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INJECTIVE CONTINUOUS IMAGES OF HAMEL BASES

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ABSTRACT. Under the assumption $L = V$ we construct a Hamel bases H_1 and H_2 of $\mathbb{R}$ and a continuous bijection $f\colon H_1 \to \mathbb{R} \setminus H_2$.

1. Notation

We use the abbreviation ND for "nowhere dense set".

We write $\forall_n^\infty$ and $\exists_n^\infty$ to mean "for almost all n" and "there exists infinitely many n", respectively.

The symbol $\mathcal{N}$ denotes the Baire space ω^ω.

We use the standard notation from descriptive set theory, namely Σ_1^1 denotes the collection of analytic subsets of $\mathbb{R}$ (i.e., continuous images of the Baire space $\mathcal{N}$) and Π_1^1 denotes the collection of all coanalytic subsets of $\mathbb{R}$ (i.e., $\Pi_1^1 = \{\mathbb{R} \setminus A\colon A \in \Sigma_1^1\}$).

Suppose that $\mathcal{P}$ is a family of pairwise disjoint sets. Denote by $\mathrm{Sel}(\mathcal{P})$ the collection of all selectors of the family $\mathcal{P}$, i.e., sets $X \subseteq \cup\mathcal{P}$ such that $\forall_{P\in\mathcal{P}} |X \cap P| = 1$. A set $X \subseteq \cup\mathcal{P}$ is called a *partial selector* of $\mathcal{P}$ iff $\forall_{P\in\mathcal{P}} |X \cap P| \leq 1$.

2. Main result

The main theorem of this paper has been motivated by the following unsolved problem:

Problem 2.1 (I. Recław, private communication)**.** Does there exist a Hamel base $H \subseteq \mathbb{R}$ such that H is homeomorphic to $\mathbb{R} \setminus H$?

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Also, we cannot even solve the following weaker problem which solution would lead to the construction of 2-continuous Hamel function (i.e., a solution to the problem from [PR]):

Problem 2.2. Does there exist a Hamel bases H_1 and H_2 of $\mathbb{R}$ and a continuous bijection $f\colon \mathbb{R}\setminus H_1 \to H_2$?

The aim of this paper is to prove that the "reverse case" (under special set-theoretic assumption) holds. Namely, we have:

Theorem 2.3. *Assume $L = V$. Then there exist a Hamel bases H_1, H_2 of $\mathbb{R}$ and a continuous bijection $f\colon H_1 \to \mathbb{R}\setminus H_2$.*

Unfortunately, it does not solve the problem from [PR].

Let us formulate the main lemma on Hamel bases. Notice, that Hamel bases as partitions were considered for example in [B], however, the author considered Hamel bases as selectors of partitions with small (countable) elements. In contrast of this, here we need the large cardinality case:

Lemma 2.4. *Assume CH. Suppose that $\mathcal{P}$ is a family of pairwise disjoint subsets of $\mathbb{R}$ with the following properties:*

(1) $\forall_{P\in\mathcal{P}} |P| = 2^\omega$;

(2) $\forall_{P\in\mathcal{P}} P$ *is nowhere dense*;

(3) $\mathbb{R}\setminus \cup\mathcal{P}$ *is meager.*

Then there exists a Hamel base $H \in \mathrm{Sel}(\mathcal{P})$.

P r o o f. Let $\{P_\alpha\}_{\alpha<2^\omega}$ be an enumeration of elements from $\mathcal{P}$. Let $\{r_\alpha\}_{\alpha<2^\omega}$ be an enumeration of all real numbers and assume that $r_\gamma = 0$.

We will construct partial selectors $(X_\gamma)_{\gamma<2^\omega}$ by induction.

Suppose that we have constructed $\{X_\alpha\}_{\alpha<\gamma}$ such that the set $X^*_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$ is: linearly independent over $\mathbf{Q}$; partial selector of $\mathcal{P}$, and, moreover, $r_\gamma \in \mathrm{span}_{\mathbf{Q}}\bigl(X^*_\gamma\bigr)$.

Let $\overline{\gamma} = \min\{\gamma' \geq \gamma\colon P_{\gamma'} \cap X^*_\gamma = \emptyset\}$ and let x_γ be any element of $P_{\overline{\gamma}} \setminus \mathrm{span}_{\mathbf{Q}}\bigl(X^*_\gamma\bigr)$. Consider two cases:

Case 1: $r_\gamma \in \mathrm{span}_{\mathbf{Q}}\bigl(X^*_\gamma \cup \{x_\gamma\}\bigr)$. Define $X_\gamma = X^*_\gamma \cup \{x_\gamma\}$.

Case 2: $r_\gamma \notin \mathrm{span}_{\mathbf{Q}}\bigl(X^*_\gamma \cup \{x_\gamma\}\bigr)$. Let us denote

$$Z = \mathrm{span}_{\mathbf{Q}}\bigl(X^*_\gamma \cup \{x_\gamma\}\bigr) \cup \bigcup\Bigl\{P \in \mathcal{P}\colon P \cap \bigl[X^*_\gamma \cup \{x_\gamma\}\bigr] \neq \emptyset\Bigr\}$$

and choose $\overline{a} \in \mathbb{R}$ such that $\forall_{q\in\mathbf{Q}\setminus\{0\}} \forall_{q'\in\mathbf{Q}}\, q\overline{a} + q' r_\gamma \in \cup\mathcal{P}\setminus Z$.

We will check that $r_\gamma - \overline{a} \notin \mathrm{span}_{\mathbf{Q}}\bigl(X^*_\gamma \cup \{x_\gamma, \overline{a}\}\bigr)$. Indeed, suppose that $r_\gamma - \overline{a} = p_0\overline{a} + \sum_{i=1}^k p_i u_i$, where $u_i \in X^*_\gamma \cup \{x_\gamma\}$. Then $r_\gamma = (p_0+1)\overline{a} + \sum_{i=1}^k p_i u_i$.

Let us consider two subcases:
SUBCASE 1: $p_0 = -1$. Hence $r_\gamma \in \operatorname{span}_{\mathbf{Q}}\big(X^*_\gamma \cup \{x_\gamma\}\big)$ which is a contradiction.
SUBCASE 2: $p_0 \neq -1$. In this case $\overline{a} = \frac{1}{p_0+1} \cdot \Big(r_\gamma - \sum_{i=1}^k p_i u_i\Big) \in \frac{1}{p_0+1} \cdot (r_\gamma + Z)$ which is impossible by the choice of $\overline{a}$.

Suppose by way of contradiction that $\forall_{w \in \mathbf{Q}\setminus\{0\}} w(r_\gamma - \overline{a}) \in P^*$, where P^* is (the unique) element from $\mathcal{P}$ such that [1] $\overline{a} \in P^*$. This is, however, impossible since P^* is nowhere dense. Therefore there exists, say $w^* \in \mathbf{Q} \setminus \{0\}$ such that $w^*(r_\gamma - \overline{a}) \notin P^*$. Let us put $X_\gamma = X^*_\gamma \cup \big\{\overline{a}, w^*(r_\gamma - \overline{a})\big\}$.

Define $H = \bigcup_{\gamma \in 2^\omega} X_\gamma$ and this Hamel base has all the required properties. $\square$

We will use the following characterization of the Baire space ω^ω which is due to Alexandrov and Urysohn:

Characterization. A topological space X is homeomorphic to ω^ω iff X is completely metrizable, separable, 0-dimensional space, and that there is no nonempty open compact subset of X.

This characterization implies:

COROLLARY 2.5. *If $U \subseteq \omega^\omega$ is a nonempty open set, then $\overline{U}$ is homeomorphic to ω^ω.*

We will need the following interesting in itself lemma. First of all let us recall the following definition (see for example [KMM], the definition before Lemma 7):

DEFINITION 2.6. A nonempty subset A of a Polish space X is locally uncountable iff every nonempty relatively open subset of A is uncountable.

Notice that in [KMM] the authors proved ([KMM, Lemma 7]) that if B is a locally uncountable Borel set, then there is a continuous bijection $\psi \colon \mathcal{N} \to B$. Following this result we prove a theorem about analytic subsets.

THEOREM 2.7. *If A is a subset of a Polish space, then the following conditions are equivalent:*

(1) *A is analytic locally uncountable.*
(2) *There exists a continuous surjection $\rho\colon \mathcal{N} \to A$ such that $\forall_{y\in A} \rho^{-1}\big[\{y\}\big]$ is nowhere dense.*
(3) *There exists a continuous surjection $\rho\colon \mathcal{N} \to A$ such that $\forall_{y\in A} \rho^{-1}\big[\{y\}\big]$ is nowhere dense set of size 2^ω.*

Proof. (2) $\Rightarrow$ (1). Let us assume that $A \subseteq \mathbb{R}$ is a Σ^1_1 subset of the real line such that there is a continuous surjection $\rho\colon \mathcal{N} \to A$ with ND fibres.

[1] I would like to thank Prof. Nikodem Mrożek for suggesting this method.

Then A is locally uncountable. Indeed, if $a \in A$ and $x < a < y$ then the preimage $\rho^{-1}\big[(x;y) \cap A\big]$ is nonempty, open, so there are uncountably many z in $(x;y) \cap A$, hence $|(x;y) \cap A| = 2^\omega$.

(1) $\Rightarrow$ (2). Suppose now that A is an analytic locally uncountable set. Let $\rho\colon \mathcal{N} \to A$ be any continuous surjection. Let $A^* = \big\{y \in A\colon \rho^{-1}[\{y\}] \notin \mathsf{ND}\big\}$. Since A^* is a countable set let us choose an enumeration without repetitions of all elements of A^*:

$$A^* = \{y_0, y_1, y_2, \ldots\}.$$

Notice that in the case $|A^*| < \aleph_0$ we can choose a finite enumeration of elements of A^*, namely:

$$A^* = \{y_0, y_1, y_2, \ldots, y_N\}.$$

For each $k \in \omega$ let us choose a homeomorphic embedding $h_k\colon 2^\omega \to A$ such that $\operatorname{diam}\big(h_k[2^\omega]\big) < \frac{1}{2^k}$ and moreover $h_k(\underline{0}) = y_k$, where $\underline{0} \in 2^\omega$ denotes a zero constant function. Denote: $E_k = \rho^{-1}\big[\{y_k\}\big]$, $U_k = \operatorname{int}(E_k)$ and $N_k = \overline{U_k}$, notice that since $E_k \notin \mathsf{ND}$, $U_k \neq \emptyset$.

By virtue of Corollary 2.5 we conclude that the space N_k is homeomorphic to ω^ω.

We will use the following (folklore?) lemma:

Lemma 2.8. *Suppose that $X \subseteq \omega^\omega$ is a ND set. There exists a homeomorphism $b\colon \omega^\omega \to \omega^\omega$ such that $b[X] \subseteq \mathrm{Even}^\omega$, where $\mathrm{Even} = \{2n\colon n \in \omega\}$.*

Since N_k is homeomorphic to ω^ω and $N_k \setminus U_k$ is a ND subsets of N_k, we conclude from the Lemma 2.8 that there is a homeomorphism $b_k\colon N_k \to \omega^\omega$ such that $b_k[N_k \setminus U_k] \subseteq \mathrm{Even}^\omega$. Let $i\colon \omega^\omega \to 2^\omega$ be a continuous surjection given by: $i(x)(k) = (x(k) \bmod 2)$.

For $x \in N_k$ denote $\rho_k(x) = h_k\big(i(b_k(x))\big)$ and define function $\rho^*\colon \omega^\omega \to A$ as follows:

$$\rho^*(x) = \begin{cases} \rho(x) & \text{if} \quad x \in \omega^\omega \setminus \bigcup_{k\in\omega} N_k, \\ \rho_k(x) & \text{if} \quad x \in N_k \quad \text{for some } k. \end{cases}$$

We will check that such defined function ρ^* has the required properties:

Surjection: Let $y \in A$. If $y \notin A^*$ then let $x \in \rho^{-1}\big[\{y\}\big]$ be arbitrary. Then $x \notin \bigcup_{k\in\omega} E_k$, hence $x \in \omega^\omega \setminus \bigcup_{k\in\omega} E_k \subseteq \omega^\omega \setminus \bigcup_{k\in\omega} N_k$, therefore $\rho^*(x) = \rho(x) = y$.

Suppose on the other hand that $y \in A^*$. Then $y = y_k$ for some $k \in \omega$. Since $h_k(\underline{0}) = y_k$ and i, b_k are surjections we conclude that $y_k \in \operatorname{ran}(\rho^*)$.

Continuity: Suppose that $\{x_m\}_{m\in\omega}$ is a sequence from ω^ω convergent to x^*. Let us consider several cases:

Case 1: $x^* \in U_k$ for some $k \in \omega$.

Then $\forall_m^\infty x_m \in U_k$, hence $\rho^*(x_m) = \rho_k(x_m) \to \rho_k(x^*) = \rho^*(x^*)$ since ρ_k is continuous.

Case 2: $x^* \in \omega^\omega \setminus \bigcup_{k\in\omega} N_k$.
Then $\forall_{k\in\omega}\forall_m^\infty x_m \notin N_k$, since N_k is a closed set, so $\forall_{\eta>0}\forall_m^\infty |\rho^*(x_m) - \rho(x_m)| < \eta$. This follows from the observation that if $x' \in N_k$ then $|\rho^*(x') - \rho(x')| < \frac{1}{2^k}$, since $\rho(x') = y_k \in h_k[2^\omega]$ and $\rho^*(x') \in h_k[2^\omega]$.

Since $\rho(x_m) \to \rho(x^*)$ and $\rho(x^*) = \rho^*(x^*)$ we conclude that $\rho^*(x_m) \to \rho^*(x^*)$, too.

Case 3: $x^* \in N_k \setminus U_k$ for some $k \in \omega$. Then $\rho^*(x^*) = h_k\big(i(b_k(x^*))\big) = h_k(\underline{0}) = y_k$.

As the sets N_j are closed pairwise disjoint, $\forall_{j\neq k}\forall_m^\infty x_m \notin N_j$. By splitting the sequence $\{x_m\}_m$ if necessary, we have to consider two subcases:

- $\forall_m x_m \in N_k$. Then $\rho^*(x_m) = \rho_k(x_m) \to \rho_k(x^*)$.
- $\forall_j \forall_m^\infty x_m \notin N_j$. Then, like in Case 2, $\forall_{\eta>0}\forall_m^\infty |\rho^*(x_m) - \rho(x_m)| < \eta$ but then $\rho(x_m) \to \rho(x^*) = y_k$, therefore $\rho^*(x_m) \to y_k = \rho^*(x^*)$.

Nowhere Dense: Let $y \in A$.

$$(\rho^*)^{-1}\big[\{y\}\big] = \left(\rho^{-1}\big[\{y\}\big] \cap \left(\omega^\omega \setminus \bigcup_{k\in\omega} N_k\right)\right) \cup \left((\rho^*)^{-1}\big[\{y\}\big] \cap \left(\bigcup_{k\in\omega} N_k\right)\right).$$

If $y \notin A^*$ then $\rho^{-1}\big[\{y\}\big] \in \mathsf{ND}$, hence $\rho^{-1}\big[\{y\}\big] \cap (\omega^\omega \setminus \bigcup_{k\in\omega} N_k) \in \mathsf{ND}$, too.
If $y \in A^*$ then there is $k \in \omega$ such that $y = y_k$. Hence

$$\rho^{-1}\big[\{y\}\big] \cap \left(\omega^\omega \setminus \bigcup_{k\in\omega} N_k\right) = E_k \setminus N_k \subseteq E_k \setminus \operatorname{int}(E_k) \in \mathsf{ND}.$$

For every $y \in A$ we have $\rho_k^{-1}\big[\{y\}\big] \in \mathsf{ND}(N_k)$, hence $\rho_k^{-1}\big[(\{y\}\big] \in \mathsf{ND}$. This shows that $(\rho^*)^{-1}\big[\{y\}\big] \cap \big(\bigcup_{k\in\omega} N_k\big) \in \mathcal{M}$, therefore $(\rho^*)^{-1}\big[\{y\}\big] \in \mathcal{M}$, hence $(\rho^*)^{-1}\big[\{y\}\big] \in \mathsf{ND}$.
(2) $\Rightarrow$ (3). Suppose that $\rho\colon \mathcal{N} \to A$ is a continuous surjection. Let $g\colon \mathcal{N} \to \mathcal{N} \times \mathcal{N}$ be any fixed homeomorphism and let $\pi_1\colon \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ denote the standard projection onto the first coordinate. It suffices to define $\rho^* = \rho \circ \pi_1 \circ g$:

$$\mathcal{N} \to_g \mathcal{N} \times \mathcal{N} \to_{\pi_1} \mathcal{N} \to_\rho A.$$

□

Let us prove Theorem 2.3:

Proof of Theorem 2.3. We need the following result.

Theorem 2.9 ([M], Theorem 9.26). *Assume $L = V$. Then there exists a Π^1_1 Hamel base for $\mathbb{R}$.*

Let H_2 be a Π^1_1 Hamel base from this theorem. Since $\mathbb{R} \setminus H_2$ is a locally uncountable (being a complement of a Hamel base) Σ^1_1 set there exists a continuous surjection $f\colon \mathcal{N} \to \mathbb{R} \setminus H_2$.

By virtue of Lemma 2.7 we may assume that $\forall_{y \in \mathbb{R} \setminus H_2} f^{-1}\big[\{y\}\big]$ is ND and of size 2^ω.

It is easy to observe that the family $\mathcal{P}$ defined by

$$\mathcal{P} = \Big\{ f^{-1}\big[\{y\}\big] \colon y \in \mathbb{R} \setminus H_2 \Big\}$$

satisfies all assumptions of Lemma 2.4. Hence we can find a Hamel base $H_1 \in \mathrm{Sel}(\mathcal{P})$ and this Hamel base has all required properties. $\square$

Question 2.10. Is it possible to prove Theorem 2.3 without the assumption $L = V$?

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