\mathrm{d}t$, for $n = 2$ we can write

$$\begin{aligned} D^{(-2)}_{\langle a,b\rangle}(f) &= \int\limits_a^b \int\limits_0^x f(t)\mathrm{d}t = \int\limits_a^b g(x)\mathrm{d}x = \int\limits_a^b g(a)\mathrm{d}x + \int\limits_a^b g(x) - g(a)\mathrm{d}x \\ &= g(a)\int\limits_a^b \mathrm{d}x + \int\limits_a^b \left(\int\limits_0^x f(t)\mathrm{d}t - \int\limits_0^a f(t)\mathrm{d}t \right) \mathrm{d}x \\ &= (b-a)\int\limits_0^a f(t)\mathrm{d}t + \int\limits_a^b \int\limits_a^x f(t)\mathrm{d}t\mathrm{d}x\,. \end{aligned} \tag{8}$$

Induction step: (7) being the induction hypothesis, it has to be proven

$$D^{(-n-1)}_{\langle a,b\rangle}(f) = \left(\int_a^b\int_0^{x_1}\cdots\int_0^{x_n} f(x_{n+1})\mathrm{d}x_{n+1}\ldots\mathrm{d}x_1\right) =$$

$$\sum_{i=1}^{n}\frac{(b-a)^i}{i!}\int_0^a\int_0^{x_1}\cdots\int_0^{x_{n-i}} f(x_{n-i+1})\mathrm{d}x_{n-i+1}\ldots\mathrm{d}x_1 +$$

$$\int_a^b\int_a^{x_1}\cdots\int_a^{x_n} f(x_{n+1})\mathrm{d}x_{n+1}\ldots\mathrm{d}x_1. \quad (9)$$

Let us denote $g(x_n) = \int_0^{x_n} f(x_{n+1})\mathrm{d}x_{n+1}$. By subsequent using (7) for $g(x_n)$ we get

$$D^{(-n-1)}_{\langle a,b\rangle}(f) = \sum_{i=1}^{n-1}\frac{(b-a)^i}{i!}\int_0^a\int_0^{x_1}\cdots\int_0^{x_{n-i-1}} g(x_{n-i})\mathrm{d}x_{n-i}\ldots\mathrm{d}x_1 +$$

$$\int_a^b\int_a^{x_1}\cdots\int_a^{x_{n-1}} g(x_n)\mathrm{d}x_n\ldots\mathrm{d}x_1, \quad (10)$$

we can proceed with the sum in (10) by using

$$g(x_{n-i}) = \int_0^{x_{n-i}} f(x_{n-i+1})\mathrm{d}x_{n-i+1}$$

and with the very last integral by using

$$g(x_n) - g(a) = \int_a^{x_n} f(x_{n+1})\mathrm{d}x_{n+1}.$$

Then from (10) we get

$$\sum_{i=1}^{n-1}\frac{(b-a)^i}{i!}\int_0^a\int_0^{x_1}\cdots\int_0^{x_{n-i}} f(x_{n-i+1})\mathrm{d}x_{n-i+1}\ldots\mathrm{d}x_1 +$$

$$\int_a^b\int_a^{x_1}\cdots\int_a^{x_n} f(x_{n+1})\mathrm{d}x_{n+1}\ldots\mathrm{d}x_1 + \int_a^b\int_a^{x_1}\cdots\int_a^{x_{n-1}} g(a)\mathrm{d}x_n\ldots\mathrm{d}x_1. \quad (11)$$

Since

$$\int_a^b\int_a^{x_1}\cdots\int_a^{x_{n-1}} g(a)\mathrm{d}x_n\ldots\mathrm{d}x_1 = g(a)\frac{(b-a)^n}{n!} = \frac{(b-a)^n}{n!}\int_0^a f(x_1)\mathrm{d}x_1,$$

we can append it as the n^{th} member to the sum in (11) to get (9). $\square$

By using Cauchy form (2) we can proceed with (7) by rearranging both its addends entirely to the single integrals

$$D^{(-n)}_{\langle a,b\rangle}(f) = \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} \frac{1}{(n-i-1)!} \int_0^a (a-t)^{n-i-1} f(t)\mathrm{d}t + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f(t)\mathrm{d}t. \quad (12)$$

But since each lower order of repetition in integration leads to a certain physical phenomenon (because the derivative of deflection yields the slope, the derivative of the slope yields bending moment, etc.), in technical practice it is worth to perform a partial rearrangement of (7):

$$D^{(-n)}_{\langle a,b\rangle}(f) = \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} \int_0^a \int_0^{x_1} \cdots \int_0^{x_{n-i-1}} f(x_{n-i})\mathrm{d}x_{n-i} \ldots \mathrm{d}x_1 + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f(t)\mathrm{d}t. \quad (13)$$

Here we leave the first member unchanged for the sake of further better handling when performing a numerical computation. For example, when a parabolic differential equation is solved by using the explicit numerical scheme when the next step is computed directly from the previous one. The validity of Theorem 1 is illustrated in the Example 1 and Example 2.

Example 1. Analytically easy solvable definite integral over a repeated integral

$$I = \int_3^7 \int_0^{x_1} \int_0^{x_2} \big(Ax_3^2 + Bx_3 + C\big)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 = \frac{4141}{15}A + \frac{290}{3}B + \frac{290}{3}C$$

can be computed by using (7) together with (12) with $n = 3, a = 3, b = 7$,

$$\begin{aligned} I &= (7-3)\int_0^3 \int_0^{x_1} \big(Ax_2^2 + Bx_2 + C\big)\mathrm{d}x_2\mathrm{d}x_1 + \frac{(7-3)^2}{2!} \int_0^3 (Ax_1^2 + Bx_1 + C)\mathrm{d}x_1 \\ &\quad + \int_3^7 \int_3^{x_1} \int_3^{x_2} \big(Ax_3^2 + Bx_3 + C\big)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 \\ &= 4\int_0^3 (3-x)\big(Ax^2 + Bx + C\big)\mathrm{d}x + 8\int_0^3 (Ax^2 + Bx + C)\mathrm{d}x \\ &\quad + \frac{1}{2}\int_3^7 (7-x)^2\big(Ax^2 + Bx + C\big)\mathrm{d}x = \frac{4141}{15}A + \frac{290}{3}B + \frac{290}{3}C. \end{aligned}$$

Example 2. While solving following integral analytically, we face the singularity problems since the computation involves $0.Log(0)$.

$$I=\int_3^7\int_0^{x_1}\int_0^{x_2}\frac{1}{(7-x_3)^2}\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1=\lim_{b\to 7^-}\left[x+(7-x_3)\ln(7-x_3)-x_1\ln 7-\frac{x_1^2}{14}\right]_3^7.$$

However, by employing L'Hospital's rule we proceed easily to get

$$I = 8/7 + 4(\ln 7 - \ln 4) \approx 3,381320295.$$

Nevertheless, we avoid this singularity by using our method, i.e., by using (7) and (12) with $n = 3, a = 3, b = 7$.

$$I=4\int_0^3(3-x)\frac{1}{(7-x_2)^2}\mathrm{d}x_2\mathrm{d}x_1+ \\ 8\int_0^3\frac{1}{(7-x_2)^2}\mathrm{d}x_1+\frac{1}{2}\int_3^7(7-x)^2\frac{1}{(7-x_2)^2}\mathrm{d}x_2\mathrm{d}x_1= \\ 4(\ln 7-\ln 4)-\frac{12}{7}+\frac{6}{7}+2\approx 3,381320295\,.$$

As illustrated in the examples above, the use of (7) together with (12) or (13) is automatic for suitable function. However, if we want to automate all computation in order to make it usable and useful for convenient engineer computation, we face the task - evaluation of definite integral within the bounds a, b and $0, a$. Let us recall that we deal with a class of functions mentioned in Theorem 1. We start with Newton-Cotes forms. Because these quadratures are not sufficiently accurate in general, we employ the most efficient tool for numerical integration, a Gaussian type integration method. It minimizes the number of nodes and ensures the result of high precision at the same time.

2. Gauss numerical integration used for the computation over the repeated integrals

As it is known, Gauss quadrature uses the approximation

$$\int_{-1}^{1}f(x)\mathrm{d}x\approx\sum_{i=1}^{m}w_if(x_i). \tag{14}$$

Though that it is set-up for interval $\langle -1, 1\rangle$, with an appropriate linear transformation

$$t(x)^{\langle a,b\rangle}=t^{\langle a,b\rangle}=\frac{(b-a)(x+1)+2a}{2} \tag{15}$$

it can be used on arbitrary interval $\langle a, b\rangle$. The nodal points x_i and the weights w_i in (14) are taken in a such manner that the best possible precision can be reached [2]. The nodal points are the roots of Legendre polynomials. They are well-known and commonly used. In Table 4 in the Appendix, we provide these values for 2, 3, 4, 8 and 16 node approximation used, as collected in [1]. For the sake of sufficient precision, as required by the technical practice, precision of 20 decimal places is used. Transformation (15) then redistributes the nodes x_i to the interval $\langle a, b\rangle$, denoted $t_i^{\langle a,b\rangle}$.

Due to the specially taken nodal points, Gauss quadrature is much more effective tool for the numerical integration than other quadratures, as the rectangle rule, the trapezoidal rule, Simpson's method, etc. By using just $m+1$ nodal points Gauss numerical integration yields the exact solution for the numerical integration of the polynomials up to the degree of $2m+1$, [2].

As it can be seen from above, the computation depends on the length of the focused interval in the case of higher order integrals. Hence, we will do the next considerations on the interval of the unit length. Later, whenever needed, all parameters can be transformed to an arbitrary interval by using (15).Without loss of generality, to simplify the derivation of the numerical scheme we consider the integral on the interval $\langle 0, 1\rangle$.

$$D^{(-n)}_{\langle 0,1\rangle}(f) = 0 + \frac{1}{(n-1)!}\int_0^1 (1-x)^{n-1} f(x)\mathrm{d}x \approx \frac{n}{n!}\sum_{j=1}^{m} \frac{w_j}{2}\left(1 - t_j^{\langle 0,1\rangle}\right)^{n-1} f\left(t_j^{\langle 0,1\rangle}\right) \quad (16)$$

for $t_j^{\langle 0,1\rangle}$ being the Gaussian nodal points redistributed by (15) to interval $\langle a, b\rangle$ and m being the number of these points. By using $t^{\langle 0,1\rangle} = \frac{x+1}{2}$ we can proceed with (16):

$$\frac{1}{n!}\sum_{j=1}^{m} \frac{nw_j}{2}\left(1 - \frac{x_j^{\langle -1,1\rangle}+1}{2}\right)^{n-1} f\left(t_j^{\langle 0,1\rangle}\right) = \frac{1}{n!}\sum_{j=1}^{m} w_{n,j} f\left(\frac{x_j^{\langle -1,1\rangle}+1}{2}\right)$$

with

$$w_{n,j} = \frac{nw_j\left(1 - x_j^{\langle -1,1\rangle}\right)^{n-1}}{2^n} \quad (17)$$

Then, using (17) in (12) we can perform

$$D^{(-n)}_{\langle a,b\rangle}(f) \approx \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!}\frac{a^{n-i}}{(n-i)!}\sum_{j=1}^{m} w_{n-i,j} f\left(t_j^{\langle 0,a\rangle}\right) + \frac{(b-a)^n}{n!}\sum_{j=1}^{m} w_{n,j} f\left(t_j^{\langle a,b\rangle}\right) \quad (18)$$

It can be seen, (18) enfolds both analytical and numerical steps of manipulation with the definite integral over a repeated integral. It represents the numerical scheme for computation of $D^{(-n)}_{\langle a,b\rangle}(f)$ together with all corresponding lower order integrals. The following examples demonstrate how this approach works; Example 3 deals with a function easily analytically integrable, Example 4 shows the increase of the computational cost, i.e., the consumption of nodes with increasing order of integration n. Finally, in Example 5 the transcendent function is dealt. In Table 4 in Appendix are the values $w_{2,j}$ provided in the third column.

Example 3. By using our analytical-numerical method developed above, let us find the solution of

$$\int_3^7\int_0^{x_1} x_2^3 - 6x_2^2 + 2x_1 - 5\ \mathrm{d}x_2\mathrm{d}x_1.$$

We employ (7) together with the Gauss type numerical integration with three points, both tools coupled in (18).

$$\begin{aligned}&\int_3^7\int_0^{x_1} x_2^3 - 6x_2^2 + 2x_2 - 5\ \mathrm{d}x_2\mathrm{d}x_1\\ &= (7-3)\sum_{j=1}^{2} w_j f\left(x_j^{\langle 0,3\rangle}\right) + \frac{(7-3)^2}{2}\sum_{j=1}^{3} w_{2,j} f\left(x_j^{\langle 3,7\rangle}\right)\\ &= -159 - 161,\bar{7} = 326,4\bar{6}\end{aligned}$$

which matches with the exact solution.

Example 4. By using (7), (13) and 4-node Gauss quadrature we compute $D^{(-4)}_{\langle 2,7\rangle}x^5$ and all lower order integrals. In each case we compare the solution with an exact one. All inputs, i.e., nodal points abscissas rescaled to the interval $\langle 2,7\rangle$ and corresponding weights $w_i - w_{4i}$ are collected in Table 1. Here we have to take into account the fact that due to utilization of normalized norm in the form (17) maximal order of Gauss polynomial for exact integration sinks one order with every increased number of repeated integration, e.g., Gauss quadrature is of order $2m+1$, cubature of the order $2m$, the third integral $2m-1$, etc. It means, as we illustrate in this example that four point Gauss integration is sufficient in the case up to $n = 3$, in the case of greater n the error occurs. Nevertheless, we use the same abscissas for each integral $D^{(-4)}_{\langle 2,7\rangle}(f)$ as provided in the very left column in Table 1. Results - both analytical and numerical solutions are provided in Table 2. Let us recall that the errors are due to Gauss integration only.

As it was mentioned above, the higher n in $D^{(-n)}_{\langle a,b\rangle}(f)$ requires the higher number of Gauss nodes. This fact is illustrated on a transcendental function in the following example. Herein we also show the comparison of two methods

TABLE 1. Example 4. Gauss quadrature inputs - nodes and weights.

Abscissas	Weights for			
	$\int_2^7 f(x)\mathrm{d}x$	$D^{(-2)}_{\langle 2,7\rangle}(f)$	$D^{(-3)}_{\langle 2,7\rangle}(f)$	$D^{(-4)}_{\langle 2,7\rangle}(f)$
x_i in $\langle 2,7\rangle$	w_i	$w_{2,i}$	$w_{3,i}$	$w_{4,i}$
2,34715922	0,34785484513	0,32370264172	0,45184105550	0,56062519697
3,65004739	0,65214515486	0,43693107259	0,43910951596	0,39226561830
5,34995261	0,65214515486	0,21521408227	0,10653403049	0,04687631975
6,65284078	0,34785484513	0,02415220341	0,00251539803	0,00023286496

TABLE 2. Results of Example 4. All lower order integrals are provided.

	Solution			
	$\int_2^7 f(x)\mathrm{d}x$	$D^{(-2)}_{\langle 2,7\rangle}(f)$	$D^{(-3)}_{\langle 2,7\rangle}(f)$	$D^{(-4)}_{\langle 2,7\rangle}(f)$
Numerical	19597,5	19605,119	17156,3839	13351,6589
Exact	19597,5	19605,119	17156,3839	13344,2774
Absolute error	0	0	0	7,38142
Relative error [%]	0	0	0	0,05532

of the Gauss type, The Lobatto and Gauss method and the comparison with exact solution is provided.

EXAMPLE 5. We use our analytical-numerical method for $D^{(-4)}_{\langle -\pi,2\pi\rangle}(\sin x)$ computation of a transcendental function $f(x) = \sin x$ which always yields only an approximate solution regardless of the number of nodes employed in the quadrature. To compare, both the Lobatto and the Gauss quadrature was used. The results for various numbers of nodal points are shown in Table 3 together with the exact solution for better comparison. The number of nodes in the Gauss type quadrature determines the accuracy of computation. The Lobatto and Gauss quadrature can be applied particularly. The comparison, the exact solution is shown as well.

It is apparent from the results gathered in Table 3 that for achieving required accuracy it is essential to use sufficient number of nodes in the quadrature. For example, if we require the precision to one decimal, the sufficient number of inner nodes is 6 (8 = 6 inner + 2 border nodes in Lobatto quadrature). In this table, it can be also seen how the number of nodes influences the accuracy of the computation.

TABLE 3. Example 5. Accuracy of the solution - dependence on the number of used nodal points. Both the Lobatto and the Gauss methods are used within the numerical step; comparison with exact solution is provided.

Solution		$\int_a^b f(x)\mathrm{d}x$	$D^{(-2)}(f)$	$D^{(-3)}(f)$	$D^{(-4)}(f)$
Exact		-2,00000000000000	9,42477796076937	16,80440660163400	37,0846370596803
Approximate:					
	Nodes				
Lobatto	4	-4,02	-0,090	5,7	55
	8	-2,000021	9,42467719	16,8059	37,092
	10	-2,0000000095	9,424777916	16,8044078	37,084643
	12	-2,0000000000017	9,4247779607614	16,8044066020	37,084637061
	15	-1,99999999999999	9,42477796076936	16,80440660163400	37,0846370596802
	18	-1,99999999999999	9,42477796076944	16,80440660163420	37,0846370596807
Gauss	2	-8,6	-22,3	-70	-135
	4	-2,19	8,5	18	50
	6	-2,00058	9,4220	16,831	37,23
	8	-2,00000045	9,4247758	16,804452	37,08486
	10	-2,00000000013	9,42477796016	16,804406623	37,08463717
	12	-2,0000000000000	9,4247779607693	16,804406601638	37,08463705970
	16	-1,99999999999998	9,42477796076941	16,80440660163420	37,0846370596806

TABLE 4. Standard Gauss nodes within $\langle -1, 1\rangle$ and the related weights.

m	$x(i)$	w_i	$w_{2,i}$
2	-0,577350269189626	1	0,788675134594813
	0,577350269189626	1	0,211324865405187
3	-0,774596669	0,555555555555555	0,492943519233745
	0	0,888888888888888	0,444444444444444
	0,774596669	0,555555555555555	0,0626120363218102
4	-0,861136311594053	0,347854845137454	0,323702641724620
	-0,339981043584856	0,652145154862546	0,436931072590761
	0,339981043584856	0,652145154862546	0,215214082271785
	0,861136311594053	0,347854845137454	0,024152203412833
8	-0,9602898564975360	0,1012285362903760	0,0992186364390584
	-0,7966664774136270	0,2223810344533740	0,1997722749074710
	-0,5255324099163290	0,3137066458778870	0,2392848277464310
	-0,1834346424956500	0,3626837833783620	0,2146062767606710
	0,1834346424956500	0,3626837833783620	0,1480775066176910
	0,5255324099163290	0,3137066458778870	0,0744218181314563
	0,7966664774136270	0,2223810344533740	0,0226087595459031
	0,9602898564975360	0,1012285362903760	0,0020098998513176

Continued on next page.

Table 4 – *Continued from previous page.*

m	$x(i)$	w_i	$w_{2,i}$
16	-9,8940093499164993259e-1	2,7152459411754094852e-2	2,7008564070533210711e-2
	-9,4457502307323257607e-1	6,2253523938647892863e-2	6,0528323874693131390e-2
	-8,6563120238783174388e-1	9,5158511682492784810e-2	8,8765344283822774006e-2
	-7,5540440835500303389e-1	1,2462897125553387205e-1	1,0938712277535655815e-1
	-6,1787624440264374844e-1	1,4959598881657673208e-1	1,2101389828213152896e-1
	-4,5801677765722738634e-1	1,6915651939500253819e-1	1,2331652166400694388e-1
	-2,8160355077925891323e-1	1,8260341504492358887e-1	1,1701259255299640986e-1
	-9,5012509837637440185e-2	1,8945061045506849628e-1	1,0372539422233855513e-1
	9,5012509837637440185e-2	1,8945061045506849628e-1	8,5725216232729941151e-2
	2,8160355077925891323e-1	1,8260341504492358887e-1	6,5590822491927179001e-2
	4,5801677765722738634e-1	1,6915651939500253819e-1	4,5839977309955594308e-2
	6,1787624440264374844e-1	1,4959598881657673208e-1	2,8582090534445203117e-2
	7,5540440835500303389e-1	1,2462897125553387205e-1	1,5241848480177313901e-2
	8,6563120238783174388e-1	9,5158511682492784810e-2	6,3931673986700108038e-3
	9,4457502307323257607e-1	6,2253523938647892863e-2	1,7252000639547614725e-3
	9,8940093499164993259e-1	2,7152459411754094852e-2	1,4389534122088414080e-4

Conclusion

The paper is devoted to the tasks involving repeated integrals. The main contribution to the field is in proving the theorem for a decomposition of definite integral over $n-1$ times repeated integral to n definite integrals with unified lower bound. After such a rearrangement, Cauchy theorem for repeated integrals can be used which reverts these integrals to the sum of single integrals. Such integrals can be solved by standard numerical methods. Though, e.g., the Newton-Cotes forms can be used as well, in order to increase the accuracy of the method with the least computational cost, we have chosen more efficient Gauss type methods. We have slightly modified Gauss quadrature in sense of weights which were rescaled to unit interval. The new weights can be prepared prior to the entire computation. In two examples we have demonstrated the linear decrease of the accuracy with increasing order of original integral. And this is a justification of using Gauss type quadrature. In such a way from relatively small number of values of ordinates provided, one can compute the integrals of several orders effectively. The table of weights for the first and the second integral are shown in Table 4.

Both analytical rearrangement and numerical integration were coupled in a unique formula that enables convenient automation and evaluation by using a computer. Possible utilization can be seen, e.g., in automation of solving Cauchy differential equations of the *nth* order when as initial condition values of derivatives up to $(n-1)th$ order can occur. The method is suitable for explicit solving of evolutionary differential equation (D'Alembert's equation). Regarding the possibility to choose the order of numerical integration, the high accuracy can be reached with quite a low computational cost.

REFERENCES

[1] ABRAMOWITZ, M.—STEGUN, I. A.: *Handbook of Mathematical functions*, Dover Publications, Inc. New York, 1972.

[2] CHENEY, R.—KINCAIUD, D.: *Numerical Mathematics*, Austin, Texas, 2004.

[3] DICKÝ, J.—MISTRÍKOVÁ, Z.—SUMEC, J.: *Elasticity and Plasticity in Civil Engineering*, Slovak University of Technology, 2005.

[4] FOLLAND, G. B.: *Advanced Calculus*, Prentice Hall, 2002.

[5] RALSTON, A.: *The Fundamentals of Numerical Mathematics*, Academia Praha, 1973.

[6] GERE, J. M.—GOODNO, B. J.: *Mechanics of Materials*, 7th edition, Cengage Learning, 2009.

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