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BOOK REVIEW

Fečkan, M.: TOPOLOGICAL DEGREE APPROACH TO BIFURCATION PROBLEMS. Topological Fixed Point Theory and its Applications Vol. 5, Springer-Verlag, Heidelberg, 2008, ISBN 978-1-4020--8723-3, e-ISBN 978-1-4020-8724-0.

The theory of bifurcations is one of the most powerful methods used in nonlinear analysis, with applications to the physics, biology, chemistry, medicine and economics, when phenomena are modeled using differential or difference equations with parameters (cf. [1, 4, 6, 8, 7, 9, 10, 11, 13]). Basic in bifurcation methods are topological, analytical and variational approaches. When nonlinearities appearing in investigated bifurcation problems are not enough smooth, topological tools are rather effective to handle those problems.

This work is devoted to the study of such kinds of bifurcation problems, ranging from non-smooth mechanical systems (cf. [2]) through ordinary differential equations on infinite lattices (cf. [15]) up to undamped abstract wave equations on Hilbert spaces with applications to nonlinear beam and string partial differential equations (cf. [11]).

The contents of the book is as follows.

In the introductory Chapter 1, an illustrative example is given by the bifurcation of chaotic solutions in an apparatus of a slender beam clamped to a rigid framework which supports two magnets. The apparatus is periodically forced using electromagnetic vibration generator. Mathematically, this leads to the well-known periodically forced and damped Duffing equation. There is derived a Melnikov bifurcation function which determines chaotic wedge-shaped regions in the parametric space, where the perturbed Duffing equation is chaotic. This example is used to outline main ideas of methods applied in the book, which are based on the perturbation method used in the theory of dynamical systems, then on the decomposition method of Lyapunov-Schmidt and on the theory of topological degrees of Brouwer and Leray-Schauder.

Chapter 2 briefly reviews some known mathematical results from the linear and nonlinear functional analysis, differential topology, the theory of multivalued mappings and dynamical systems, which are applied in proofs of basic results of the book.

The main contributions of the book start from Chapter 3. This chapter studies bifurcations of periodic and subharmonic solutions from either periodic or homoclinic trajectories for several types of differential systems: differential equations with dry frictions, weakly coupled nonlinear oscillators and forced systems with relay hysteresis as well. First, there is proved the existence of infinitely many subharmonics with periods tending to infinity for weakly discontinuous differential equations. This is the first step to find chaos in discontinuous systems. Then singularly perturbed discontinuous systems are studied. Next, among others, classical saddle-node and Poincaré-Andronov bifurcations of periodic solutions (cf. [9, 14]) are extended to nonsmooth differential equations. Many concrete examples illustrate the theory.

Chapter 4 brings chaotic solutions for discontinuous differential equations by extension of the method of Chapter 3. The unperturbed systems are supposed to have either single or manifolds of homoclinic solutions to hyperbolic equilibria. Melnikov type conditions are established to prove the existence of chaos for perturbed systems. Almost-periodically and quasi-periodically forced differential inclusions are also studied. The chapter ends with a review of recent homoclinic bifurcation results for other types of discontinuous differential equations.

Chapter 5 proceeds with the study of chaos for diffeomorphisms when intersections of stable and unstable manifolds of hyperbolic fixed points are only topologically transversal. To handle this problem, again topological degree methods are necessary to be used. The classical Smale-Birkhoff homoclinic theorem (cf. [9]) is extended to the case of the existence of topologically transversal homoclinic and heteroclinic points of diffeomorphisms. The bifurcation of such points is also studied. The rest of this chapter deals with accumulation of periodic points of reversible diffeomorphisms on homoclinic points with extensions of this phenomenon to chains of reversible oscillators. This phenomenon is known as the blue sky catastrophe (cf. [8]).

Chapter 6 continues with investigation of equations on infinite lattices which are spatially discretized partial differential equations (cf. [16]). The persistences of kink traveling waves of partial differential equations under discretization are investigated. The idea is to consider a traveling wave equation of the lattice equation as an evolution equation on some Banach space. Then the known center manifold method (cf. [9, 10, 14]) is applied to lower the dimension of this infinite dimensional evolution equation. Next, a bifurcation result for periodic solutions of certain singularly perturbed ordinary differential equations are derived with an application to the reduced traveling wave equation on the center manifold. The singular parameter is the discretization step size. The both sine — Gordon and Klein-Gordon discretized lattice equations are studied [16]. The chapter is completed with a review of existence results of traveling waves for differential equations on two-dimensional lattices.

Chapter 7 is devoted to the existence of periodics and subharmonics of undamped abstract wave equations. Infinitely many resonant terms known as the problem of small divisors (cf. [12]) appear automaticaly in this study. To avoid these resonances, Diophantine-type inequalities are introduced. Then using analytical and topological arguments, bifurcations of subharmonic solutions from homoclinic ones, and bifurcations of periodic solutions from periodic ones are established with applications to several types of periodically forced nonlinear beam equations. Then weakly nonlinear wave equations are studied, too. The final part of this chapter is devoted to the investigation of those Diophantine-type inequalities using some tools from the number theory (cf. [5]).

The final Chapter 8 studies discontinuous wave equations with infinitely many resonances. First, a topological degree is developed to monotone multi-valued mappings which makes up a combination of the multivalued Browder-Skrypnik topological degree [3, 17] with Mawhin's coincidence index theory [13]. Then this topological degree is used to extend two classical bifurcation results: the Krasnoselskii bifurcation theorem and bifurcations from infinity [1, 6, 13], to multivalued operator equations in Hilbert spaces. These abstract bifurcation theorems are applied to the study of bifurcations of weak forced periodic solutions with large amplitudes of discontinuous undamped semilinear wave equations. This chapter ends up with an outline of a possibility to use methods of this book for show chaos into weakly discontinuous and periodically forced semilinear wave equations.

Results of this book are based on the papers earlier published by the author, and they have not appeared yet in any other book. So the book is original.

To proceed further study and better understanding of the significance of bifurcation theory, the author lists some additional results he has obtained, and adds further specific research topics and open problems. Many other interesting and promising research objects in bifurcation theory are presented as further research topics. The most significant objects would have to develop theory for continuous time dynamical systems. The bifurcation theory provides a theoretical framework to unify many natural phenomena, and allows us to see problems in several mathematical fields from a new perspective. This book will open a way for future discussion and research in this theory and their relationships to other sciences.

Hence the book under review is suitable both for graduate students and researchers with interest in theoretical biology, genetics, and other applications. The text contains a clear, detailed and self-contained exposition of bifurcation theory. While reading this book we learn a large variety of highly interesting ideas and methods and enjoy beautiful results that introduce us into a fast developing and interesting new mathematical area.

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