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COMMUTATIVITY THEOREMS AND PROJECTION ON THE CENTER OF A BANACH ALGEBRA

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ABSTRACT. Let $\mathcal{X}$ be a Banach algebra. In this article, on the one hand, we proved some results concerning the continuous projection from $\mathcal{X}$ to its center. On the other hand, we investigate the commutativity of $\mathcal{X}$ under specific conditions. Finally, we included some examples and applications to prove that various restrictions in the hypotheses of our theorems are necessary.

1. Introduction

All in this paper the symbol $\mathcal{R}$ denote an associative ring and $\mathcal{X}$ denotes a Banach algebra with center $Z(\mathcal{X})$. For any $x, y \in \mathcal{X}$, the symbol $x \circ y$ (respectively $[x, y]$) will denote the anticommutator $xy + yx$ (respectively the commutator $xy - yx$). Recall that $\mathcal{X}$ is prime if for all $x, y \in \mathcal{X}$, $x\mathcal{X}y = 0$ yields $x = 0$ or $y = 0$. The additive mapping $d : \mathcal{X} \longrightarrow \mathcal{X}$ is a derivation if $d(xy) = d(x)y + xd(y)$ for every x and y in $\mathcal{X}$. In particular, d defined by $d(x) = [a, x]$ for all $x \in \mathcal{X}$ is a derivation, called the inner derivation induced by an element $a \in \mathcal{X}$. An additive subgroup L of $\mathcal{R}$ is known to be a Lie ideal of $\mathcal{R}$ if $[l, r] \in L$, for all $l \in L$ and $r \in \mathcal{R}$. Let M be a subspace of $\mathcal{X}$, the linear operator $P : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be a projection of $\mathcal{X}$ on M, if $P(x) \in M$ for all $x \in \mathcal{X}$ and $P(x) = x$ for all $x \in M$. Let M and N two subspaces of a the Banach space $\mathcal{X}$ such that $M \oplus_{al} N$ is an algebraic direct sum of $\mathcal{X}$, we provide the two subspace M and N with the induced topology of $\mathcal{X}$ and $M \times N$ by the product topology. We say that $M \oplus_t N$ is a topological direct sum of $\mathcal{X}$, if more, the mapping $\varphi : M \times N \longrightarrow \mathcal{X}$, define by $\varphi(x, y) = x + y$ is a homeomorphism, and we say that M is complemented in $\mathcal{X}$ and N is his topological complement. In this case, there is a unique continuous projection P from $\mathcal{X}$ to M.

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In 1941, Sobszyk [1] proved that if $\mathcal{X}$ is separable and M is a subspace of $\mathcal{X}$ isomorphic to C_0 (the space of sequences in $\mathbb{C}$ which converge to 0), then M is complemented in $\mathcal{X}$. Following, Lindenstrans and Tzafriri [4], every infinite-dimensional Banach space that is not isomorphic to a Hilbert space contains a closed uncomplemented subspace. For more information on these notions, see [7].

Numerous results in the literature concerning the commutativity of a prime and semi-prime Banach algebra are proved, for example, in 2022, M. Moumen, L. Taoufiq and L. Oukhtite [5] showed that if $\mathcal{X}$ is a prime Banach algebra having two non-void open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ such that for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ there is $(n, m) \in {\mathbb{N}^*}^2$ such that $d([x^n, y^m]) + x^n \circ y^m = 0$, then $\mathcal{X}$ is commutative (where d is continuous non-injective derivation).

Motivated by these theorems, the main objective of this work is to prove some results concerning the non-zero continuous projection from a Banach algebra to its center. Our topological approach is based on Baire's category theorem and some properties of functional analysis. Among our results. We have shown that if $\mathcal{X}$ is a real or complex prime Banach algebra admitting a non-zero continuous projection P from $\mathcal{X}$ to $Z(\mathcal{X})$ such that for all $x \in \mathcal{H}$ there is $n \in \mathbb{N}^*$ such that $P(x^n) = x^n$, where $\mathcal{H}$ is a non-void open subset of $\mathcal{X}$, then P is the identity application of $\mathcal{X}$ and then $\mathcal{X}$ is commutative.

2. Some preliminaries

In this section, we quote some basic lemmas that will be used as tools in subsequent section. The following lemma, due to Bonsall and Duncan [2], is crucial for developing the proof of our main results.

Lemma 2.1. *Let $\mathcal{X}$ be a real or complex Banach algebra and $P(t) = \sum_{k=0}^{n} b_k t^k$ a polynomial in the real variable t with coefficients in $\mathcal{X}$. If for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathcal{X}$, then every b_k lies in M.*

It's worth noting that the following complementability comparable result can be obtained (see [3]).

Lemma 2.2. *Let $\mathcal{X}$ be a Banach space. If a subspace M of $\mathcal{X}$ is closed and there is a closed subspace N of $\mathcal{X}$ such that $M \cap N = \{0\}$ and $M + N = \mathcal{X}$, it is complemented in $\mathcal{X}$.*

Lemma 2.3. *Let M be a closed subspace of a Banach space $\mathcal{X}$. M is complemented if there is a continuous projection P of $\mathcal{X}$ on M, and its complement is $(I - P)(\mathcal{X})$, where I is the identity mapping on $\mathcal{X}$.*

P r o o f. Let $x \in \mathcal{X}$ we can write $x = P(x) + \big(x - P(x)\big)$, since $P(x) \in M$ and $x - P(x) \in (I-P)(\mathcal{X})$, we conclude that $\mathcal{X} = M + (I-P)(\mathcal{X})$. Now, we will show that $M \cap (I-P)(\mathcal{X}) = \{0\}$, indeed: Let $x \in M \cap (I-P)(\mathcal{X})$ then $x \in (I-P)(\mathcal{X})$, therefore there is $a \in \mathcal{X}$ such that $x = (I-P)(a) = a - P(a)$, since $x \in M$ and $P(a) \in M$ we conclude that $a \in M$ then $P(a) = a$, we obtain $x = 0$. Therefore $M \oplus_t (I-P)(\mathcal{X})$ is an algebraic direct sum of the Banach space $\mathcal{X}$. It remains for us to show that $(I-P)(\mathcal{X})$ is closed in $\mathcal{X}$. For this, we consider a sequence $(y_n)_{n\in\mathbb{N}} \subset (I-P)(\mathcal{X})$ converge to $y \in \mathcal{X}$ as $n \to \infty$. Since $M \oplus_t (I-P)(\mathcal{X})$ is an algebraic direct sum of $\mathcal{X}$, then there is $y_1 \in M$ and $y_2 \in (I-P)(\mathcal{X})$ such that $y = y_1 + y_2$, therefore $P(y) = y_1$ (1). On the other hand, we have $P(y_n) = 0$ for all $n \in \mathbb{N}$, as P is continuous then $P(\lim y_n) = 0$ that is to say $P(y) = 0$ (2). According to (1) and (2) we obtain $y = y_2 \in (I-P)(\mathcal{X})$, then $(I-P)(\mathcal{X})$ is closed. Consequently, $M \oplus_t (I-P)(\mathcal{X})$ is a topological direct sum of the Banach space $\mathcal{X}$. □

3. Main results

Theorem 3.1. *Let $\mathcal{X}$ be a real or complex prime Banach algebra admiting a continuous projection P from $\mathcal{X}$ to $Z(\mathcal{X})$ if there is $n \in \mathbb{N}^*$ such that*

$$P(x^n) = x^n \quad \text{for all } x \in \mathcal{H},$$

where $\mathcal{H}$ is a nonvoid open subset of $\mathcal{X}$. Then P is the identity mapping on $\mathcal{X}$ and $\mathcal{X}$ is commutative.

P r o o f. First we observe that if P satisfies

$$P(x^n) = x^n \ \forall x \in \mathcal{H}, \tag{1}$$

then the projection P is not zero. Since $Z(\mathcal{X})$ is a closed subspace of $\mathcal{X}$ and P is a continuous projection onto $Z(\mathcal{X})$. By Lemma 2.3, we conclude that $Z(\mathcal{X})$ is complemented in $\mathcal{X}$ and

$$\mathcal{X} = Z(\mathcal{X}) \oplus_t (I-P)(\mathcal{X}). \tag{2}$$

Now let $x_0 \in \mathcal{H}$ and $x \in \mathcal{X}$, then $x_0 + tx \in \mathcal{H}$ for any sufficiently small real t, so $P\big((x_0+tx)^n\big) - (x_0+tx)^n = 0$. We can write

$$(x_0 + tx)^n = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n.$$

We put $Q(t) = P\big((x_0+tx)^n\big) - (x_0+tx)^n$, since P is a continuous maps, we write

$$Q(t) = \big(P(A_0) - A_0\big) + \big(P(A_1) - A_1\big)t + \big(P(A_2) - A_2\big)t^2 + \cdots + \big(P(A_n) - A_n\big)t^n.$$

The coefficient of t^n in above polynomial is $P(A_n) - A_n$. Since $A_n = x^n$, then this coefficient is $P(x^n) - x^n$, according to Lemma 2.1 we obtain $P(x^n) - x^n = 0$. Consequently, $P(x^n) = x^n$ for all $x \in \mathcal{X}$. Since $P(x) \in Z(\mathcal{X})$ for all $x \in \mathcal{X}$, we conclude that $x^n \in Z(\mathcal{X})$ for all $x \in \mathcal{X}$. Therefore, $x^n y^n \in Z(\mathcal{X})$ for all $(x, y) \in \mathcal{X} \times \mathcal{X}$, so the required result follows from [8]. That is $\mathcal{X} = Z(\mathcal{X})$, according to (2) necessary $(I - P)(\mathcal{X}) = \{0\}$ and $P = I$. □

We immediately get the following corollary from the above Theorem 3.1.

Corollary 3.2. *Let D be a part dense in a prime Banach algebra $\mathcal{X}$. If $\mathcal{X}$ admits a continuous projection P and there is $n \in \mathbb{N}^*$ such that*

$$P(x^n) = x^n \ \ \forall x \in \mathcal{D},$$

then $P = I$ and $\mathcal{X}$ is commutative where I is the identity application of $\mathcal{X}$.

Proof. Let $x \in \mathcal{X}$, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{D}$ converging to x. Since $(x_k)_{k\in\mathbb{N}} \subset \mathcal{D}$, so

$$P\big((x_k)^n\big) - (x_k)^n = 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

By the continuity of P, we conclude that $P(x^n) - x^n = 0$. Consequently,

$$\exists n \in \mathbb{N}^* \ : P(x^n) - x^n = 0 \ (\forall x \in \mathcal{X}).$$

By Theorem 3.1, we get the desired conclusion. □

The following result is a generalization of the previous theorem.

Theorem 3.3. *Let $\mathcal{X}$ be a real or complex prime Banach algebra and $\mathcal{H}$ a nonvoid open subset of $\mathcal{X}$. If there is a continuous projection P of $\mathcal{X}$ on $Z(\mathcal{X})$ such that*

$$(\forall x \in \mathcal{H})(\exists n \in \mathbb{N}^*) \ : \ P(x^n) = x^n,$$

then $P = I$, where I is the identity mapping on $\mathcal{X}$ and $\mathcal{X}$ is commutative.

Proof. For any $n \in \mathbb{N}^*$ we define the following two parts subset of $\mathcal{X}$:

$$O_n = \{x \in \mathcal{X} \mid P(x^n) - x^n \neq 0\} \quad \text{and} \ \ F_n = \{x \in \mathcal{X} \mid P(x^n) - x^n = 0\}.$$

Now we claim that each O_n is open in $\mathcal{X}$. That is, we have to show that F_n the complement of O_n is closed. For this, we consider a sequence $(x_k)_{k\in\mathbb{N}} \subset F_n$ converge to $x \in \mathcal{X}$. Since $(x_k)_{k\in\mathbb{N}} \subset F_n$, so

$$P\big((x_k)^n\big) - (x_k)^n = 0 \quad \text{for all} \ \ k \in \mathbb{N}.$$

According to the continuity of P, we conclude that $P(x^n) - x^n = 0$, therefore $x \in F_n$ and F_n is closed (i.e., O_n is open). If every O_n is dense, we know that their intersection is also dense by Baire category theorem which contradict with of $(\cap O_n) \cap \mathcal{H} = \varnothing$. There is $m \in \mathbb{N}^*$, such that O_m is not dense and there exists a non-void open subset O in F_m, such that :

$$P(x^m) - x^m = 0 \quad \text{for all} \ \ x \in O.$$

By Theorem 3.1, we get the result of Theorem 3.3. □

Application 3.4. Let $\mathbb{R}$ be the field of real numbers, $\mathcal{X} = \mathcal{M}_2(\mathbb{R})$ endowed with usual matrix addition and multiplication and the norm $\|\cdot\|_1$ defined by

$$\|A\|_1 = \sum_{1\leq i,j\leq 2} |a_{i,j}| \quad \text{for all } A = (a_{i,j})_{1\leqslant i,j\leqslant 2} \in \mathcal{X},$$

is a real prime Banach algebra.
We put

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

it is easy to verify that $\mathcal{B} = \{E_1, E_2, E_3, E_4\}$ is a basis of $\mathcal{X}$ and $Z(\mathcal{X} = \operatorname{span}(E_1)$ (the subspace of $\mathcal{X}$ generated by E_1). So we can write $\mathcal{X} = Z(\mathcal{X}) \oplus_t \operatorname{span}(E_2, E_3, E_4)$ ($\oplus_t$ means the direct topological sum). The mapping P define from $\mathcal{X}$ to $Z(\mathcal{X})$ by $P(M) = a_1E_1$ for all $M = \sum_{i=0}^4 a_iE_i \in \mathcal{X}$, which is the non-zero continuous projection of $\mathcal{X}$ on $Z(\mathcal{X})$. Now let $\mathcal{H}$ be an open subset of $\mathcal{X}$ included in $Z(\mathcal{X})$.

Suppose that $\mathcal{H}$ is not empty. For all $A \in \mathcal{H}$ there is $a \in \mathbb{R}$ such that

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

therefore

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} = a^nE_1, \ \forall n \in \mathbb{N}^*,$$

that is,

$$P(A^n) = A^n, \quad \forall n \in \mathbb{N}^*.$$

By Theorem 3.3, we conclude that $P = I$, but $P(E_4) = 0 \neq E_4$ (in view of defined map P), which is impossible as P is the identity mapping of $\mathcal{X}$. Consequently, $\mathcal{H} = \varnothing$ and we conclude that the only open subset included in $Z(\mathcal{X})$ is the empty set.

In this theorem, the two symbols "T" and "$*$" represent either the Lie product "$[.,.]$", or the Jordan product "$\circ$", or the multiplicative law "$\cdot$".

Theorem 3.5. *Let $\mathcal{X}$ be a real or complex prime Banach algebra $\mathcal{H}_1$ and $\mathcal{H}_2$ are two nonvoid open subsets of $\mathcal{X}$. If there is a non-zero continuous projection P of $\mathcal{X}$ on $Z(\mathcal{X})$ such that*

$$\big(\forall (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2\big)\big(\exists (n,m) \in \mathbb{N}^{*2}\big) \ : \ P(x^nTy^m) = x^n * y^m,$$

then P is the identity mapping on $\mathcal{X}$ and $Z(\mathcal{X}) = \mathcal{X}$.

Proof. Since $Z(\mathcal{X})$ is a closed subspace of $\mathcal{X}$ and P is a continuous projection onto $Z(\mathcal{X})$. By Lemma 2.3, $Z(\mathcal{X})$ is complemented in $\mathcal{X}$. We can write

$$\mathcal{X} = Z(\mathcal{X}) \oplus_t (I - P)(\mathcal{X}). \tag{**}$$

Now, for any $(n,m) \in \mathbb{N}^{*2}$ we define two set :

$$O_{n,m} = \{(x,y) \in \mathcal{X}^2 \mid P(x^nTy^m) \neq x^n * y^m\}$$

and

$$F_{n,m} = \{(x,y) \in \mathcal{X}^2 \mid P(x^nTy^m) = x^n * y^m\}.$$

We observe that

$$(\cap O_{n,m}) \cap (\mathcal{H}_1 \times \mathcal{H}_2) = \varnothing,$$

indeed: if there exists $(x,y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that $(x,y) \in O_{n,m}$ for all $(n,m) \in \mathbb{N}^{*2}$. Then

$$P(x^nTy^m) \neq x^n * y^m \quad \text{for all } (n,m) \in \mathbb{N}^{*2}$$

which is absurd with the hypotheses of the theorem.

Now, we claim that each $O_{n,m}$ is open in $\mathcal{X} \times \mathcal{X}$. That is, we have to show that $F_{n,m}$ the complement of $O_{n,m}$ is closed. For this, we consider a sequence $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{n,m}$ converge to $(x,y) \in \mathcal{X} \times \mathcal{X}$. Since $((x_k, y_k))_{k\in\mathbb{N}} \subset F_{n,m}$, so

$$P\big((x_k)^n T (y_k)^m\big) = (x_k)^n * (y_k)^m \quad \text{for all} \quad k \in \mathbb{N}.$$

We conclude that

$$d(x^nTy^m) = x^n * y^m.$$

Therefore,

$$(x,y) \in F_{n,m} \quad \text{and} \quad F_{n,m}$$

is closed (i.e., $O_{n,m}$ is open).
If every $O_{n,m}$ is dense, we know that their intersection is also dense by Baire category theorem, which contradict with $(\cap O_{n,m}) \cap (\mathcal{H}_1 \times \mathcal{H}_2) = \varnothing$. Therefore, there is $(p,q) \in \mathbb{N}^{*2}$ such that $O_{p,q}$ is not a dense set and there exists a non-void open subset $O \times O'$ in $F_{p,q}$, such that :

$$P(x^pTy^q) = x^p * y^q \quad \text{for all} \quad (x,y) \in O \times O'.$$

Since $P(x) \in Z(\mathcal{X})$ for all $x \in \mathcal{X}$, we conclude that

$$x^p * y^q \in Z(\mathcal{X}) \quad \text{for all} \quad (x,y) \in O \times O'.$$

So, the required result follows from [8]. Then $Z(\mathcal{X}) = \mathcal{X}$, according to (**) necessary $(I - P)(\mathcal{X}) = \{0\}$. Therefore, $P = I$. □

Remark 1. We get the same results as Theorem 3.5 when we change the order of x^n and y^m on one side, that is, $P(x^nTy^m) = y^m * x^n$ or $P(y^mTx^n) = x^n * y^m$.

Application 3.6. Let $\mathcal{X}$ the non-commutative prime Banach algebra of square matrices of size $r \in \mathbb{N}^* \backslash \{1\}$ with coefficinets in $\mathbb{C}$ or $\mathbb{R}$ with usual matrix addition and multiplication, the norm is defined by

$$\|A\|_1 = \sum_{1\leq i,j\leq r} |a_{i,j}| \quad \text{for all } A = (a_{i,j})_{1\leqslant i,j\leqslant r} \in \mathcal{X}.$$

We show that the interior of the closed subspace $Z(\mathcal{X})$ is empty, indeed: Since $\mathcal{X}$ is of finite dimensional and $Z(\mathcal{X})$ is closed. Then $Z(\mathcal{X})$ is complemented, therefore there is a continuous projection P onto $Z(\mathcal{X})$. Suppose that $G = \operatorname{int}\big(Z(\mathcal{X})\big)$ (the interior of $Z(\mathcal{X})$) is not empty. For all $(n,m) \in \mathbb{N}^{*2}$ and for all $(A,B) \in G^2$ we have

$$A^n \in Z(\mathcal{X}) \quad \text{and} \quad B^m \in Z(\mathcal{X}),$$

then

$$[A^n, B^m] = 0,$$

therefore,

$$P([A^n, B^m]) = [A^n, B^m].$$

Since $P(I_r) = I_r \neq 0_{\mathcal{X}}$ (I_r is the matrix unit of size r), we get P is a non-zero projection, by Theorem 3.5 we conclude that $P = I$, which is absurd because $\mathcal{X} \neq Z(\mathcal{X})$. Consequently the interior of $Z(\mathcal{X})$ is empty.

The necessary and sufficient condition for the commutativity of a normed algebra.

Theorem 3.7. *The normed algebra $\mathcal{X}$ over $\mathbb{C}$ or $\mathbb{R}$ is commutatative if and only if the interior of his center is not empty.*

Proof.

$\Rightarrow$) If $\mathcal{X}$ is commutative then $\mathcal{X} = Z(\mathcal{X})$, therefore the interior of $Z(\mathcal{X})$ is itself because $\mathcal{X}$ is open in $\mathcal{X}$.

$\Leftarrow$) If $\operatorname{int}\big(Z(\mathcal{X})\big)$ (the interior of $Z(\mathcal{X})$) is a non-void open subset of $\mathcal{X}$, then there is $x \in \operatorname{int}\big(Z(\mathcal{X})\big)$.

Let $z \in \mathcal{X}$, we have $x + tz \in \operatorname{int}\big(Z(\mathcal{X})\big)$ for all sufficiently small non zero real t, therefore,

$$[x + tz, y] = 0 \ \ (\forall y \in \mathcal{X}),$$

that is,

$$[x, y] + t[z, y] = 0 \ \ (\forall y \in \mathcal{X}).$$

Since

$$[x, y] = 0 \ \ (\forall y \in \mathcal{X}),$$

we obtain

$$t[z, y] = 0 \ \ (\forall y \in \mathcal{X}),$$

then

$$[z, y] = 0 \ \ (\forall y \in \mathcal{X}),$$

which implies that $z \in Z(\mathcal{X})$. Consequently, $\mathcal{X}$ is commutative. $\square$

Application 3.8. Let E be a normed space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), the space $\mathcal{X} = 1\mathcal{L}_c(E)$ of continuous linear applications from E to E endowed with usual application addition and composition and the norm defined by

$$\| T \| = \sup_{\|x\| \leq 1} \| T(x) \|$$

for all $T \in \mathcal{X}$ is a normed algebra over $\mathbb{F}$. Let G be the subspace of $\mathcal{X}$ defined by $G = \{\lambda I_E \mid \lambda \in \mathbb{F}\}$ where I_E is the identity of E, we observe that $G \subset Z(\mathcal{X})$, according to Theorem 3.7, we conclude that the interior of G is empty because $\mathcal{X}$ is not commutative and $\operatorname{int}(G) \subset \operatorname{int}\big(Z(\mathcal{X})\big) = \varnothing$.

We immediately get the following corollary from the above Theorem 3.7.

Corollary 3.9. *If $Z(\mathcal{X})$ contain an isolated point of $\mathcal{X}$, then $\mathcal{X}$ is commutative.*

Proof. Let a be an isolated point of $\mathcal{X}$ belongs to $Z(\mathcal{X})$. We have $\{a\} \subset Z(\mathcal{X})$, therefore

$$\operatorname{int}(\{a\}) \subset \operatorname{int}\big(Z(\mathcal{X})\big).$$

Since a is isolated point in $\mathcal{X}$, we get $\{a\}$ is an open subset of $\mathcal{X}$, that is $\operatorname{int}(\{a\}) = \{a\}$, therefore $\operatorname{int}(Z(\mathcal{X})) \neq \varnothing$, according to theorem 3.7, we conclude that $\mathcal{X}$ is commutative. □

In particular, we get the following result.

Corollary 3.10. *Let $\mathcal{X}$ be a normed algebra over $\mathbb{R}$ or $\mathbb{C}$, if 0 is an isolated point in $\mathcal{X}$, then $\mathcal{X}$ is commutative.*

Remark 2. Let $\mathcal{X}$ be a non-commutative normed algebra over $\mathbb{R}$ or $\mathbb{C}$.

1. The only open subset of $\mathcal{X}$ included in $Z(\mathcal{X})$ is the empty set .
2. The center $Z(\mathcal{X})$ contains no isolated points of $\mathcal{X}$.

Remark 3. It is easy to see that Theorem 3.7, Corollary 3.10 and Corollary 3.9 are true if $\mathcal{X}$ is considered as a normed algebra over $\mathbb{Q}$ (the field of the rational numbers).

The following example show that the hypotheses "$\mathcal{H}_1$ and $\mathcal{H}_2$ are nonvoid opens" is not superfluous in Theorem 3.3 and in Theorem 3.5 (in the case where T and $*$ represent the same thing).

Example.
Let $\mathbb{R}$ be the field of real numbers, $\mathcal{X} = \mathcal{M}_2(\mathbb{R})$ endowed with usual matrix addition and multiplication and the norm defined by $\|A\|_1 = \sum_{1 \leq i,j \leq 2} |a_{i,j}|$ for all $A = (a_{i,j})_{1 \leqslant i,j \leqslant 2} \in \mathcal{X}$, is a real prime Banach algebra. Let

$$\mathcal{F}_1 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \mid t > 0 \right\} \quad \text{and} \quad \mathcal{F}_2 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \mid t \in \mathbb{R}^* \right\}.$$

$\mathcal{F}_1$ is not open in $\mathcal{X}$, indeed, we have to show that the complement of $\mathcal{F}_1$ is not closed. For this, we consider the sequence

$$\left(\begin{pmatrix} 1+\frac{1}{n} & \frac{-1}{n} \\ \frac{1}{n} & 1+\frac{1}{n} \end{pmatrix}\right)_{n\in\mathbb{N}^*}$$

in $\mathcal{F}_1^c$ complememt of $\mathcal{F}_1$ who converge to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \mathcal{F}_1^c,$$

then $\mathcal{F}_1^c$ is not closed, that is, $\mathcal{F}_1$ is not open in $\mathcal{X}$ (in same way we prove that $\mathcal{F}_2$ is not open). We put

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we observe that the family $\mathcal{B} = \{E_1, E_2, E_3, E_4\}$ is a basis of $\mathcal{X}$ and $Z(\mathcal{X}) = \text{span}(E_1)$, so we can write $\mathcal{X} = Z(\mathcal{X}) \oplus_t \text{span}(E_2, E_3, E_4)$. The mapping P defined from $\mathcal{X}$ to $Z(\mathcal{X})$ by

$$P(M) = a_1 E_1 \quad \text{for all} \quad M = \sum_{i=0}^{4} a_i E_i \in \mathcal{X}$$

is a continuous projection of $\mathcal{X}$ on $Z(\mathcal{X})$. For all

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{F}_1, \qquad\qquad B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in \mathcal{F}_2,$$

and for all $(m,n) \in \mathbb{N}^2$, we have

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} \qquad \text{and} \qquad B^m = \begin{pmatrix} b^m & 0 \\ 0 & b^m \end{pmatrix}.$$

So,

$$A^n B^m = \begin{pmatrix} a^n b^m & 0 \\ 0 & a^n b^m \end{pmatrix}, \qquad A^n o B^m = \begin{pmatrix} 2a^n b^m & 0 \\ 0 & 2a^n b^m \end{pmatrix}$$

and

$$[A^n, B^m] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We conclude that:

$$A^n B^m = a^n b^m E_1, \qquad A^n o B^m = 2a^n b^m E_1, \qquad [A^n, B^m] = 0 E_1.$$

Therefore,

1. $P(A^n o B^m) = A^n o B^m$,
2. $P(A^n B^m) = A^n B^m$,
3. $P([A^n, B^m]) = [A^n, B^m]$,
4. $P(A^n) = A^n$.

But $P \neq I$ (because $P(E_3) = 0_{\mathcal{X}} \neq E_3$).

The following example show that we can't replace $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ in the hypotheses of Theorem 3.3 and Theorem 3.5 (in the cases where T and $*$ represent the same thing).

Example. Let $\mathcal{X}$ be the prime Banach algebra of square matrices of size 2 with coefficients in $\mathbb{Z}/3\mathbb{Z}$, with usual matrix addition and matrix multiplication, the norm is defined by

$$\|A\|_1 = \sum_{1\leq i,j\leq 2} |a_{i,j}| \quad \text{for all } A = (a_{i,j})_{1\leqslant i,j\leqslant 2} \in \mathcal{X}$$

with $|\cdot|$ is the norm defined on $\mathbb{Z}/3\mathbb{Z}$ by

$$|\overline{0}| = 0, \quad |\overline{1}| = 1 \quad \text{and} \quad |\overline{2}| = 2.$$

Observe that

$$\mathcal{H} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}/3\mathbb{Z} \right\} \quad \text{is open in} \quad \mathcal{X}.$$

Indeed, let $A \in \mathcal{H}$ the open ball

$$B(A,1) = \{X \in \mathcal{X} \quad \text{such that} \quad \|A - X\|_1 < 1\} = \{A\} \subset \mathcal{H},$$

then $\mathcal{H}$ is nonvoid open subset of $\mathcal{X}$. We put

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we observe that the family $\mathcal{B} = \{E_1, E_2, E_3, E_4\}$ is a basis of $\mathcal{X}$ and $Z(\mathcal{X}) = \operatorname{span}(E_1)$, therefore, we can write

$$\mathcal{X} = Z(\mathcal{X}) \oplus_t \operatorname{span}(E_2, E_3, E_4).$$

Define P from $\mathcal{X}$ to $Z(\mathcal{X})$ by $P(M) = a_1 E_1$ for all $M = \sum_{i=0}^{4} a_i E_i \in \mathcal{X}$, the mapping P is a non-zero continuous projection of $\mathcal{X}$ on $Z(\mathcal{X})$. For all

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{H}, \qquad B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \in \mathcal{H},$$

and for all $(m,n) \in \mathbb{N}^2$, we have

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} \qquad \text{and} \qquad B^m = \begin{pmatrix} b^m & 0 \\ 0 & b^m \end{pmatrix}.$$

Therefore,

$$A^n B^m = \begin{pmatrix} a^n b^m & 0 \\ 0 & a^n b^m \end{pmatrix}, \qquad A^n o B^m = \begin{pmatrix} 2a^n b^m & 0 \\ 0 & 2a^n b^m \end{pmatrix}$$

and

$$[A^n, B^m] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We can write

$$A^nB^m = a^nb^mE_1\ ,\quad A^noB^m = 2a^nb^mE_1\ ,\quad [A^n, B^m] = 0E_1.$$

We conclude that:

1. $P(A^noB^m) = A^noB^m$,
2. $P(A^nB^m) = A^nB^m$,
3. $P([A^n, B^m]) = [A^n, B^m]$,
4. $P(A^n) = A^n$.

But $P \neq I$ (because $P(E_2) = 0_{\mathcal{X}} \neq E_2$).

Remark 4. The example 3 show that the Theorem 3.7 is false if $\mathbb{F}_3$ replace $\mathbb{R}$ or $\mathbb{C}$.

P r o o f. In example 3, if we put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{we have} \qquad \{A\} = B(A, 1)$$

(the open ball of center A and radius 1), therefore the singleton $\{A\}$ is an open subset of $\mathcal{X}$ included in $Z(\mathcal{X})$, that is,

$$\operatorname{int}\big(Z(\mathcal{X})\big) \neq \varnothing,$$

but $\mathcal{X}$ is not commutative. □

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