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TWO DISJOINT AND INFINITE SETS OF SOLUTIONS FOR AN ELLIPTIC EQUATION WITH CRITICAL HARDY-SOBOLEV-MAZ'YA TERM AND CONCAVE-CONVEX NONLINEARITIES

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ABSTRACT. In this paper, we consider the following critical Hardy-Sobolev-Maz'ya problem

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(t)-2}u}{|y|^t} + \mu|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$, which contains some points $(0, z^*)$, $\mu > 0$, $1 < q < 2$, $2^*(t) = \frac{2(N-t)}{N-2}$, $0 \leq t < 2$, $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k < N$. We prove that if $N > 2\frac{q+1}{q-1} + t$, then the above problem has two disjoint and infinite sets of solutions. Here, we give a positive answer to one open problem proposed by Ambrosetti, Brezis and Cerami in [1] for the case of the critical Hardy-Sobolev-Maz'ya problem.

1. Introduction

We are concerned with the problem

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(t)-2}u}{|y|^t} + \mu|u|^{q-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ that contains some points $(0, z^*)$, $\mu > 0$, $1 < q < 2$, $0 \leq t < 2$, $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k < N$ and

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$2^*(t)=\frac{2(N-t)}{N-2}$. The corresponding energy functional to (1.1) is

$$I(u):=\frac{1}{2}\int\limits_\Omega|\nabla u|^2\,\mathrm{d}x-\frac{1}{2^*(t)}\int\limits_\Omega\frac{|u|^{2^*(t)}}{|y|^t}\,\mathrm{d}x-\frac{\mu}{q}\int\limits_\Omega|u|^q\,\mathrm{d}x\,.$$

When $t=0$ and $q=2$ the problem (1.1) reduces to the following problem

$$\begin{cases}-\Delta u=|u|^{2^*-2}u+\mu u & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega,\end{cases}\tag{1.2}$$

G. Devillanova and S. Solimini in [7] considered the problem (1.2) and they established the existence of infinitely many solutions if $N\geq 7$. Their crucial idea is to show the strong convergence of approximating solutions of (1.2). The main ingredient used to achieve this goal is to obtain some estimates for approximating solutions of (1.2) in a carefully defined safe region, and then a local Pohozaev identity is used to obtain the result. P. Han in [8] invested the similar approaches to show for $t=0$ that, if $N>\frac{2(q+1)}{q-1}$, then problem (1.1) admits an infinite sets of solutions with positive energy, which can be viewed as one of the positive answer to the above open problem. When $q=2$, the method of our paper was used by Shuangjie Peng and Chunhua Wang in [10] to establish that if $N>6+t$, then the problem (1.1) has infinitely many solutions. For more similar results, we refer the reader to [5, 6, 12]. It seems that there is no similar result concerning (1.1) or the concave case, i.e., $1<q<2$. The main result of this paper is the following

Theorem 1.1. *If we assume that $N>2\frac{q+1}{q-1}+t$, then*

- i) *There exists a sequence of solutions $(v_k)_k$ of (1.1) such that $I(v_k)>0$ and $I(v_k)\to+\infty$ as $k\to+\infty$.*
- ii) *There exists a sequence of solutions $(u_k)_k$ of (1.1) such that $I(u_k)<0$ and $I(u_k)\to 0$ as $k\to+\infty$.*

This paper is organized as follows. Section 2 is devoted to the strong convergence of approximating solutions in $H_0^1(\Omega)$ of (1.1). Unlike [10], some technical difficulties arise in applying the Moser iteration since we do not have an reverse Hölder inequality when $q<2$. To overcome this difficulty we employ an argument used by Trudinger in [11] and our key result in this way is Proposition 2.4 below. By applying the Fountain theorem and its dual form [3, 13], we prove Theorem 1.1 in Section 3. To conclude this introduction, we explain some notations used in what follows. Denote the norms of the spaces $H_0^1(\Omega), L^p(\Omega)(1\leq p<\infty)$ by

$$\|u\|:=\Bigg(\int\limits_\Omega|\nabla u|^2\,\mathrm{d}x\Bigg)^{\frac{1}{2}},\qquad |u|_{L_t^p(\Omega)}:=\Bigg(\int\limits_\Omega\frac{|u|^p}{|y|^t}\,\mathrm{d}x\Bigg)^{\frac{1}{p}},$$

respectively. By symbol C we denote a generic constant whose value may change from line to line.

2. Strong convergence of approximating solutions in $H_0^1(\Omega)$

We consider the following perturbed problem:

$$\begin{cases} -\Delta u = \frac{|u|^{2^*(t)-2-\varepsilon}u}{|y|^t} + \mu|u|^{q-2}u & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $\varepsilon > 0$ is a small constant, For brevity of notations, in the sequel we denote $2_\epsilon^*(t) = 2^*(t) - \epsilon$. A function $u \in H_0^1(\Omega)$ is said to be a weak solution of problem (2.1) if

$$\int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x - \int_\Omega \frac{|u|^{2_\epsilon^*(t)-2}}{|y|^t} u\varphi \,\mathrm{d}x - \int_\Omega \mu|u|^{q-2}u\varphi \,\mathrm{d}x = 0,$$

for any $\varphi \in H_0^1(\Omega)$.

The corresponding energy functional to problem (2.1) is defined in $H_0^1(\Omega)$ by

$$I_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2_\epsilon^*(t)}\int_\Omega \frac{|u|^{2_\epsilon^*(t)}}{|y|^t} \,\mathrm{d}x - \frac{1}{q}\int_\Omega \mu|u|^q \,\mathrm{d}x \,.$$

We first introduce some notations and terminologies which will be used in the sequel. Let u be a solution of problem (2.1), set $\tilde{u} := |u|$ (extended by zero out of Ω). Then $\tilde{u} \in H^1\left(\mathbb{R}^N\right)$, with $\varphi \geq 0$

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla\tilde{u}\nabla\varphi &= \int_\Omega \nabla|u| \cdot \nabla\varphi \,\mathrm{d}x \\ &= \int_{\partial\Omega} \varphi\frac{\partial|u|}{\partial n} \,\mathrm{d}s - \int_\Omega |u|^{-1}u \operatorname{div}(\nabla u)\varphi \,\mathrm{d}x \\ &\leq \int_\Omega u|u|^{-1}\left(\frac{|u|^{2_\epsilon^*(t)-2}}{|y|^t}u + \mu|u|^{q-2}u\right)\varphi \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left(\frac{\tilde{u}^{2_\epsilon^*(t)-1}}{|y|^t} + \mu\tilde{u}^{q-1}\right)\varphi \,\mathrm{d}x, \end{aligned}$$

which implies in the sense of distribution

$$-\Delta\tilde{u} \leq \frac{\tilde{u}^{2_\epsilon^*(t)-1}}{|y|^t} + \mu\tilde{u}^{q-1}.$$

An easy computation shows that, for $A>0$ a large constant,

$$-\Delta \tilde{u} \leq \frac{2\tilde{u}^{2^*(t)-1}}{|y|^t} + \frac{A}{|y|^t}. \tag{2.2}$$

So in next section we can only consider the estimates of solutions to (2.2) in $H^1\left(\mathbb{R}^N\right)$, and this also makes us free from caring about the sign of u and the bounded domain Ω.

Definition 2.1. Let $(u_n)_{n\in\mathbb{N}}$ be a given sequence. We shall say that $(u_n)_{n\in\mathbb{N}}$ is a controlled sequence if each u_n is a solution to problem (2.2).

For any $\lambda>0$ and $x\in\mathbb{R}^N$, we define

$$\rho_{x,\lambda}(u) = \lambda^{\frac{N-t}{2^*(t)}} u\big(\lambda\big(\cdot - x\big)\big), \qquad u \in H_0^1(\Omega).$$

We have the following decomposition of approximating solutions.

Proposition 2.2 ([10] Proposition C.1). *Suppose that* $N\geq 3$. *Let* u_n *be a solution of* (2.1) *with* $\epsilon=\epsilon_n\to 0$, *satisfying* $\|u_n\|\leq C$ *for some constant* C. *Then,* u_n *can be decomposed as*

$$u_n = u_0 + \sum_{j=1}^{h} \rho_{x_{n,j},\lambda_{n,j}}\left(U_j\right) + \omega_n, \tag{2.3}$$

where $\omega_n\to 0$ *in* $H^1(\Omega)$, u_0 *is a solution for* (1.1) *and* U_j *is a solution of*

$$-\Delta u = \frac{|u|^{2^*(t)-2}u}{|y|^t}, \qquad u \in D^{1,2}\left(\mathbb{R}^N\right).$$

In order to prove the strong convergence of u_n in $H_0^1(\Omega)$, we only need to show that the bubbles $\rho_{x_{n,j},\lambda_{n,j}}\left(U_j\right)$ will not appear in the decomposition of u_n.

Among all the bubbles $\rho_{x_{n,j},\lambda_{n,j}}\left(U_j\right)$, we can choose a bubble, such that this bubble has the slowest concentration rate. That is, the corresponding λ is the lowest order infinity among all the λ appearing in the bubbles. For simplicity, we denote λ_n the slowest concentration rate and x_n the corresponding concentration point. Because the number of the bubbles of u_n is finite, we may always choose a constant $\bar{C}>0$ such that the region

$$\mathcal{A}_n^1 := \left(B_{(\bar{C}+5)\lambda_n^{-\frac{1}{2}}}(x_n) \Big\backslash B_{\bar{C}\lambda_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega,$$

does not contain any concentration point of u_n for every n. We call this region a safe region for u_n. We consider two thinner subsets as follows

$$\mathcal{A}_n^2 := \left(B_{(\bar{C}+4)\lambda_n^{-\frac{1}{2}}}(x_n) \Big\backslash B_{(\bar{C}+1)\lambda_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega,$$

and

$$\mathcal{A}_n^3 := \left(B_{(\bar{C}+3)\lambda_n^{-\frac{1}{2}}}(x_n) \Big\backslash B_{(\bar{C}+2)\lambda_n^{-\frac{1}{2}}}(x_n)\right) \cap \Omega.$$

Lemma 2.3 ([10] Lemma 3.2)**.** *Let w_n be a controlled sequence. Then there is a constant $C > 0$ independent of n, such that*

$$\left(r^{t-N} \int\limits_{B_r(\hat{x})\cap\Omega} \frac{w_n^\tau}{|y|^t}\,\mathrm{d}x \right)^{\frac{1}{\tau}} \le C, \quad \forall \hat{x} \in \mathbb{R}^N,$$

for all $r \in \left[\bar{C}\lambda_n^{-\frac{1}{2}}, (\bar{C}+5)\lambda_n^{-\frac{1}{2}}\right]$, where $\tau = \frac{2(N-t)}{2N-t-2}$.

In this section, we will prove the following technical result:

Proposition 2.4. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled sequence. Then there is a positive constant C independent of n such that*

$$\int\limits_{\mathcal{A}_n^2} \frac{|u_n|^{2\beta^2}}{|y|^t}\,\mathrm{d}x \le C\lambda_n^{-\frac{N-t}{2}}, \quad \text{where} \quad \beta := \frac{2^*(t)}{2}.$$

Proof. We set

$$v_n(x) := |u_n|\left(\lambda_n^{-1/2}x\right),\ x \in \Omega_n, \quad \text{where} \quad \Omega_n := \left\{x : \lambda_n^{-1/2}x \in \Omega\right\}.$$

Using the inequality (2.2), it is easy to check that v_n (extended by zero out of Ω) satisfies

$$-\Delta v_n \le \lambda_n^{\frac{t}{2}-1}\left(\frac{2v_n^{2^*(t)-1}}{|y|^t} + \frac{A}{|y|^t}\right) \text{ in } \mathbb{R}^N. \tag{2.4}$$

For a fixed $l > 0$ we consider the two following functions defined on $[0,+\infty)$ by

$$F(u) := \begin{cases} u^\beta & \text{if } u \le l, \\ \beta l^{\beta-1}(u-l) + l^\beta & \text{if } u > l, \end{cases}$$

and

$$G(u) := \begin{cases} u^{2\beta-1} & \text{if } u \le l, \\ \beta[2\beta-1]l^{2(\beta-1)}(u-l) + l^{2\beta-1} & \text{if } u > l. \end{cases}$$

An easy argument shows that

(i) $G(u) \le uG'(u)$,

(ii) $C\,[F'(u)]^2 \le G'(u)$,

(iii) $uG(u) \le C[F(u)]^2$,

(iv) If $u \in H_0^1(\Omega)$, then $F(u), G(u) \in H_0^1(\Omega)$.

For fixed $\hat{x} \in \mathcal{A}_n^2$ and $0 < r < R \leq 1$, we set

$$z_n := \lambda_n^{1/2} \hat{x} \quad \text{and} \quad \xi := \eta^2 G\left(v_n\right),$$

where $\eta \in \mathcal{C}_0^\infty\left(B\left(z_n, R\right)\right)$ is a non-negative cut-off function such that $\eta = 1$ in $B\left(z_n, r\right)$ and $|\nabla \eta| \leq \frac{2}{R-r}$.

Using (2.4), it follows that

$$\int\limits_{\mathbb{R}^N} \nabla v_n \nabla\left(\eta^2 G\left(v_n\right)\right) \mathrm{d}x \leq \lambda_n^{\frac{t}{2}-1} \int\limits_{\mathbb{R}^N} f\left(v_n\right) \eta^2 G\left(v_n\right) \mathrm{d}x,$$

where

$$f(h) := 2 \frac{h^{2^*(t)-1}}{|y|^t} + \frac{A}{|y|^t}, \qquad h \geq 0.$$

Using (i) and Young's inequality, to get

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} \left|\nabla v_n\right|^2 \eta^2 G'\left(v_n\right) \mathrm{d}x &= \int\limits_{\mathbb{R}^N} \nabla v_n \nabla\big(\eta^2 G(v_n)\big) \mathrm{d}x - 2 \int\limits_{\mathbb{R}^N} \nabla v_n G\left(v_n\right) \eta \nabla \eta \, \mathrm{d}x \\
&\leq 2 \int\limits_{\mathbb{R}^N} \left|\nabla v_n\right| \eta \big(G(v_n)\big)^{1/2} \big(G(v_n)\big)^{1/2} |\nabla \eta| \, \mathrm{d}x + \lambda_n^{\frac{t}{2}-1} \int\limits_{\mathbb{R}^N} f\left(v_n\right) \eta^2 G\left(v_n\right) \mathrm{d}x \\
&\leq 2 \int\limits_{\mathbb{R}^N} \left|\nabla v_n\right| \left(G'\left(v_n\right)\right)^{1/2} \eta v_n^{1/2} \big(G(v_n)\big)^{1/2} |\nabla \eta| \, \mathrm{d}x + \lambda_n^{\frac{t}{2}-1} \int\limits_{\mathbb{R}^N} f\left(v_n\right) \eta^2 G\left(v_n\right) \mathrm{d}x \\
&\leq \frac{1}{2} \int\limits_{\mathbb{R}^N} \left|\nabla v_n\right|^2 \eta^2 G'\left(v_n\right) \mathrm{d}x + C \int\limits_{\mathbb{R}^N} |\nabla \eta|^2 v_n G\left(v_n\right) \mathrm{d}x + \lambda_n^{\frac{t-2}{2}} \int\limits_{\mathbb{R}^N} f\left(v_n\right) \eta^2 G\left(v_n\right) \mathrm{d}x .
\end{aligned}$$

This implies that

$$\int\limits_{\mathbb{R}^N} \left|\nabla v_n\right|^2 \eta^2 G'\left(v_n\right) \mathrm{d}x \leq C \int\limits_{\mathbb{R}^N} |\nabla \eta|^2 v_n G\left(v_n\right) \mathrm{d}x + 2 \lambda_n^{\frac{t-2}{2}} \int\limits_{\mathbb{R}^N} f\left(v_n\right) \eta^2 G\left(v_n\right) \mathrm{d}x .$$

It follows from (iii) that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} \left|\nabla v_n\right|^2 \eta^2 G'\left(v_n\right) \mathrm{d}x \leq{} & C \int\limits_{\mathbb{R}^N} |\nabla \eta|^2 \left[F\left(v_n\right)\right]^2 \mathrm{d}x \\
& + C \lambda_n^{\frac{t-2}{2}} \int\limits_{\mathbb{R}^N} \eta^2 \frac{v_n^{2^*(t)-2}}{|y|^t} \left[F\left(v_n\right)\right]^2 \mathrm{d}x + C \int\limits_{\mathbb{R}^N} \frac{\eta^2 G\left(v_n\right)}{|y|^t} \mathrm{d}x .
\end{aligned}$$

Combining this with (ii), then we deduce that

$$\int\limits_{\mathbb{R}^N} \left|\nabla\left(\eta F\left(v_n\right)\right)\right|^2 \mathrm{d}x \leq C \int\limits_{\mathbb{R}^N} |\nabla\eta|^2 \left[F\left(v_n\right)\right]^2 \mathrm{d}x$$

$$+ C\lambda_n^{\frac{t-2}{2}} \int\limits_{\mathbb{R}^N} \eta^2 \frac{v_n^{2^*(t)-2}}{|y|^t} \left[F\left(v_n\right)\right]^2 \mathrm{d}x + C \int\limits_{\mathbb{R}^N} \frac{\eta^2 G\left(v_n\right)}{|y|^t} \mathrm{d}x\,.$$

Applying the Hardy-Sobolev embedding theorem and Hölder's inequality, it follows that

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^{\mathbb{N}}} \eta^{2^*(t)} \frac{F\left(v_n\right)^{2^*}(t)}{|y|^t} dx\right)^{2/2^*(t)} &\leq C \int\limits_{\mathbb{R}^{\mathbb{N}}} |\nabla\eta|^2 \left[F\left(v_n\right)\right]^2 \mathrm{d}x \\ + C\lambda_n^{\frac{t-2}{2}} \left(\int\limits_{\mathbb{R}^N} \frac{\eta^{2^*(t)}}{|y|^t} \left[F\left(v_n\right)\right]^{2^*(t)} \mathrm{d}x\right)^{\frac{2}{2^*(t)}} &\left(\int\limits_{B(z_n,1)} \frac{v_n^{2^*(t)}}{|y|^t} \mathrm{d}x\right)^{\frac{2-t}{N-t}} \\ + C \int\limits_{\mathbb{R}^N} \frac{\eta^2 G\left(v_n\right)}{|y|^t} \mathrm{d}x\,.\end{aligned} \tag{2.5}$$

Since $y \in \mathcal{A}_n^1$, then it is easy to verify that $B\left(y, \lambda_n^{-1/2}\right) \subset \mathcal{A}_n^1$. From $\mathcal{A}_n^1$ does not contain any concentration point of u_n, we can deduce that

$$\lambda_n^{\frac{t-2}{2}} \left[\int\limits_{B(z_n,1)} \frac{v_n^{2^*(t)}}{|y|^t} \mathrm{d}x\right]^{\frac{2-t}{N-t}} = \left[\int\limits_{B\left(y,\lambda_n^{-\frac{1}{2}}\right)} \frac{|u_n|^{2^*(t)}}{|y|^t} \mathrm{d}x\right]^{\frac{2-t}{N-t}} \to 0,$$

as $n \to +\infty$. It follows that

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^{\mathbb{N}}} \eta^{2^*(t)} \frac{F\left(v_n\right)^{2^*(t)}}{|y|^t} \mathrm{d}x\right)^{\frac{2}{2^*(t)}} \leq\; & C \int\limits_{\mathbb{R}^{\mathbb{N}}} |\nabla\eta|^2 \left[F\left(v_n\right)\right]^2 \mathrm{d}x \\ & + \frac{1}{2} \left(\int\limits_{\mathbb{R}^{\mathbb{N}}} \eta^{2^*(t)} \frac{F\left(v_n\right)^{2^*(t)}}{|y|^t} \mathrm{d}x\right)^{2/2^*(t)} \\ & + C \int\limits_{\mathbb{R}^N} \frac{\eta^2 G\left(v_n\right)}{|y|^t} \mathrm{d}x\,.\end{aligned}$$

Thus

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^{\mathbb{N}}} \eta^{2^*(t)} \frac{F\left(v_n\right)^{2^*(t)}}{|y|^t}\,\mathrm{d}x\right)^{\frac{2}{2^*(t)}} &\leq C \int\limits_{\mathbb{R}^N} |\nabla\eta|^2 \left[F\left(v_n\right)\right]^2 \mathrm{d}x + C\int\limits_{\mathbb{R}^N} \frac{\eta^2 G\left(v_n\right)}{|y|^t}\,\mathrm{d}x \\ &\leq \frac{C}{(R-r)^2}\int\limits_{\mathbb{R}^N} \left[F\left(v_n\right)\right]^2 \mathrm{d}x + C\int\limits_{\mathbb{R}^N} \frac{\eta^2 G\left(v_n\right)}{|y|^t}\,\mathrm{d}x\,.\end{aligned} \tag{2.6}$$

Letting $l \to +\infty$ in (2.6), we obtain

$$\begin{aligned}\left(\int\limits_{B(z_n,r)} \frac{v_n^{2^*(t)\beta}}{|y|^t}\,\mathrm{d}x\right)^{\frac{2}{2^*(t)}} &\leq \frac{C}{(R-r)^2}\int\limits_{B(z_n,R)} v_n^{2\beta}\,\mathrm{d}x + C\int\limits_{B(z_n,R)} \frac{v_n^{2\beta-1}}{|y|^s}dx \\ &\leq \frac{CR^t}{(R-r)^2}\int\limits_{B(z_n,R)} \frac{v_n^{2\beta}}{|y|^t}\,\mathrm{d}x + C\int\limits_{B(z_n,R)} \frac{v_n^{2\beta-1}}{|y|^t}dx \\ &\leq \frac{C}{(R-r)^2}\int\limits_{B(z_n,R)} \frac{v_n^{2\beta}}{|y|^t}\,\mathrm{d}x + C\int\limits_{B(z_n,R)} \frac{v_n^{2\beta-1}}{|y|^t}\,\mathrm{d}x\,.\end{aligned}$$

Then the above inequality can be written as

$$\begin{aligned}\left(\int\limits_{B(z_n,r)} \frac{v_n^{2\beta^2}}{|y|^t}\,\mathrm{d}x\right)^{\frac{1}{2\beta^2}} &\leq \frac{C}{(R-r)^{\frac{1}{\beta}}}\left(\int\limits_{B(z_n,R)} \frac{v_n^{2^*(t)}}{|y|^t}\,\mathrm{d}x\right)^{1/2^*(t)} \\ &\quad + C\left(\int\limits_{B(z_n,R)} \frac{v_n^{2\beta-1}}{|y|^t}\,\mathrm{d}x\right)^{1/2^*(t)}.\end{aligned} \tag{2.7}$$

Since $2\beta - 1 < 2^*$, by Young's inequality we have that

$$\int\limits_{B(z_n,R)} \frac{v_n^{2\beta-1}}{|y|^t}\,\mathrm{d}x \leq C\int\limits_{B(z_n,R)} \frac{1}{|y|^t}\,\mathrm{d}x + C\int\limits_{B(z_n,R)} \frac{v_n^{2^*(t)}}{|y|^t}dx \leq C + C\int\limits_{B(z_n,R)} \frac{v_n^{2^*(t)}}{|y|^t}\,\mathrm{d}x\,.$$

Together with (2.7), this implies that

$$\left(\int\limits_{B(z_n,r)} \frac{v_n^{2\beta^2}}{|y|^t}\,\mathrm{d}x\right)^{\frac{1}{2\beta^2}} \leq \left(\frac{C}{(R-r)^{\frac{1}{\beta}}} + C\right)\left(\int\limits_{B(z_n,R)} \frac{v_n^{2^*(t)}}{|y|^t}\,\mathrm{d}x\right)^{1/2^*(t)} + C.$$

Let $k \in (0,1)$ and $\tau = \frac{2(N-t)}{2N-t-2}$, since $0 < \tau < 2^*(t) < 2\beta^2$ by Hölder's inequality and Young's inequality we obtain

$$\begin{aligned} |v_n|_{L_t^{2\beta^2}(B(z_n,r))} &\le \left(\frac{C}{(R-r)^{\frac{1}{\beta}}} + C\right) |v_n|^{k}_{L_t^{\tau}(B(z_n,R))} |v_n|^{1-k}_{L_t^{2\beta^2}(B(z_n,R))} + C \\ &\le \frac{1}{2} |v_n|_{L_t^{2\beta^2}(B(z_n,R))} + \left(\frac{C}{(R-r)^{\frac{1}{k\beta}}} + C\right) |v_n|_{L_t^{\tau}(B(z_n,R))} + C. \end{aligned} \tag{2.8}$$

By using iteration argument, we deduce from (2.8) that

$$|v_n|_{L_t^{2\beta^2}\left(B\left(z_n,\frac{1}{2}\right)\right)} \le C\, |v_n|_{L_t^{\tau}(B(z_n,1))} + C. \tag{2.9}$$

On the other hand, it is easy to see from Lemma 2.3 that for any $y \in A_n^2$

$$|v_n|_{L_t^{\tau}(B(z_n,1))} \le C.$$

Combining this with (2.9) and using the definition of v_n, we obtain then the desired result. ☐

As a consequence of the previous proposition we have the following estimates which play a crucial role in the proof of Proposition 2.7 below.

Lemma 2.5. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled sequence. For any $\gamma \le 2^*(t)$ there exists a positive constant C such that for any n*

$$\int\limits_{\mathcal{A}_n^2} \frac{|u_n|^\gamma}{|y|^t}\, \mathrm{d}x \le\; C\lambda_n^{-\frac{N-t}{2}}.$$

Proof. By Hölder's inequality and Proposition 2.4 we obtain for any $\gamma \le 2^*(t)$,

$$\begin{aligned} \int\limits_{\mathcal{A}_n^2} \frac{|u_n|^\gamma}{|y|^t}\, \mathrm{d}x &\le\; C\left(\int\limits_{\mathcal{A}_n^2} \frac{|u_n|^{2\beta^2}}{|y|^t}\right)^{\frac{\gamma}{2\beta^2}} \lambda_n^{-\frac{N-t}{2}\left(1-\frac{\gamma}{2\beta^2}\right)} \\ &\le\; C\lambda_n^{-\frac{N-t}{2}\frac{\gamma}{2\beta^2}} \lambda_n^{-\frac{N-t}{2}+\frac{(N-t)\gamma}{4\beta^2}} \\ &\le\; C\lambda_n^{-\frac{N-t}{2}}. \end{aligned}$$

☐

Proposition 2.6. *We have,*

$$\int\limits_{\mathcal{A}_n^3} |\nabla u_n|^2\, \mathrm{d}x \;\le\; C\int\limits_{\mathcal{A}_n^2} \frac{|u_n|^{2^*(t)}+1}{|y|^t}\, \mathrm{d}x + C\lambda_n \int\limits_{\mathcal{A}_n^2} \frac{|u_n|^q}{|y|^t}\, \mathrm{d}x\,. \tag{2.10}$$

Particularly,

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \,\mathrm{d}x \;\leq\; C\lambda_n^{\frac{2-(N-t)}{2}}. \tag{2.11}$$

Proof. Let $\phi_n \in C_0^\infty\left(\mathcal{A}_n^2\right)$ be a function with $\phi_n = 1$ in $\mathcal{A}_n^3, 0 \leq \phi_n \leq 1$ and $|\nabla \phi_n| \leq C\lambda_n^{\frac{1}{2}}$ From

$$\int_{\Omega} \nabla u_n \nabla\left(\phi_n^2 u_n\right) \mathrm{d}x \leq C \int_{\Omega} \left(\frac{|u_n|^{2^*(t)-1}+1}{|y|^t}\right) \phi_n^2 |u_n| \,\mathrm{d}x\,.$$

we obtain (2.10). From (2.10) and Lemma 2.5, we have

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \,\mathrm{d}x \leq C\lambda_n^{-\frac{N-t}{2}} + C\lambda_n\lambda_n^{-\frac{(N-t)}{2}} \leq C\lambda_n^{\frac{2-(N-t)}{2}}. \qquad \square$$

Proposition 2.7. *For any u_n witch is a solution of (2.1) with $\varepsilon = \varepsilon_n \to 0$ as $n \to +\infty$, satisfying $\|u_n\| \leq C$ for some constant independent of n, the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly in $H_0^1(\Omega)$.*

Proof. Take a $t_n \in [\bar{C}+2, \bar{C}+3]$, satisfying

$$\begin{aligned} &\int_{\partial B_{t_n\lambda_n^{-1/2}(x_n)}} \left(\lambda_n^{-\frac{t}{2}} \frac{u_n^{2^*_{\epsilon_n}(t)}}{|y|^t} + |u_n|^q + \lambda_n^{-1}|\nabla u_n|^2\right) \\ &\leq C\lambda_n^{1/2} \int_{\mathcal{A}_n^3} \left(\lambda_n^{-\frac{t}{2}} \frac{u_n^{2^*_{\epsilon_n}(t)}}{|y|^t} + |u_n|^q + \lambda_n^{-1}|\nabla u_n|^2\right). \end{aligned} \tag{2.12}$$

Using Lemma 2.5, (2.11) and (2.12), we obtain

$$\int_{\partial B_{t_n\lambda_n^{-1/2}(x_n)}} \left(\lambda_n^{-\frac{t}{2}} \frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t} + |u_n|^q + \lambda_n^{-1}|\nabla u_n|^2\right) \leq C\lambda_n^{\frac{1}{2}-\frac{N-t}{2}}. \tag{2.13}$$

We have two different cases:

(i) $B_{t_n\lambda_n^{-\frac{1}{2}}}(x_n) \cap \left(\mathbb{R}^N \backslash \Omega\right) \neq \emptyset$,

(ii) $B_{t_n\lambda_n^{-\frac{1}{2}}}(x_n) \subset \Omega$.

Recall that $2^*_{\epsilon_n}(t) = 2^*(t) - \varepsilon_n$. We have the following local Pohozaev identity for u_n on $B_n = B_{t_n\lambda_n^{-1/2}}(x_n) \cap \Omega$:

$$\begin{aligned}
&\left(\frac{N-t}{2^*_{\epsilon_n}(t)} - \frac{N-2}{2}\right) \int\limits_{B_n} \frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t}\,\mathrm{d}x + \mu\left(\frac{N}{q} - \frac{N-2}{2}\right)\int\limits_{B_n} |u_n|^q\,\mathrm{d}x \\
&= \frac{N-2}{2}\int\limits_{\partial B_n} (\nabla u_n\cdot\nu)\,u_n\,\mathrm{d}\sigma + \frac{1}{2}\int\limits_{\partial B_n} |\nabla u_n|^2\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \qquad (2.14)\\
&\quad + \frac{1}{2^*_{\epsilon_n}(t)}\int\limits_{\partial B_n} \frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t}\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma + \frac{\mu}{q}\int\limits_{\partial B_n} |u_n|^q\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma,
\end{aligned}$$

where x_0 is a point in $\mathbb{R}^N$ and where ν is the outward normal to ∂B_n. The point x_0 in (2.14) is chosen as follows

i) we take $x_0 \in \mathbb{R}^N\backslash\Omega$ with

$$|x_0 - x_n| \le 2t_n\lambda_n^{-\frac{1}{2}} \quad \text{and} \quad (x-x_0)\cdot\nu \le 0 \ \text{ in } \ \partial\Omega\cap B_n,$$

ii) we take a point $x_0 = x_n$.

Due that

$$2^*_{\epsilon_n}(t) < 2^*(t), \qquad \frac{N-t}{2^*_{\epsilon_n}(t)} - \frac{N-2}{2} > 0.$$

Hence, the first term in the left-hand side of (2.14) is nonnegative, and (2.14) can be rewritten as

$$\begin{aligned}
\mu\left(\frac{N}{q} - \frac{N-2}{2}\right)\int\limits_{B_n}|u_n|^q\,\mathrm{d}x \le{}& \frac{N-2}{2}\int\limits_{\partial B_n} (\nabla u_n\cdot v)\,u_n\,\mathrm{d}\sigma \\
&+ \frac{1}{2}\int\limits_{\partial B_n} |\nabla u_n|^2\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \\
&+ \frac{1}{2^*_{\epsilon_n}(t)}\int\limits_{\partial B_n} \frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t}\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \qquad (2.15)\\
&+ \frac{\mu}{q}\int\limits_{\partial B_n} |u_n|^q\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma.
\end{aligned}$$

Now, we decompose ∂B_n into $\partial B_n = \partial_i B_n \cup \partial_e B_n$, where

$$\partial_i B_n = \partial B_n \cap \Omega \quad \text{and} \quad \partial_e B_n = \partial B_n \cap \partial\Omega.$$

Observing that $u_n = 0$ on $\partial\Omega$, we have

$$\begin{aligned}\frac{N-2}{2}\int\limits_{\partial_e B_n} (\nabla u_n \cdot \nu)\, u_n \,\mathrm{d}\sigma + \frac{1}{2}\int\limits_{\partial_e B_n} |\nabla u_n|^2\, (x-x_0)\cdot\nu \,\mathrm{d}\sigma \\ + \frac{1}{2^*_{\epsilon_n}(t)}\int\limits_{\partial_e B_n} \frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t}\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \\ + \frac{\mu}{q}\int\limits_{\partial_e B_n} |u_n|^q\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \\ = \frac{1}{2}\int\limits_{\partial_e B_n} |\nabla u_n|^2\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \le 0.\end{aligned}$$

Hence, we can rewrite (2.15) as

$$\begin{aligned}\mu\left(\frac{N}{q}-\frac{N-2}{2}\right)\int\limits_{B_n}|u_n|^2\,\mathrm{d}x \le \frac{N-2}{2}\int\limits_{\partial_i B_n}(\nabla u_n\cdot\nu)\,u_n\,\mathrm{d}\sigma \\ + \frac{1}{2}\int\limits_{\partial_i B_n}|\nabla u_n|^2\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \\ + \frac{1}{2^*_{\epsilon_n}(t)}\int\limits_{\partial_i B_n}\frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t}\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma \\ + \frac{\mu}{q}\int\limits_{\partial_i B_n}|u_n|^q\,(x-x_0)\cdot\nu\,\mathrm{d}\sigma.\end{aligned} \tag{2.16}$$

From (2.13), noting that $|x-x_0| \le C\lambda_n^{-\frac{1}{2}}$ for $x \in \partial_i B_n$, we have RHS of (2.16)

$$\begin{aligned}\le C\lambda_n^{-\frac{1}{2}}\int\limits_{\partial_i B_n}\left(|\nabla u_n|^2 + \frac{|u_n|^{2^*_{\epsilon_n}(t)}}{|y|^t} + |u_n|^q\right)\mathrm{d}\sigma \\ + C\int\limits_{\partial_i B_n}|\nabla u_n||u_n|\,\mathrm{d}\sigma \le C\lambda_n^{1-\frac{N-t}{2}}.\end{aligned} \tag{2.17}$$

On the other hand, using the same argument as in [8, 10], we have

$$\int\limits_{B_n}|u_n|^q \ge C\lambda_n^{-q-N+\frac{Nq}{2}}. \tag{2.18}$$

Combing (2.17) and (2.18), we obtain

$$\lambda_n^{-q-N+\frac{Nq}{2}} \le C\lambda_n^{\frac{2-(N-t)}{2}}. \tag{2.19}$$

which is a contradiction since $N \ge 2\frac{q+1}{q-1} + t$. □

3. The proof of main results

In this section, we demonstrate our main result, following the ideas in [3, 9, 13]. Since $H_0^1(\Omega)$ is a Hilbert space, then there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n, \ldots\}$ of $H_0^1(\Omega)$. For any $i = 1, 2, \ldots$ we denote $X_i = \mathbb{R}e_i$. We have $H_0^1(\Omega) = \overline{\bigoplus_{i=1}^{\infty} X_i}$. Following the notations used by Bartsch (see Theorem 2.5 in [2]), for any $k \in \mathbb{N}$, we put

$$Y_k := \operatorname{span}\{e_1, \ldots, e_k\}, \quad \text{and} \quad Z_k := \overline{\operatorname{span}\{e_k, e_{k+1}, \ldots\}}.$$

Define

$$B_k := \{u \in Y_k : \|u\| \le \rho_k\}, \qquad N_k := \{u \in Z_k : \|u\| = r_k\},$$

where $\rho_k > r_k > 0$. Let $2^*_{\varepsilon_n}(t) := 2^*(t) - \varepsilon_n$, where $(\varepsilon_n)_n$ is a decreasing sequence with $0 < \varepsilon_n < 2^*(t) - 2$ and $\varepsilon_n \to 0$ as $n \to \infty$.

The proof of (i) of Theorem 1.1. First, we claim that for every $k \in \mathbb{N}$, there exist $\rho_k > \tau_k > 0$ such that $\rho_k \to +\infty$ as $k \to +\infty$ and

$$a_k^n := \max_{\substack{u \in Y_k \\ \|u\| = \rho_k}} I_{\varepsilon_n}(u) \le 0, \; b_k^n := \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} I_{\varepsilon_n}(u) \to \infty \;\text{ as }\; k \to +\infty.$$

We can choose p_t such that $2 < p_t < 2^*_{\varepsilon_n}(t)$, for all n. It follows from the Hölder inequality that for any $u \in Y_k$,

$$I_{\varepsilon_n}(u) \le \frac{1}{2}\|u\|^2 - C|u|_{L_t^{p_t}}^{2^*_{\varepsilon_n}(t)} - \frac{\mu}{q}|u|_{L^q}^q.$$

Since all norms on the finite dimensional space are equivalent, it follows that

$$I_{\varepsilon_n}(u) \le \frac{1}{2}\|u\|^2 - C\|u\|^{p_t} - C\|u\|^q, \tag{3.1}$$

provide that $\|u\| \ge 1$. On the other hand, using the Hölder inequality and the Sobolev embedding, we obtain that for any $u \in Z_k$

$$I_{\varepsilon_n}(u) \ge \frac{1}{2}\|u\|^2 - C\|u\|^{2^*(t)} - C\|u\|^q. \tag{3.2}$$

From (3.1) and (3.2) we obtain the existence of $\rho_k > r_k > 0$, independent of n, such that $a_k^n < b_k^n$.

So by [13, Theorem 3.5 (Fountain theorem)], we conclude that I_{ε_n} has a sequence of critical points, denoted by $(v_k^n)_n$. Moreover, $c_k^n = I_{\varepsilon_n}(v_k^n)$, where

$$c_k^n := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_{\varepsilon_n}\big(\gamma(u)\big), \quad \text{and} \quad \Gamma_k := \Big\{\gamma \in \mathcal{C}\left(B_k, H_0^1(\Omega)\right) : \gamma_{|\partial B_k} = id\Big\}.$$

We claim that for any $k \in \mathbb{N}$,

$$c_k^n \to c_k := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I\big(\gamma(u)\big) \;\text{ as }\; n \to +\infty.$$

Indeed, for $\gamma \in \Gamma_k$, the functionals $I_{\varepsilon_n}(\gamma)$ are equicontinuous on the compact set B_k, we derive that

$$\lim_{n\to\infty} \sup_{u\in B_k} I_{\varepsilon_n}\big(\gamma(u)\big) \to \sup_{u\in B_k} I\big(\gamma(u)\big).$$

Passing to the limit as $n \to +\infty$, we deduce that for any $k \in \mathbb{N}$

$$\overline{\lim_{n\to\infty}}\, c_k^n \leq \overline{\lim_{n\to\infty}} \lim_{u\in B_k} I_{\varepsilon_n}\big(\gamma(u)\big) = \sup_{u\in B_k} I\big(\gamma(u)\big).$$

Since γ is arbitrary, then

$$\overline{\lim_{n\to\infty}}\, c_k^n \leq c_k. \tag{3.3}$$

On the other hand, for every $u \in H_0^1(\Omega)$, we have

$$I(u) = I_{\varepsilon_n}(u) + \int\limits_{\Omega} \frac{1}{|y|^t} g(u),$$

where $g(u) := \frac{|u|^{2^*_{\varepsilon_n}(t)}}{2^*_{\varepsilon_n}(t)} - \frac{|u|^{2^*(t)}}{2^*(t)}$. The function $g(r) = \frac{r^{2^*_{\varepsilon_n}(t)}}{2^*_{\varepsilon_n}(t)} - \frac{r^{2^*(t)}}{2^*(t)}$, $r > 0$, get it maximum value in $r = 1$, which implies that

$$g(r) \leq \frac{1}{2^*_{\varepsilon_n}(t)} - \frac{1}{2^*(t)} \quad \text{for all} \quad r > 0.$$

Let $\gamma \in \Gamma_k$,

$$\begin{aligned} I\big(\gamma(u)\big) &\leq I_{\varepsilon_n}\big(\gamma(u)\big) + \left(\frac{1}{2^*_{\varepsilon_n}(t)} - \frac{1}{2^*(t)}\right) \int\limits_{\Omega} \frac{1}{|y|^t} \\ &\leq I_{\varepsilon_n}\big(\gamma(u)\big) + C\left(\frac{1}{2^*_{\varepsilon_n}(t)} - \frac{1}{2^*(t)}\right). \end{aligned}$$

It follows from this that

$$c_k \leq c_k^n + C\left(\frac{1}{2^*_{\varepsilon_n}(t)} - \frac{1}{2^*(t)}\right).$$

We get for any $k \in \mathbb{N}$

$$c_k \leq \underline{\lim_{n\to\infty}}\, c_k^n. \tag{3.4}$$

Combining (3.3) with (3.4), we infer that

$$\lim_{n\to\infty} c_k^n = c_k. \tag{3.5}$$

We have

$$I_{\varepsilon_n}(v_k^n) = c_k^n \quad \text{and} \quad I'_{\varepsilon_n}(v_k^n)\, v_k^n = 0.$$

From this we obtain

$$\left(\frac{1}{2} - \frac{1}{2^*_{\varepsilon_n}(t)}\right) \int\limits_{\Omega} |\nabla v_k^n|^2 \, dx - \mu \left(\frac{1}{q} - \frac{1}{2^*_{\varepsilon_n}(t)}\right) \int\limits_{\Omega} |v_k^n|^q \, dx < c_k^n.$$

Since $(c_k^n)_n$ is bounded, then for all n

$$\left(\frac{1}{2}-\frac{1}{2^*_{\varepsilon_n}(t)}\right)\int_\Omega |\nabla v_k^n|^2 \,\mathrm{d}x < \mu\left(\frac{1}{q}-\frac{1}{2^*_{\varepsilon_n}(t)}\right)\int_\Omega |v_k^n|^q \,\mathrm{d}x + C.$$

By Sobolev's embedding and the fact that $q < 2$, we get that $(v_k^n)_n$ is bounded in $H_0^1(\Omega)$. Applying Proposition 2.7 we can find a subsequence of $(v_k^n)_n$, still denoted by $(v_k^n)_n$, such that $v_k^n \to v_k$ strongly in $H_0^1(\Omega)$, for some $v_k \in H_0^1(\Omega)$ and $I(v_k) = c_k$. Therefore, (v_k) is solution of (1.1).
It follows from (3.5) that for every $k \in \mathbb{N}$, there exists $n_k > k$ such that

$$|c_k^{n_k} - c_k| < \frac{1}{k}. \tag{3.6}$$

Let $\delta \in (0, \delta_0)$ be a fixed number, where

$$\delta_0 := \inf_{u\in H_0^1(\Omega), |u|_{L^2}=1} \int_\Omega |\nabla u|^2 \,\mathrm{d}x > 0.$$

Define

$$\alpha_k := \inf_{u\in Z_k, |u|_{L_t^{2^*_{\varepsilon_{n_k}}(t)}}=1} \int_\Omega \left(|\nabla u|^2 - \delta |u|^q\right) \mathrm{d}x, \tag{3.7}$$

We will show that, up to a subsequence, $\alpha_k \to +\infty$ as $k \to \infty$. Since $2^*_{\varepsilon_{n_k}}(t) < 2^*(t)$, then the scalar α_k can be achieved by a function $w_k \in Z_k$, which satisfies

$$-\Delta w_k = \alpha_k \frac{|w_k|^{2^*_{\varepsilon_{n_k}}(t)-2} w_k}{|y|^t} + \delta |w_k|^{q-2} w_k.$$

If $\alpha_k \not\to \infty$ as $k \to \infty$, then $\int_\Omega |\nabla w_k|^2 \,\mathrm{d}x \le C$ by the choice of δ. From Proposition 2.7, we conclude that $(w_k)_k$ converges strongly in $H_0^1(\Omega)$. Since $w_k \in Z_k$, up to a subsequence, we may assume that

$$w_k \to 0 \ \text{ in } \ H_0^1(\Omega).$$

By using Hölder's inequality, we deduce that $\displaystyle\lim_{k\to\infty} \int_\Omega |w_k|^{2^*_{\varepsilon_{n_k}}(t)} \,\mathrm{d}x = 0$, which is a contradiction due to $\int_\Omega |w_k|^{2^*_{\varepsilon_{n_k}}(t)} \,\mathrm{d}x = 1$. Thus

$$\alpha_k \to \infty \text{ as } k \to \infty.$$

By the Young inequality and Sobolev's embedding we obtain

$$\begin{aligned} I_{\varepsilon_{n_k}}(u) &\ge \frac{1}{2}\|u\|^2 - C\left[\alpha_k\right]^{-\frac{2^*_{\varepsilon_{n_k}}(t)}{2}} \|u\|^{2^*_{\varepsilon_{n_k}}(t)} - \frac{1}{4}\|u\|^2 - C \\ &= \frac{1}{4}\|u\|^2 - C\left[\alpha_k\right]^{-\frac{2^*_{\varepsilon_{n_k}}(t)}{2}} \|u\|^{2^*_{\varepsilon_{n_k}}(t)} - C. \end{aligned}$$

Choosing

$$r_k := \left(\frac{\alpha_k^{\frac{2^*_{\varepsilon_{n_k}}(t)}{2}}}{2C2^*_{\varepsilon_{n_k}}(t)}\right)^{\frac{1}{2^*_{\varepsilon_{n_k}}(t)-2}}.$$

We obtain that if $u \in Z_k$ and $\|u\| = r_k$,

$$I_{\varepsilon_{n_k}}(u) \geq \frac{1}{4}\left(1 - \frac{1}{2^*_{\varepsilon_{n_k}}(t)}\right)\left(\frac{\alpha_k^{\frac{2^*_{\varepsilon_{n_k}}(t)}{2}}}{2C2^*_{\varepsilon_{n_k}}(t)}\right)^{\frac{2}{2^*_{\varepsilon_{n_k}}(t)-2}} - C.$$

Since we have that $\alpha_k \to \infty$ as $k \to +\infty$, then $b_k^{n_k} \to \infty$ as $k \to \infty$. By [13, Theorem 3.5], we have that $c_k^{n_k} \geq b_k^{n_k}$ and so from (3.6), we get that

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_k^{n_k} = +\infty.$$

The conclusion of (i) of Theorem 1.1 is now obvious. ☐

The proof of (ii) of Theorem 1.1. Using the arguments similar to those of Theorem 3.20 in [13], we will show that for every $k \geq k_0$, there exist $\rho_k > r_k > 0$, independent of n, such that $\rho_k \to 0$ as $k \to +\infty$ and

(a) $$a_k^n := \inf_{\substack{u\in Z_k\\ \|u\|=\rho_k}} I_{\varepsilon_n}(u) \geq 0,$$

(b) $$b_k^n := \max_{\substack{u\in Y_k\\ \|u\|=r_k}} I_{\varepsilon_n}(u) < 0,$$

(c) $$b_k := \max_{\substack{u\in Y_k\\ \|u\|=r_k}} I(u) < 0,$$

(d) $$d_k^n := \inf_{\substack{u\in Z_k\\ \|u\|\leq\rho_k}} I_{\varepsilon_n}(u) \to 0 \text{ as } k \to +\infty.$$

In the interest of keeping this paper self-contained, we sketch here the proof of the above assertions. First we prove (a). Using Hölder's inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L_t^{2^*_{\varepsilon_n}(t)}(\Omega)$, there exists $0 < R < 1$ such that if $u \in H_0^1(\Omega)$ and $\|u\| \leq R$, then

$$\frac{1}{2^*_{\varepsilon_n}(t)}|u|^{2^*_{\varepsilon_n}(t)}_{L_t^{2^*_{\varepsilon_n}(t)}} \leq \frac{1}{4}\|u\|^2.$$

For any $k \in \mathbb{N}^*$, we define

$$\beta_k := \sup_{\substack{u\in Z_k\\ \|u\|=1}} |u|_q.$$

It is easy to see that $\beta_k \to 0$ as $k \to \infty$ (for details see [13, Lemma 3.8]). Then if $u \in Z_k$ satisfies $\|u\| \leq R$, we have

$$I_{\varepsilon_{n_k}}(u) \geq \frac{\|u\|^2}{4} - \frac{\mu}{q}\beta_k^q\|u\|^q. \tag{3.8}$$

We choose $\rho_k := \left(\frac{4\mu\beta_k^q}{q}\right)^{1/(2-q)}$. Since $\beta_k \to 0$ as $k \to \infty$, it follows that $\rho_k \to 0$ as $k \to \infty$. Let $k_0 \in \mathbb{N}^*$ such that $\rho_k \leq R$, for any $k \geq k_0$. Thus, for $k \geq k_0$, $u \in Z_k$ and $\|u\| = \rho_k$, we have $I_{\varepsilon_n}(u) \geq 0$ and (a) is proved.

Next we show (b), note that for $u \in H_0^1(\Omega)$, by the fact that on the finite dimensional space Y_k all norms are equivalent, we have

$$I_{\varepsilon_{n_k}}(u) \leq \frac{1}{2}\|u\|^2 - \frac{\mu}{q}\|u\|^q. \tag{3.9}$$

As a consequence of (3.9), for any $u \in Y_k$ with $\|u\| = r_k$, we get that $I_{\varepsilon_{n_k}}(u) \leq 0$, provided $r_k > 0$ is small enough, which gives (b). In the same way we obtain also (c). The proof of (d) follows from the combination of (3.8) and (3.9). On the other hand, a standard argument shows that the function I_{ε_n} satisfies the $(\mathrm{PS})_c^*$ condition with respect to (Y_k) (see [13, Theorem 3.20]). So from [13, Theorem 3.18 (Dual fountain theorem)] that I_{ε_n} has a sequence of critical points, denoted by $(u_k^n)_n$, moreover

$$I_{\varepsilon_n}(u_k^n) = c_k^n \in [d_k^n, b_k^n].$$

Since c_k^n is negatif, we get

$$\left(\frac{1}{2} - \frac{1}{2^*_{\varepsilon_n}(t)}\right)\int_\Omega |\nabla u_k^n|^2 \,\mathrm{d}x < \mu\left(\frac{1}{q} - \frac{1}{2^*_{\varepsilon_n}(t)}\right)\int_\Omega |u_k^n|^q \,\mathrm{d}x.$$

By using Sobolev's embedding, we deduce that $(u_k^n)_n$ is bounded in $H_0^1(\Omega)$. It follows from Proposition 2.7 that we can find a subsequence of $(u_k^n)_n$, which strongly converges to a solution u_k of (1.1) at level c_k, with $c_k := \lim_{n\to+\infty} c_k^n$. We claim first that for any $k \geq k_0,\ c_k < 0$. Indeed, since ∂B_k is compact and the functionals $(I_{\varepsilon_n})_n$ are equicontinuous, we derive that

$$b_k^n \to b_k.$$

It follows that

$$c_k \leq b_k < 0.$$

Secondly, we claim that $\lim\limits_{k\to+\infty} c_k = 0$. In fact, it follows from $c_k^n \to c_k$, for any every positive integer $k \geq k_0$, there exists $n_k > k$ such that

$$|c_k^{n_k} - c_k| < \frac{1}{k}. \tag{3.10}$$

By the Sobolev embedding, we have for any $u \in Z_k$ and $\|u\| \leq \rho_k$

$$\begin{aligned} I_{\varepsilon_{n_k}}(u) &\geq \frac{1}{2}\|u\|^2 - C\,[\alpha_k]^{-\frac{2^*_{\varepsilon_{n_k}}(t)}{2}}\,\|u\|^{2^*_{\varepsilon_{n_k}}(t)} - C\,\|u\|^q \\ &\geq \frac{1}{2}{\rho_k}^2 - C\,[\alpha_k]^{-\frac{2^*_{\varepsilon_{n_k}}(t)}{2}}\,{\rho_k}^{2^*_{\varepsilon_{n_k}}(t)} - C{\rho_k}^q, \end{aligned}$$

where C is a positive constant. Then for k large enough, we get

$$c_k^{n_k} \geq d_k^{n_k} \geq \frac{1}{2}{\rho_k}^2 - C\,[\alpha_k]^{-\frac{2^*_{\varepsilon_{n_k}}(t)}{2}}\,{\rho_k}^{2^*_{\varepsilon_{n_k}}(t)} - C{\rho_k}^q.$$

Since $\rho_k \to 0$ and $\alpha_k \to +\infty$ as $k \to \infty$, it follows (3.10) that

$$\lim_{k\to+\infty} c_k = \lim_{k\to+\infty} c_k^{n_k} = 0.$$

As results, we get that $I(u_k) \to 0$ as $k \to \infty$ and $I(u_k) < 0$ for any positive integer $k \geq k_0$. Finally we conclude that problem (1.1) has infinitely many solutions $(u_k)_k$ with negative energy converging to 0 as $k \to +\infty$. ☐

REFERENCES

[1] AMBROSETTI, A.—BREZIS, H.—CERAMI, G.: *Combined effects of concave-convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.

[2] BARTSCH, T.: *Infnitely many solutions of a symmetric Dirichlet problem*, Nonlinear Anal. **20** (1993), 1205–1216.

[3] T. BARTSCH, T.—WILLEM, M.: *On an elliptic equation with concave and convex nonlinearities*, Proc. Am. Math. Soc. **123** (1995), 3555–3555.

[4] BREZIS, H.—NIRENBERG, L.: *Positive solutions of nonlinear elliptic equations involving critical sobolev exponents*, Commun. Pure Appl. Math. **36** (1983), 437–477.

[5] CAO, D.—PENG, S.—YAN, S.: *Infinitely many solutions for p-Laplacian equation involving critical Sobolev growth*, J. Funct. Anal. **262** (2012), 2861–2902.

[6] CAO, D.—YAN, S.: *Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential*, Calc. Var. Partial Differ. Equ. **38** (2010), 471–501.

[7] G. Devillanova, S. Solimini, *Concentration estimates and multiple solutions to elliptic problems at critical growth*, Adv. Differ. Equations **7** (2002), 1257–1280.

[8] HAN, P.: *Many solutions for elliptic equations with critical exponents*, Israel J. Math. **164** (2008), 125–152.

[9] LIU, Z.—HAN, P.: *Infinitely many solutions for elliptic systems with critical exponents*, J. Math. Anal. Appl. **353** (2009), 544–552.

[10] PENG, S.—WANG, C.: *Infnitely many solutions for a Hardy–Sobolev equation involving critical growth*, Math. Methods in the Appl. Sci. **38** (2014), no. 2, 197–220.

[11] TRUDINGER, N.: *Remarks concerning the conformal deformation of riemannian structures on compact manifolds*, Ann. Della Sc. Norm. Super. Di Pisa-Cl. Di Sci. **22** (1968), 265–274.

[12] WANG, C.—YANG, J.: *Infinitely many solutions for a Hardy–Sobolev equation involving critical growth*, Math. Meth. Appl. Sci. **38** (2013), 197–220.

[13] WILLEM, M.: *Minimax Theorems.* Birkhäuser, Boston, Boston, MA, 1996, 5–70.

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