sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2023-0007
Tatra Mt. Math. Publ. **83** (2023), 87–102

ON THE GEOMETRIC DETERMINATION OF EXTENSIONS OF NON-ARCHIMEDEAN ABSOLUTE VALUES

Mohamed Faris — Lhoussain El Fadil

Sidi Mohamed Ben Abdellah University, Fez, MOROCCO

ABSTRACT. Let $|\ |$ be a discrete non-archimedean absolute value of a field K with valuation ring $\mathcal{O}$, maximal ideal $\mathcal{M}$ and residue field $\mathbb{F} = \mathcal{O}/\mathcal{M}$. Let L be a simple finite extension of K generated by a root α of a monic irreducible polynomial $F \in \mathcal{O}[x]$. Assume that $\overline{F} = \overline{\phi}^l$ in $\mathbb{F}[x]$ for some monic polynomial $\phi \in \mathcal{O}[x]$ whose reduction modulo $\mathcal{M}$ is irreducible, the ϕ-Newton polygon $N_{\phi}^{-}(F)$ has a single side of negative slope λ, and the residual polynomial $R_{\lambda}(F)(y)$ has no multiple factors in $\mathbb{F}_{\phi}[y]$. In this paper, we describe all absolute values of L extending $|\ |$. The problem is classical but our approach uses new ideas. Some useful remarks and computational examples are given to highlight some improvements due to our results.

1. Introduction

Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field with $\alpha \in \mathbb{Z}_K$ the ring of algebraic integers of K. Let F be the minimal polynomial of α over the field $\mathbb{Q}$. The determination of the prime ideal decomposition in $\mathbb{Z}_K$ of any rational prime p is one of the most important problems in algebraic number theory and is related to the factorization of the polynomial $\overline{F}$ in $\mathbb{F}_p[x]$. Let $\overline{F} = \prod_{i=1}^{r} \overline{\phi_i}^{e_i}$ be the factorization of $\overline{F}$ in $\mathbb{F}_p[x]$, where $\overline{\phi}_1, \dots, \overline{\phi}_r$ are distinct irreducible polynomials over $\mathbb{F}_p$ and $\phi_i \in \mathbb{Z}[x]$ monic. In 1878, Dedekind proved that if p does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$, then $p\mathbb{Z}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are distinct prime ideals of $\mathbb{Z}_K$ with $\mathfrak{p}_i = p\mathbb{Z}_K + \phi_i(\alpha)\mathbb{Z}_K$ having residual degree equal to $\deg \phi_i$ (See [2, Theorem 4.8.13]). Dedekind also gave a criterion to test whether p divides the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ (see [2, Theorem 6.1.4], [7]). In 1894,

---

2020 Mathematics Subject Classification: 12E25,12E99, 13A18.
Keywords: extensions of non-archimedean absolute value, Newton polygon, residual polynomial.

Hensel developed a powerful approach by showing that the prime ideals of $\mathbb{Z}_K$ lying over p are in one-to-one correspondence with the monic irreducible factors of F over the field $\mathbb{Q}_p$ of p-adic numbers and that the ramification index together with the residual degree of a prime ideal of $\mathbb{Z}_K$ lying over p are the same as those of a simple extension of $\mathbb{Q}_p$ obtained by adjoining a root of the corresponding irreducible factor of F belonging to $\mathbb{Q}_p[x]$. Keeping in view Hensel's result in 1928, Ore [13] introduced a new technique which generalizes Dedekind's criterion; Namely, Newton polygon techniques which enables us to get the factorization of $p\mathbb{Z}_K$. By virtue of Hensel's Lemma, the factorization of $\overline{F}$ in $\mathbb{F}_p[x]$ leads to a factorization $F = F_1 \cdots F_r$ over the ring $\mathbb{Z}_p$ of p-adic integers with $\overline{F}_i = \overline{\phi}_i^{e_i}$ in $\mathbb{F}_p[x]$. For this purpose, he considered the ϕ_i-Newton polygon of F_i for each i, having t_i sides with negative slope which leads to a factorization of F_i into t_i factors, say $F_i = F_{i_1} \cdots F_{it_i}$ in $\mathbb{Z}_p[x]$. Moreover, to each side S of slope λ of the ϕ_i-Newton polygon of F_i, he associated a polynomial $R_\lambda(F_i)(y)$ over the finite field $\mathbb{F}_{\phi_i} := \frac{\mathbb{F}_p[x]}{(\overline{\phi}_i)}$ in an indeterminate y. The factorization of the associated polynomial $R_\lambda(F_i)(y)$ over $\mathbb{F}_{\phi_i}$ provides a further factorization of the factor of F_i corresponding to the side S (for more details on Newton polygon see below and [5]). Finally, Ore showed that if for some i, all these polynomials $R_{\lambda_j}(F_i)(y)$ corresponding to various sides S_j, $1 \le j \le t_i$, of the ϕ_i-Newton polygon of F_i have no multiple factor, say $R_{\lambda_j}(F_i)(y)$ splits into n_{ij} distinct irreducible factors over $\mathbb{F}_{\phi_i}$, then all the $\sum_{j=1}^{t_i} n_{ij}$ factors of F_i obtained in this way are irreducible over $\mathbb{Q}_p$. Further, the slopes of the sides of the ϕ_i-Newton polygon of F_i and the degrees of the irreducible factors of $R_\lambda(F_i)(y)$ over $\mathbb{F}_{\phi_i}$ for S ranging over all the sides of such a polygon lead to the explicit determination of the residual degrees and the ramification indices of all those prime ideals of $\mathbb{Z}_K$ lying over p which correspond to the irreducible factors of F_i.

Non-archimedean absolute values are useful in non-archimedean analysis, p-adic differential equations, p-adic series, p-adic analytic number theory, and p-adic analytic geometry ([8], [9], [10], [14], [11]). Several authors studied the extensions of any rank one discrete absolute value. In this paper, our aim is to extend the scope of non-archimedean absolute value when the base field is an arbitrary field K with a discrete non-archimedean absolute value $| \ |$, where $\mathcal{O}$ is the valuation ring of $| \ |$, $\mathcal{M}$ is its maximal ideal, and $\mathbb{F} = \mathcal{O}/\mathcal{M}$ is its residue field. Let $F \in \mathcal{O}[x]$ be a monic irreducible polynomial such that $\overline{F} = \overline{\phi}^l$ in $\mathbb{F}[x]$ for some monic polynomial $\phi \in \mathcal{O}[x]$ whose reduction modulo $\mathcal{M}$ is irreducible, the ϕ-Newton polygon $N_\phi^-(F)$ has a single side of negative slope λ, and the residual polynomial $R_\lambda(F)(y)$ has no multiple factors in $\mathbb{F}_\phi[y]$. The main motivation behind this work is the result given in Bourbaki (see [12, N^o : 7, page: 149, Proposition: 10]) which shows that a non-archimedean absolute value extends, to any Galois extension L of finite degree, in a unique way when the

the base field is complete and non-discrete for $|\ |$. This absolute value is given by

$$|\beta|_L = \left(|N_{L/K}(\beta)|\right)^{\frac{1}{n}}$$

for every $\beta \in L$, where $n = [L : K]$ and $N_{L/K}$ is the norm of L over K. The main goal of this paper is to study the case where K is not necessarily Henselian and L/K is a simple algebraic extension which is not necessarily Galois. Some illustrating examples are also given, too.

2. Preliminaries

Recall that a valued field is the given of a pair $(K, |\ |)$, where K is a field and $|\ |$ is an absolute value of K, that is a mapping $|\ | : K \longrightarrow \mathbb{R}^+$ satisfying the following properties:

(1) $|x| = 0$ if and only if $x = 0$,

(2) $|xy| = |x||y|$,

(3) $|x + y| \leq |x| + |y|$

for every x, y in K.

If the third property is replaced by an ultrametric one, namely; $|x + y| \leq \max\{|x|, |y|\}$, then the absolute value is called non-archimedean.

In this paper, we fix a valued field $(K, |\ |)$ with $|\ |$ a non-archimedean absolute value which we simply call in the rest of the article absolute value. Let L be a field extension of K, and $|\ |_L$ an absolute value of L extending $|\ |$. Consider the sets $\Gamma = |K^*| = \{|x|,\ x \in K^*\}$ and $\Gamma_{|\ |_L} = |L^*|_L = \{|x|_L,\ x \in L^*\}$. These sets are abelian totally ordered groups where Γ is a subgroup of $\Gamma_{|\ |_L}$. The index of Γ in $\Gamma_{|\ |_L}$, denoted $\mathfrak{e}(|\ |_L/|\ |) = (\Gamma_{|\ |_L} : \Gamma)$, is called the ramification index of the extension $|\ |_L$ above $|\ |$. In the same context the residue degree of $|\ |_L$ over $|\ |$ is the degree $[\mathbb{F}_{|\ |_L} : \mathbb{F}]$ denoted by $\mathfrak{f}(|\ |_L/|\ |)$.

Consider also the following sets: $\mathcal{O} = \{x \in K,\ |x| \leq 1\}$, $\mathcal{M} = \{x \in K,\ |x| < 1\}$. It is well Known that that $\mathcal{O}$ is a ring of valuation called the valuation ring of $(K, |\ |)$ and $\mathcal{M}$ is its maximal ideal, hence $\mathbb{F} = \mathcal{O}/\mathcal{M}$ is a field, called the residue field of $(K, |\ |)$. When $\mathcal{M}$ is a principal ideal generated by an element π, the absolute value $|\ |$ is called discrete, and if the Krull dimension of $\mathcal{O}$ is 1, we say that $|\ |$ is of rank one.

Remarks.

(1) Let $|\ | : K \longrightarrow \mathbb{R}^+$ be an absolute value and $\nu : K \longrightarrow \mathbb{R}$ the map defined by

$$\nu(x) = -\ln(|x|) \quad \text{for all } x \in K^*.$$

Then ν satisfies the first 2 axioms of a valuation but not necessarily the third one. We say that ν is a Krull valuation of K if and only if $|\ |$ is non-archimedean absolute value. In this case ν is called the Krull valuation associated to $|\ |$. Moreover, if ν of rank one discrete valuation we say also that $|\ |$ is of rank one discrete absolute value. In treating non-archimedean absolute value $|\ |$, it is convenient to replace $|a|$ by the related "exponential" value $e^{-\nu(a)}$, for every $a \in K$.

(2) Every absolute value $|\ |$ on K induces a topology on K. The completion of $(K, |\ |)$ will be denoted by $(\hat{K}, |\ |)$.

(3) Every rank one valued field $(K, |\ |)$ allows a unique algebraic extension, up to value-preserving isomorphism, that satisfies Hensel's Lemma. This extension is denoted by K^h and called the henselization of the given valued field. Further, K^h is the separable closure of K in the completion $\hat{K}$ with respect to $|\ |$.

Let ν be the discrete Krull valuation associated to $|\ |$, $\mathcal{O}_\nu$ its valuation ring and M_ν its maximal ideal, then $\mathcal{O}_\nu = \mathcal{O}$, $M_\nu = \mathcal{M}$, and $\mathbb{F}_\nu = \mathbb{F}$. By normalization, we can assume that $\nu(K^*) = \mathbb{Z}$, and $\nu(\pi) = 1$. Hence $|K^*| = \{\dots, e^{-2}, e^{-1}, 1, e, e^2, \dots\}$.

Let $(\hat{K}, \hat{\nu})$ be the completion of (K, ν), $\mathcal{O}_{\hat{\nu}}$ its valuation ring and $M_{\hat{\nu}}$ its maximal ideal. It is well known that $\Gamma_{\hat{\nu}} = \Gamma_\nu$, $M_{\hat{\nu}}$ is a principal ideal of $\mathcal{O}_{\hat{\nu}}$ generated by π, and $\mathbb{F}_{\hat{\nu}} \simeq \mathbb{F}_\nu$. Denote also by $\hat{\nu}$ the Gauss's extension of ν to the field $\hat{K}(x)$ defined by

$$\hat{\nu}(P) = \min\{\hat{\nu}(a_i),\ i = 0, \dots, n\}$$

for every polynomial $P = \sum_{i=0}^n a_i x^i \in \hat{K}[x]$, and extend $\hat{\nu}$ to $\hat{K}(x)^*$ by $\hat{\nu}(A/B) = \hat{\nu}(A) - \hat{\nu}(B)$ for every $(A, B) \in \hat{K}[x] \times \hat{K}[x]^*$. The corresponding absolute value of the Gauss's valuation is called the infinite absolute value and it is defined by

$$|\ |_\infty : \hat{K}[x] \longrightarrow [0, +\infty[,$$

$$P = \sum_{i=0}^{n} p_i x^i \mapsto |P|_\infty = \max\{|p_i|,\ i = 0, \dots, n\}$$

and extend $|\ |_\infty$ to $hatK(x)^*$ by

$$\left|\frac{P}{Q}\right|_\infty = \frac{|P|_\infty}{|Q|_\infty}, \quad \text{for every} \quad (P, Q) \in \hat{K}[x] \times \hat{K}[x]^*.$$

Let $\phi \in \mathcal{O}_{\hat{\nu}}[x]$ be a monic polynomial whose reduction $\overline{\phi}$ modulo $M_{\hat{\nu}}$ is irreducible. Let $\mathbb{F}_\phi = \mathcal{O}_\nu[x]/(\pi, \phi) \cong \mathbb{F}_\nu[x]/(\overline{\phi})$ be the associated residue field. For every polynomial $P \in \mathcal{O}_{\hat{\nu}}[x]$, let $P = p_n\phi^n + p_{n-1}\phi^{n-1} + \cdots + p_1\phi + p_0$ be the ϕ-expansion of P. This is reached by the Euclidean division of P by successive powers of ϕ. So $p_i \in \mathcal{O}_{\hat{\nu}}[x]$ with $\deg(p_i) < \deg(\phi)$ for $i = 0, \dots, n$.

If $p_n \neq 0$, then the integer n is called the ϕ-degree of P. The ϕ-Newton polygon $N_\phi(P)$ of P with respect to the valuation ν is the polygonal path consisting of the lower edges of positive lengths $S_1, \dots, S_t$ of the convex hull of the set of points $(i, \nu(p_i))$ in the Euclidean plane with $\nu(a_i) < \infty$, $i = 0, \dots, n$, where the edges S_j are ordered by increasing slopes. We call each edge a side of $N_\phi(P)$ and write $N_\phi(P) = S_1 + \cdots + S_t$. For every $j = 1, \dots, t$, let l_j be the length of the projection of S_j on the x-axis (which is called *the length of* S_j), $H_j = H(S_j)$ the length of the projection of S_j on the y-axis (which we call *the height of* S_j), and $d_j = d(S_j) = \gcd(l_j, H_j)$ is called *the degree of* S_j. Letting $\mathtt{e}_j = \frac{l_j}{d_j}$ and $\mathtt{h}_j = \frac{H_j}{d_j}$. It follows that $\mathtt{e}_j$ and $\mathtt{h}_j$ are two coprime positive integers and $\lambda_j = -\frac{\mathtt{h}_j}{\mathtt{e}_j}$ is the slope of S_j. The part of $N_\phi(P)$ consisting of the polygon whose sides are those consecutive sides of $N_\phi(P)$ of negative slopes is called the principal ϕ-Newton Polygon of P denote $N_\phi^-(P)$. For every $\lambda \in \mathbb{Q}^-$, we call the largest segment of $N_\phi(P)$ of slope λ the λ-component of P. It is reduced to the end point of S_t if $\lambda > \lambda_j$ for every $j = 1, \dots, t$, to the initial point of S_1 if $\lambda_j > \lambda$ for every $j = 1, \dots, t$, and to the end point of S_{j_*} (which coincides with the initial point of S_{j_*+1}) if $\lambda_{j_*} < \lambda < \lambda_{j_*+1}$, where $j_* = \max\{j = 1, \dots, t-1 \mid \lambda_j < \lambda\}$. Let $\lambda = -\frac{\mathtt{h}}{\mathtt{e}} \in \mathbb{Q}^-$ for some coprime integers $\mathtt{e}$ and $\mathtt{h}$, and S be the λ-component of $N_\phi(P)$. Let (s, u) be the initial point of S, l its length, and $d = \frac{l}{\mathtt{e}}$ its degree. For every $i = 0, \dots, l$, define the residue coefficient $t_i \in \mathbb{F}_\phi$ associated to S, by $t_i = 0$ if $\big(i, \nu(p_i)\big)$ lies strictly above S, and $t_i = \big(\frac{p_i}{\pi^{\nu(p_i)}}\big)$ if $\big(i, \nu(p_i)\big)$ lies on S. Remark that the only points of integer coordinates are $(s, u), (s + \mathtt{e}, u - \mathtt{h}), \dots, (s + d\mathtt{e}, u - d\mathtt{h})$. We attach to S the residual polynomial $R_\lambda(P)(y) \in \mathbb{F}_\phi[y]$ defined by $R_\lambda(P)(y) = \sum_{i=0}^d c_i y^i$ with $c_i = t_{s+ie}$, for every $i = 0, \dots, d$. For more details we refer to [6] for Newton Polygon $\mathbb{Z}_p$ and [5] for rank one discrete valuation.

The following Theorem plays a key role to prove our main results. It establishes a one-to-one correspondence between extensions of $|\ |$ to L and the irreducible factors of F in $\hat{K}[x]$. In particular, if $(K, |\ |)$ is a complete field then there is a unique extension of $|\ |$ to any algebraic extension of K.

Theorem 2.1 ([3], Theorem 2.1). *Let $L = K(\alpha)$ be a simple extension generated by a root $\alpha \in \overline{K}$ of a monic irreducible polynomial $F \in K[x]$, and let $F = \prod_{i=1}^t F_i^{l_i}$ be the factorization into powers of monic irreducible factors in $K^h[x]$. Then $l_i = 1$ for every $i = 1, \dots, t$ and there are exactly t distinct absolute values $|\ |_1, \dots,$ and $|\ |_t$ of L extending $|\ |$. Furthermore, for every absolute value $|\ |_i$ of L associated to the irreducible factor F_i, we have*

$$|P(\alpha)|_i = |P(\alpha_i)|_{\overline{K^h}},$$

where $|\ |_{\overline{K^h}}$ is the unique absolute value of an algebraic closure $\overline{K^h}$ of K^h extending $|\ |$ and $\alpha_i \in \overline{K}$ is a root of F_i.

3. Main results

Lemma 3.1. *Let $(K, |\ |)$ be a valued field provided by a discrete non-archimedean absolute value $|\ |$. Let $F \in \mathcal{O}[x]$ be a monic irreducible polynomial. Assume that $\overline{F} = \overline{\phi}^l \in \mathbb{F}[x]$ for some monic polynomial $\phi \in \mathcal{O}[x]$ whose reduction is irreducible in $\mathbb{F}[x]$ and for some natural integer l, $N_\phi^-(F) = S$ has a single side of a negative finite slope λ, and $R_\lambda(F)(y)$ is a power of a monic irreducible polynomial $\psi(y)$ in $\mathbb{F}_\phi[y]$. Let $\gamma = \frac{\phi(\alpha)^{\mathtt{e}}}{\pi^{\mathtt{h}}}$, where $\lambda = -\dfrac{\mathtt{h}}{\mathtt{e}}$ for some coprime integers $\mathtt{e}$ and $\mathtt{h}$. Then the polynomial $\psi(y)$ is the minimal polynomial of the element $\overline{\gamma}$ over $\mathbb{F}_\phi$.*

Theorem 3.2. *Under the hypotheses of Lemma 3.1. Let $|\ |_L$ be an absolute value of L extending $|\ |$, then*

$$|P(\alpha)|_L \le \max\{|p_i(x)|_\infty \cdot e^{i\lambda}, \quad i = 0, \ldots, n\} \tag{1}$$

for any polynomial $P = \sum_{i=0}^n p_i\phi^i \in \mathcal{O}[x]$ with $\deg(p_i) < \deg(\phi)$ for every $i = 0, \ldots, n$. The equality holds if and only if $\psi(y)$ does not divide $R_\lambda(P)(y)$.

Corollary 3.3. *Under the above hypotheses of Theorem 3.2 assume that $R_\lambda(F)(y)$ is a monic irreducible polynomial of $\mathbb{F}_\phi[y]$, then there is a unique absolute value $|\ |_L$ of L extending $|\ |$ such that*

$$|P(\alpha)|_L = \max\{|p_i(x)|_\infty \cdot e^{i\lambda}, \quad i = 0, \ldots, n\} \tag{2}$$

for every polynomial $P = \sum_{i=0}^n p_i\phi^i \in K[x]$ such that $\deg p_i < \deg \phi$ and $\deg P < \deg F$.

Theorem 3.4. *Under the hypotheses of the Theorem 3.2, we have the following*

(1) *For every absolute value $|\ |_L$ of L extending $|\ |$, $\mathtt{e}$ divides the ramification index $\mathfrak{e}(|\ |_L/|\ |)$ and $m = \deg \phi$ divides the residue degree $\mathfrak{f}(|\ |_L/|\ |)$.*
(2) *If $R_\lambda(F)(y)$ is irreducible over $\mathbb{F}_\phi$, then the ramification index and the residual degree of the absolute value $|\ |_L$ of L extending $|\ |$ satisfy : $\mathfrak{e}(|\ |_L/|\ |) = \mathtt{e}$ and $\mathfrak{f}(|\ |_L/|\ |) = d \cdot m$ where $d = \frac{l}{\mathtt{e}}$.*

Corollary 3.5. *Under the assumptions of Theorem 3.2, assume that $R_\lambda(F)(y) = \prod_{i=1}^t \psi_i(y)$ is the factorization of $R_\lambda(F)(y)$ into a product of distinct irreducible polynomials $\psi_i(y)$ in $\mathbb{F}_\phi[y]$. Then there exists exactly t absolute values $|\ |_1, \ldots, |\ |_t$ of L extending $|\ |$. Moreover,*

(i) *For every polynomial P with ϕ-expansion $\sum_{j=0}^n p_j\phi^j \in K[x]$ such that $\psi_i(y)$ does not divide $R_\lambda\left(\frac{P}{\pi^{\nu(P)}}\right)(y)$, we have*

$$|P(\alpha)|_i = \max\left\{|p_j(x)|_\infty \cdot e^{j\lambda}, \quad j = 0, \ldots, n\right\}. \tag{3}$$

(ii) $\mathfrak{e}(|\ |_i/|\ |) = \mathtt{e}$ *and* $\mathfrak{f}(|\ |_i/|\ |) = \deg \psi_i \cdot \deg \phi$ *for every* $i = 1, \ldots, t$.

4. Proofs of the main results

Proof of Lemma 3.1. By [4, Lemma 3.6] the homomorphism $\mathcal{O}[x] \hookrightarrow \mathbb{F}_{|\ |_L}$ defined by $P \mapsto \overline{P(\alpha)}$ induces the following injective homomorphism $\mathbb{F}_\phi \hookrightarrow \mathbb{F}_{|\ |_L}$ defined by $\overline{P} \mapsto \overline{P(\alpha)}$. Hence, $\mathbb{F}_\phi$ is identified with a subfield of $\mathbb{F}_{|\ |_L}$, and so we can say that any residual polynomial has its residual coefficients in the field $\mathbb{F}_{|\ |_L}$.

Let $F = \sum_{i=0}^{l} a_i\phi^i \in \mathcal{O}[x]$ be the ϕ-expansion of the polynomial F, then

$$\sum_{i=0}^{l} a_i(\alpha)\phi(\alpha)^i = 0.$$

So

$$\sum_{(i,\nu(a_i))\in S} a_i(\alpha)\phi(\alpha)^i \quad + \sum_{(i,\nu(a_i))\ \text{above}\ S} a_i(\alpha)\phi(\alpha)^i = 0$$

(see Figure 1).

The unique points with integer coordinates are $\big(0,\nu(a_0)\big), (\mathtt{e}, \nu(a_0)-\mathtt{h}), \ldots$ $\ldots, (d\mathtt{e} = l, 0)$. Then

$$\sum_{\substack{i=0\\ \mathtt{e}\ \text{divides}\ i}}^{l} a_i(\alpha)\phi(\alpha)^i \quad + \sum_{\substack{i=0\\ \mathtt{e}\ \text{does not divide}\ i}}^{l} a_i(\alpha)\phi(\alpha)^i = 0.$$

As $d = \frac{l}{\mathtt{e}}$, we get

$$\sum_{i=0}^{d} a_{i\mathtt{e}}(\alpha)(\phi(\alpha)^{\mathtt{e}})^i \quad + \sum_{\substack{i=1\\ i\notin \mathtt{e}\cdot\mathbb{N}}}^{l-1} a_i(\alpha)\phi(\alpha)^i = 0.$$

By factoring by $\pi^{d\mathtt{h}}$, we get

$$\sum_{i=0}^{d} \frac{a_{i\mathtt{e}}(\alpha)}{\pi^{(d-i)\mathtt{h}}}\left(\frac{\phi(\alpha)^{\mathtt{e}}}{\pi^{\mathtt{h}}}\right)^i \quad + \sum_{\substack{i=1\\ i\notin \mathtt{e}\cdot\mathbb{N}}}^{l-1} \frac{a_i(\alpha)\phi(\alpha)^i}{\pi^{d\mathtt{h}}} = 0 \tag{4}$$

Since $N_\phi^-(F) = S$ is a single side of slope $\lambda = -\frac{\mathtt{h}}{\mathtt{e}}$, $\nu\big(a_{ie}(x)\big) \ge \lambda(i\mathtt{e}-l)$ and so $|a_{i\mathtt{e}}(x)|_\infty \le e^{\lambda(l-ie)}$, thus $|a_{ie}(\alpha)|_L \le e^{\lambda(l-i\mathtt{e})}$. Hence $\left|\frac{a_{i\mathtt{e}}(\alpha)}{\pi^{(d-i)\mathtt{h}}}\left(\frac{\phi(\alpha)^{\mathtt{e}}}{\pi^{\mathtt{h}}}\right)^i\right|_L \le 1$. Therefore

$$\frac{a_{i\mathtt{e}}(\alpha)}{\pi^{(d-i)\mathtt{h}}}\left(\frac{\phi(\alpha)^{\mathtt{e}}}{\pi^{\mathtt{h}}}\right)^i \in \mathcal{O}_{|\ |_L}, \tag{5}$$

for every $i = 0, \ldots, d$. If $\mathtt{e}$ does not divide i, then $\nu(a_i) > \lambda(i-l)$ and so $|a_i(x)|_\infty < e^{\lambda(l-i)}$. Thus $\left|\frac{a_i(\alpha)\phi(\alpha)^i}{\pi^{d\mathtt{h}}}\right|_L < 1$. Therefore

$$\frac{a_i(\alpha)\phi(\alpha)^i}{\pi^{d\mathtt{h}}} \in M_{|\ |_L}, \tag{6}$$

From equations (4), (5), and (6), one deduces that

$$\sum_{i=0}^{d} \frac{a_{i\mathsf{e}}(\alpha)}{\pi^{(d-i)\mathtt{h}}} \left(\frac{\phi(\alpha)^{\mathsf{e}}}{\pi^{\mathtt{h}}}\right)^{i} = 0 \pmod{M_{|\ |_L}}.$$

Hence $R_\lambda(F)\left(\overline{\gamma}\right) = 0$, and so $\psi\left(\overline{\gamma}\right) = 0$.

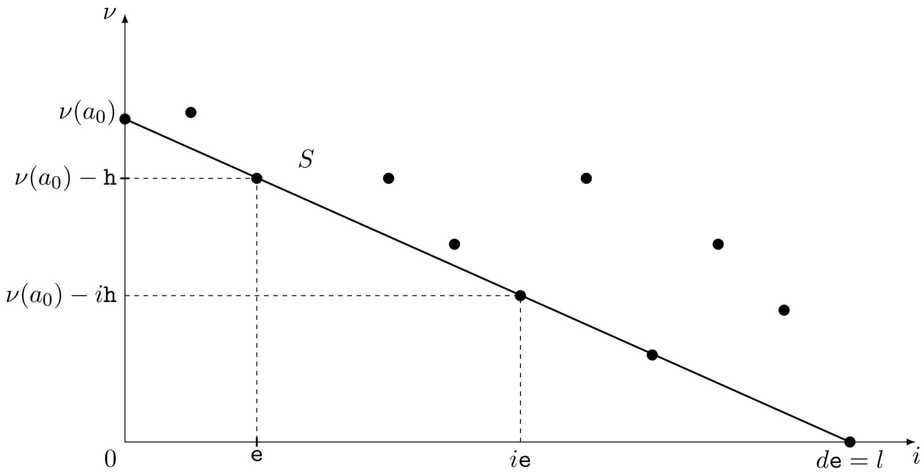

Figure 1. ϕ-Newton polygon of F.

□

P r o o f o f T h e o r e m 3.2. As $P = \sum_{j=0}^{n} p_j \phi^j \in \mathcal{O}[x]$, we have

$$|P(\alpha)|_L \leq \max\left\{|p_j(\alpha)|_L \cdot |\phi(\alpha)|_L^j,\ j = 0, \ldots, n\right\}.$$

By [4, Lemma 3.6] and [4, Theorem 3.10], we have

$$|P(\alpha)|_L \leq \max\left\{|p_j(x)|_\infty \cdot e^{j\lambda},\ \ j = 0, \ldots, n\right\}. \tag{7}$$

Now, suppose that $|P(\alpha)|_L < \max\left\{|p_j(x)|_\infty \cdot e^{j\lambda},\ j = 0, \ldots, n\right\}$, and show that $R_\lambda(P)(\overline{\gamma}) = 0$ where $\gamma = \frac{\phi(\alpha)^{\mathsf{e}}}{\pi^{\mathtt{h}}}$. Let (s, u) be the initial point of the λ--component T of the ϕ-Newton polygon of P, $t = l(T)$ its length, $\delta = d(T) = \frac{t}{\mathsf{e}}$ its degree, and $(s + \delta\mathtt{h}, u - \delta\mathtt{h})$ its end point (see Figure 2). Then the polynomial P can be written as: $P = Q + R$ such that

$$Q = \sum_{(j,\nu(p_j))\in T} p_j\phi^j \qquad \text{and} \qquad R = \sum_{(j,\nu(p_j))\ \text{above}\ T} p_j\phi^j.$$

It is well-known that the side T is the set of points $(a, b) \in N_\phi^-(P)$ in the Euclidean plane such that $b - \lambda a$ is minimal. Since (s, u) lies on T, then

$\min\{\nu(p_j) - j\lambda \ / \ j = 0,\ldots,n\} = u - s\lambda$, and so $\max\{|p_j(x)|_\infty \cdot e^{j\lambda} \ / \ j = 0,\ldots,n\} = e^{-u+s\lambda}$.

If $\big(j,\nu(p_j)\big)$ lies on T, then $\nu(p_j) - j\lambda = u - s\lambda$, thus

$$|Q(\alpha)|_L \leq e^{-u+s\lambda}.$$

If $\big(j,\nu(p_j)\big)$ lies strictly above T, then $\nu(p_j) - j\lambda > u - s\lambda$, thus

$$|R(\alpha)|_L < e^{-u+s\lambda}.$$

So

$$\left|\sum_{(j,\nu(p_j))\in T} p_j(\alpha)\phi(\alpha)^j\right|_L < e^{-u+s\lambda}. \tag{8}$$

If $\big(j,\nu(p_j)\big)$ lies on T, then $j = s + i\mathtt{e}$ for some $i = 0,\ldots,\delta$, and so $\nu(p_{s+i\mathtt{e}}) = u - i\mathtt{h}$. Therefore

$$\left|\sum_{i=0}^{\delta} p_{s+i\mathtt{e}}(\alpha)\phi(\alpha)^{s+i\mathtt{e}}\right|_L < e^{-u+s\lambda}.$$

Hence

$$|\pi^u \cdot \phi(\alpha)^s|_L \cdot \left|\sum_{i=0}^{\delta} \frac{p_{s+i\mathtt{e}}(\alpha)}{\pi^u}(\phi(\alpha)^{\mathtt{e}})^i\right|_L < e^{-u+s\lambda}.$$

Since

$$|\pi^u.\phi(\alpha)^s|_L = e^{-u+s\lambda}$$

by [4, Theorem 3.10], we conclude that

$$\left|\sum_{i=0}^{\delta} \frac{p_{s+i\mathtt{e}}(\alpha)}{\pi^{u-i\mathtt{h}}}\gamma^i\right|_L < 1.$$

Thus

$$\sum_{i=0}^{\delta}\left(\overline{\frac{p_{s+i\mathtt{e}}}{\pi^{u-i\mathtt{h}}}}\right)(\overline{\gamma})^i = 0 \pmod{\mathcal{M}}.$$

So $R_\lambda(P)\,(\overline{\gamma}) = 0$. Hence by Lemma 3.1 $\psi(y)$ divides $R_\lambda(P)(y)$ in $\mathbb{F}_\phi[y]$. Therefore, one deduces that when $\psi(y)$ does not divide $R_\lambda(P)(y)$ in $\mathbb{F}_\phi[y]$, then $|P(\alpha)|_L = \max\big\{|p_j(x)|_\infty \cdot e^{j\lambda},\ j = 0,\ldots,n\big\}$. If $\psi(y)$ divides $R_\lambda(P)(y)$ in $\mathbb{F}_\phi[y]$, then $R_\lambda(P)\,(\overline{\gamma}) = 0$. By the same above process we conclude that

$$|Q(\alpha)|_L < e^{-u+s\lambda} \quad \text{and so} \quad |P(\alpha)|_L < \max\big\{|p_j(x)|_\infty \cdot e^{j\lambda},\ j = 0,\ldots,n\big\}.$$

From where the equivalence.

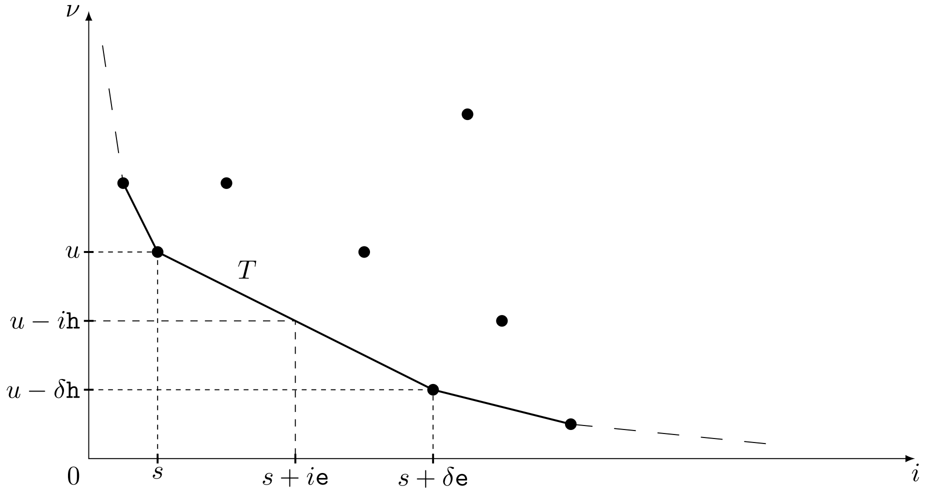

FIGURE 2. ϕ-Newton polygon of P.

□

Proof of Corollary 3.3. Let $|\ |_L$ be an absolute value of L extending $|\ |$, and let $P \in K[x]$ be a polynomial of degree less than the degree of F. The ϕ-expansion of F has the following form $F = a_0 + a_1\phi + \cdots + \phi^l \in \mathcal{O}[x]$. Let $P = b_0 + \cdots + b_t\phi^t \in K[x]$ be the ϕ-expansion of P with $t < l$. Let $P_0 = \frac{P}{\pi^{\nu(P)}} \in \mathcal{O}[x]$. Then $\deg R_\lambda(P_0)(y) < \deg R_\lambda(F)(y)$ and so $R_\lambda(F)(y)$ does not divide $R_\lambda(P_0)(y)$ in $\mathbb{F}_\phi[y]$. According to Theorem 3.2, we have

$$|P_0(\alpha)|_L = \max\left\{\left|\frac{p_i(x)}{\pi^{\nu(P)}}\right|_\infty \cdot e^{i\lambda},\ i = 0, \ldots, n\right\}.$$

So

$$|P(\alpha)|_L = \max\left\{|p_i(x)|_\infty \cdot e^{i\lambda},\ i = 0, \ldots, n\right\}. \tag{9}$$

Let $|\ |_1$ and $|\ |_2$ be two absolute values of L extending $|\ |$, then $|\ |_1$ and $|\ |_2$ have the same expression (9) on L. Therefore, there is a unique absolute value $|\ |_L$ of L extending $|\ |$. □

Proof of Theorem 3.4.

(1) By [4, Theorem 3.10] we have $|\phi(\alpha)|_L = e^\lambda$. Then $\Gamma \subset \Gamma[e^\lambda] \subset \Gamma_{|\ |_L}$. Thus $(\Gamma[e^\lambda] : \Gamma)$ divides $(\Gamma_{|\ |_L} : \Gamma)$. On the other hand, $\mathtt{e}$ is the smallest integer such that $(e^\lambda)^{\mathtt{e}} = e^{-h} \in \Gamma$. So $\Gamma[e^{-\lambda}]/\Gamma$ is a torsion group of order $\mathtt{e}$. Therefore $\mathtt{e}$ divides $\mathfrak{e}(|\ |_L/|\ |)$. In the beginning of the proof of Lemma 3.1, we showed that $\mathbb{F}_\phi$ is a subfield of $\mathbb{F}_{|\ |_L}$. Then $\mathbb{F} \subset \mathbb{F}_\phi \subset \mathbb{F}_{|\ |_L}$. Then $[\mathbb{F}_\phi : \mathbb{F}]$ divides $[\mathbb{F}_{|\ |_L} : \mathbb{F}]$. As $[\mathbb{F}_\phi : \mathbb{F}] = \deg\phi = m$, m divides $\mathfrak{f}(|\ |_L/|\ |)$.

(2)(a) We show that $(\Gamma_{|\ |_L} : \Gamma) = \mathsf{e}$. Let $\mathbb{N}_{\mathsf{e}-1} = \{0, \ldots, \mathsf{e}-1\}$ and consider the mapping $\sigma : \mathbb{N}_{\mathsf{e}-1} \to \Gamma_{|\ |_L}/\Gamma$ defined by $i \mapsto \left(\overline{e^{i\lambda}}\right)$.
Let us show that σ is injective. Let $i, j \in \mathbb{N}_{\mathsf{e}-1}$ such that

$$\overline{e^{-i\frac{\mathsf{h}}{\mathsf{e}}}} = \overline{e^{-j\frac{\mathsf{h}}{\mathsf{e}}}} \pmod{\Gamma}.$$

Then $\overline{e^{(j-i)\frac{\mathsf{h}}{\mathsf{e}}}} = 1 \pmod{\Gamma}$, thus e divides $j-i$ and so $i = j$.
Let us show that σ is surjective. Let $\mu \in \Gamma_{|\ |_L}/\Gamma$. Then there exists a polynomial $P \in K[x]$ with $\deg P < \deg F$ such that $\mu = |P(\alpha)|_L \cdot \Gamma$, then $\deg R_\lambda(P)(y) < \deg R_\lambda(F)(y)$ and so $\psi(y)$ does not divide $R_\lambda(F)(y)$. Let $\sum_{i=0}^n p_i\phi^i$ be the ϕ-expansion of P. By Theorem 3.2, there exists an integer $i_0 = 0, \ldots, n$ such that

$$\begin{aligned} P(\alpha)|_L &= \max\{|p_i(x)|_\infty \cdot e^{i\lambda},\ i = 0, \ldots, n\} \\ &= e^{-u_{i_0}} e^{i_0\lambda} = e^{-u_{i_0} - i_0\frac{\mathsf{h}}{\mathsf{e}}}, \end{aligned}$$

where $u_{i_0} = \nu(p_{i_0}) \in \mathbb{Z}$. By the Euclidean division, there exists a unique pair (q, j) of integers such that $i_0 = q\mathsf{e} + j$ with $0 \le j \le \mathsf{e}-1$, then

$$|P(\alpha)|_L = e^{-j\frac{\mathsf{h}}{\mathsf{e}}} \cdot e^{-u_{i_0} - qh} \in e^{-j\frac{\mathsf{h}}{\mathsf{e}}} \cdot \Gamma.$$

Thus there exists an element $j \in \mathbb{N}_{\mathsf{e}-1}$ such that $\sigma(j) = \mu$ and so σ is a one-to-one correspondence between $\mathbb{N}_{\mathsf{e}-1}$ and $\Gamma_{|\ |_L}/\Gamma$. Therefore

$$(\Gamma_{|\ |_L} : \Gamma) = \mathsf{e}.$$

(b) We show that $\mathfrak{f}(|\ |_L/|\ |) = md$. Since $[\mathbb{F}_{|\ |_L} : \mathbb{F}] = [\mathbb{F}_{|\ |_L} : \mathbb{F}_\phi][\mathbb{F}_\phi : \mathbb{F}]$, and $\mathbb{F}_\phi = \frac{\mathbb{F}[x]}{(\overline{\phi})}$, we get $\quad [\mathbb{F}_\phi : \mathbb{F}] = \deg\phi = m.$

Then it remains to show that $[\mathbb{F}_{|\ |_L} : \mathbb{F}_\phi] = d$. Consider the following ring homomorphism

$$\tau : \mathbb{F}_\phi[y] \to \mathbb{F}_{|\ |_L}$$

defined by

$$Q(y) \mapsto Q(\overline{\gamma}) \quad \text{with} \quad \gamma = \frac{\phi(\alpha)^{\mathsf{e}}}{\pi^{\mathsf{h}}} \in \mathcal{O}^*_{|\ |_L}.$$

Then:

(i) We claim that $\ker\tau = (R_\lambda(F)(y))$. By Lemma 3.1, $R_\lambda(F)(y)$ is the minimal polynomial of $\overline{\gamma}$ over $\mathbb{F}_\phi$. Thus we conclude that $\ker\tau$ is the principal ideal of $\mathbb{F}_\phi[y]$ generated by $R_\lambda(F)(y)$.

(ii) We show that τ is a surjective. Let ω be an element of $\mathbb{F}_{|\ |_L}$, then $\omega = \overline{P(\alpha)}$ for some polynomial $P \in K[x]$ with $\deg P < \deg F$ and $|P(\alpha)|_L \le 1$. If $|P(\alpha)|_L < 1$ (i.e, $\omega = 0$), then by Lemma 3.1, we have $R_\lambda(F)(\overline{\gamma}) = 0$. Hence $\omega = \tau\big(R_\lambda(F)(y)\big)$. If $|P(\alpha)|_L = 1$ (i.e; $\omega \ne \overline{0}$), then $|P(\alpha)|_L = 1$. Let $\sum_{i=0}^n p_i\phi^i$ be the ϕ-expansion of P, $P_0 = \frac{P}{\pi^{\nu(P)}} \in \mathcal{O}[x]$, and $N_\phi^-(P_0)$ the ϕ-Newton polygon of P_0.

Let (s,u) be the initial point of T; the λ-component of $N_\phi^-(P_0)$, $t=l(T)$ its x-length, and let $\delta = d(T) = \frac{t}{\mathtt{e}}$ be its degree. As we have seen in the proof of the Theorem 3.2, we have

$$P(\alpha)=\pi^{u+\nu(P)}\phi(\alpha)^s\left[\sum_{(j,\nu(p_{0j}))\in T}\frac{p_{0j}(\alpha)\phi(\alpha)^j}{\pi^u\cdot\phi(\alpha)^s} + \sum_{(j,\nu(p_{0j}))\text{ above }T}\frac{p_{0j}(\alpha)\phi(\alpha)^j}{\pi^u\cdot\phi(\alpha)^s}\right] \text{ in } \mathcal{O}_{|\ |_L},$$

with $p_{0j}=\frac{p_j}{\pi^{\nu(P)}}\in\mathcal{O}[x]$, for every $j=0,\dots,n$.
If $\big(j,\nu(p_{0j})\big)$ lies above T, then $\left|\frac{p_{0j}(\alpha)\phi(\alpha)^j}{\pi^u\cdot\phi(\alpha)^s}\right|_L<1$. Then

$$\frac{p_{0j}(\alpha)\phi(\alpha)^j}{b\cdot\pi^u\cdot\phi(\alpha)^s}\equiv 0 \pmod{M_{|\ |_L}},$$

and so

$$P(\alpha)\equiv\pi^{u+\nu(P)}\cdot\phi(\alpha)^s\left[\sum_{(j,\nu(p_{0j}))\in T}\frac{p_{0j}(\alpha)\phi(\alpha)^j}{\pi^u\cdot\phi(\alpha)^s}\right] \pmod{M_{|\ |_L}}.$$

As $\big(j,\nu(p_{0j})\big)$ lies on T, there is a unique $i=0,\dots,\delta$ such that $j=s+i\mathtt{e}$ and $\nu(p_{0s+i\mathtt{e}})=u-i\mathtt{h}$. Thus

$$P(\alpha)\equiv\pi^{u+\nu(P)}\cdot\phi(\alpha)^s\left[\sum_{i=0}^{\delta}\left(\frac{p_{0s+i\mathtt{e}}(\alpha)}{\pi^{u-i\mathtt{h}}}\right)\left(\frac{\phi(\alpha)^{\mathtt{e}}}{\pi^{\mathtt{h}}}\right)^i\right] \pmod{M_{|\ |_L}}.$$

On the other hand, since the point (s,u) lies on T, by Theorem 3.2 we have $|P(\alpha)|_L=e^{-u-\nu(P)}\cdot e^{s\lambda}$. Since $|P(\alpha)|_L=1$, $e^{-\nu(P)-u-s\frac{\mathtt{h}}{\mathtt{e}}}=1$. Then $\nu(P)+u+s\frac{\mathtt{h}}{\mathtt{e}}=0$, thus $\mathtt{e}$ divides s, and $s=\mathtt{e}a$ for some rational integer a. Hence $\pi^{u+\nu(P)}\cdot\phi(\alpha)^s=\pi^{u+\nu(P)+a\mathtt{h}}\cdot\gamma^a=c\cdot\gamma^a$, where $c=\pi^{u+\nu(P)+a\mathtt{h}}\in\mathbb{F}^*$ because $|c|=e^{-\nu(P)-u-a\mathtt{h}}=1$. Then

$$P(\alpha)\equiv c\cdot\gamma^a\sum_{i=0}^{\delta}\left(\frac{p_{0s+i\mathtt{e}}(\alpha)}{\pi^{u-i\mathtt{h}}}\right)\gamma^i \pmod{M_{|\ |_L}}.$$

Thus, there exists a polynomial $Q(y)=cy^aR_\lambda(P_0)(y)\in\mathbb{F}_\phi[y]$ such that $\overline{P(\alpha)}=Q(\overline{\gamma})$ in $\mathbb{F}_{|\ |_L}$. Therefore, τ is a surjective ring homomorphism. As $R_\lambda(F)(y)$ is the minimal polynomial of $\overline{\gamma}$ over $\mathbb{F}_\phi$, we get

$$\mathbb{F}_{|\ |_L}\cong\frac{\mathbb{F}_\phi[y]}{(R_\lambda(F)(y))}.$$

Thus

$$[\mathbb{F}_{|\ |_L}:\mathbb{F}_\phi]=\deg R_\lambda(F)(y)=d \quad\text{and}\quad \mathfrak{f}(|\ |_L/|\ |)=md.$$

□

Proof of Corollary 3.5.

(1) Let $F = F_1 \cdots F_t$ be the factorization of the polynomial F in $K^h[x]$. For every $i = 1, \dots, t$, we have by [5, Theorem 3.7] that $\overline{F_i} = \overline{\phi}^{l_i} \in \mathbb{F}_\phi[x]$ for some integer $l_i \geq 1$, $N_\phi^-(F_i) = S_i$ has a single side of slope λ, and $R_\lambda(F_i)(y) \equiv \psi_i(y) \in \mathbb{F}_\phi[y]$, then F_i is irreducible in $K^h[x]$. Let $\alpha_i \in \overline{K^h}$ be a root of F_i, and $L_i = K^h(\alpha_i)$. Then there is a unique absolute value $|\ |_{L_i}$ of L_i extending $|\ |$. Moreover, by Theorem 3.2,

$$|P(\alpha_i)|_{L_i} = \max\{|p_j(x)|_\infty \cdot e^{j\lambda},\ j = 0, \dots, n\}$$

for every polynomial

$$P = \sum_{j=0}^{n} p_j \phi^j \in K[x] \quad \text{with} \quad \psi_i(y)$$

does not divide

$$R_\lambda\left(\frac{P}{\pi^{\nu(P)}}\right)(y).$$

By Theorem 2.1, there are exactly t absolute values $|\ |_1, \dots, |\ |_t$, of $L = K(\alpha)$ extending $|\ |$ such that

$$|P(\alpha)|_i = |P(\alpha_i)|_{\overline{K^h}} = |P(\alpha_i)|_{L_i}$$

for every such a polynomial P.

(2) By Theorem 3.4, for every $i = 1, \dots, t$ we have

$$\mathfrak{e}(|\ |_i/|\ |) = \mathfrak{e}(|\ |_{L_i}/|\ |) = \mathsf{e}$$

and

$$\mathfrak{f}(|\ |_i/|\ |) = \mathfrak{f}(|\ |_{L_i}/|\ |) = \deg \psi_i \cdot \deg \phi. \qquad \square$$

5. Examples

Example 1. Let $(\mathbb{Q}, |\ |_3)$ be the non-archimedean valued field with $|\ |_3$ the 3-adic absolute value associated to the valuation ν_3. Let $F = x^9 + 54x^3 + 45 \in \mathbb{Z}[x]$. Then

$$\overline{F} = \phi^9 \in \mathbb{F}_3[x] \quad \text{with} \quad \phi = x \quad \text{and} \quad N_\phi^-(F) = S$$

has a single side joining the points $(0,2)$ and $(9,0)$ of slope $\lambda = -\frac{2}{9}$ (see Figure 3). Also we have $R_\lambda(F)(y) = 1 + y \in \mathbb{F}_\phi[y]$ is irreducible. Then by [5, Corollary 3.2] F is an irreducible polynomial over $\mathbb{Q}_3$. Let α be a root of F and $L = \mathbb{Q}(\alpha)$. By Corollary 3.3, there exists a unique absolute value of L extending $|\ |_3$ such that

$$\left|a_0 + a_1\alpha + \cdots + a_8\alpha^8\right|_L = \max\left\{|a_0|_3, |a_1|_3 \cdot e^{\frac{-2}{9}}, \dots, |a_8|_3 \cdot e^{\frac{-16}{9}}\right\}$$

for every $a_0, \dots, a_8 \in \mathbb{Q}$.

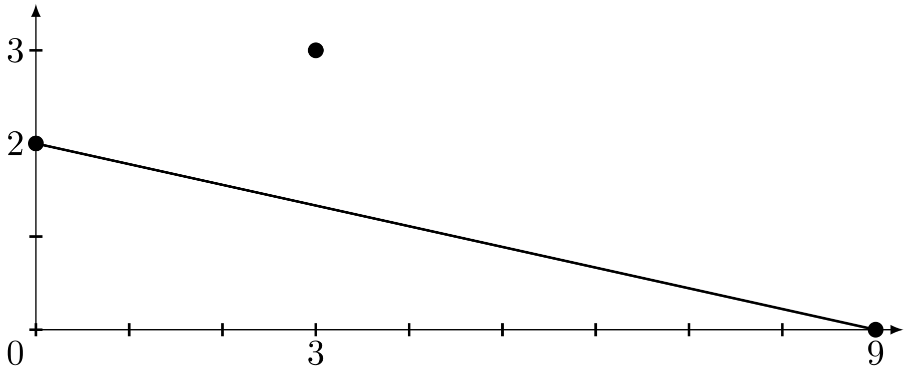

FIGURE 3. ϕ-Newton polygon of F.

EXAMPLE 2. Let $K = \mathbb{F}_3((x))$ be the field of formal power series over $\mathbb{F}_3$. Consider the following non-archimedean absolute value defined

$$|f| = \max\left\{e^{-i},\ x^i \text{ divides } f \text{ in } \mathbb{F}_3[[x]]\right\}.$$

For every $f \in \mathbb{F}_3[[x]]$ and extend $|\ |$ to $\mathbb{F}_3((x))$ by $\left|\frac{f}{g}\right| = \dfrac{|f|}{|g|}$, for every

$$(f,g) \in \mathbb{F}_3[[x]] \times \mathbb{F}_3[[x]]^*.$$

Let

$$F(y) = y^4 + x^2y^2 + x^3y + x^3 \in \mathcal{O}[y],$$

where $\mathcal{O} = \mathbb{F}_3[[x]]$ is the valuation ring of $(K, |\ |)$ with maximal ideal $\mathcal{M} = x \cdot \mathbb{F}_3[[x]]$. Then $F(y) \equiv y^4 \pmod{x}$. Let $\phi = y$, then $N_\phi^-(F) = S$ is a single side of slope $\lambda = \frac{-3}{4}$, joining the points $(0,3)$ and $(4,0)$ (see Figure 4). Since the side S is of degree $\gcd(3,4) = 1$ and

$$R_\lambda(F)(T) = 1 + T \in \mathbb{F}_\phi[T] \cong \mathbb{F}_3[T].$$

By [5, Corollary 3.2] we conclude that $F(y)$ is irreducible over K. Let α be a root of the polynomial $F(y)$. By Corollary 3.3, there is a unique absolute value $|\ |_L$ of $L = K(\alpha)$ extending $|\ |$ such that

$$\left|a_0(x) + a_1(x)\alpha + a_2(x)\alpha^2 + a_3(x)\alpha^3\right|_L = \max\left\{|a_0(x)|, |a_1(x)| \cdot e^{-3/4}, |a_2(x)| \cdot e^{-3/2}, |a_3(x)| \cdot e^{-9/4}\right\}$$

for every $a_0(x), \ldots, a_3(x) \in \mathbb{F}_3((x))$. Moreover,

$$\mathfrak{e}(|\ |_L/|\ |) = 4 \quad \text{and} \quad \mathfrak{f}(|\ |_L/|\ |) = 1.$$

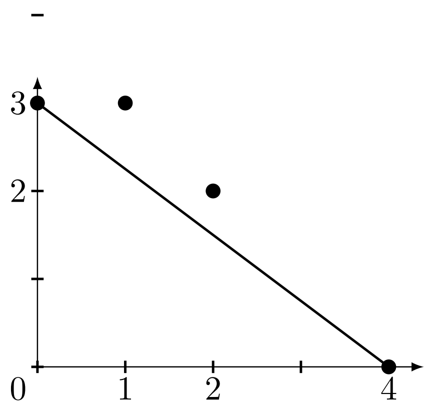

FIGURE 4. ϕ-Newton polygon of F.

Acknowledgement.
The authors are very grateful to Professor Abdulaziz Deajim for his valuable comments and suggestions, which have considerably improved the quality of this paper.

REFERENCES

[1] BROWN, R.: *Roots of generalized Schönemann polynomials in henselian extension fields*, Indian J. Pure Appl. Math. **39** (2008), 403–410.

[2] COHEN, H.: *A Course in Computational Algebraic Number Theory*, Springer-Verlag, Berlin, 1993.

[3] DEAJIM, A.—EL FADIL, L.—NAJIM, A.: *On a theorem of Dedekind* (submitted).

[4] EL FADIL, L.—FARIS, M.: *On the Irreducible Factors of a Polynomial and Applications to Extensions of Absolute Values*, IntechOpen, `https://www.intechopen.com/online-first/on-the-irreducible-factors-of-a-polynomial-and-applications-to-extensions-of-absolute-values`

[5] EL FADIL, L.: *On Newton polygon echniques and factorization of polynomials over Henselian fields*, J. Algebra Appl. **19** (2020), no. 10, doi: 10.1142/S0219498820501881.

[6] GUÀRDIA, J.— MONTES, J.—NART, E.: *Newton polygons of higher order in algebraic number theory*, Trans. Amer. Math. Soc. **364** (2012), no. 1, 361–416.

[7] KHANDUJA, S. K.—KUMAR, M.: *A generalization of Dedekind criterion*, Comm. Algebra **35** (2007), 1479–1486.

[8] MANJRA, S.: *A note on non-Robba p-adic differential equations*, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), no. 3, 40–43.

[9] MANJRA, S.—REMMAL, S. E.: *Equations différentielles p-adiques et Séries Gevrey arithmétiques*, Math. Ann. **334** (2006), 37–64.

[10] MANJRA, S.: *Arithmetic differential equations and E-functions*, Illinois J. Math. **49** (Winter 2005), no. 4, 1061–1092. DOI: 10.1215/ijm/1258138127.

[11] KEDLAYA, K. S.: *p-adic Differential Equations*, Cambridge University Press, New York, 2010.

[12] BOURBAKI, N.: *Algébre Commutative, Chapitres* 5 à 7. Springer-Verlag, Berlin, Heidelberg, 2006.

[13] ORE, O.: *Newtonsche Polygone in der Theorie der algebraischen Körper*, Math. Ann. **99** (1928), 84–117.

[14] MURTY, M. RAM: *Introduction to p-adic Analytic Number Theory*. AMS/IP Stud. Adv. Math. Vol. 27. American Mathematical Society (AMS), Providence, RI: International Press, Cambridge, MA, 2002. `https://doi.org/10.1090/amsip/027`

Received Novembeer 11, 2022

Mohamed Faris
Lhoussain El Fadil
Department of Mathematics
Faculty of Sciences
Sidi Mohamed Ben Abdellah University
Fez, MOROCCO
E-mail: mohamedfaris9293@gmail.com
lhouelfadil2@gmail.com