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THE STABLE POINTS AND THE ATTRACTORS OF DARBOUX FUNCTIONS

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ABSTRACT. A special kind of stable points for an established family of functions which are continuous in suitable topologies introduced in $\mathbb{R}$ is considered.

Introduction

The family of Darboux functions contains many important classes of mappings (for example: derivatives, approximately continuous functions, etc.). Recently, a lot of interesting results connected with dynamical systems generated by discontinuous functions have also been connected with the Darboux functions ([6], [14], [19], [20]), and moreover, many considerations connected with various topics close to dynamical systems lead us to the situation when the base of these considerations are Darboux functions, e.g., [13]. On the other hand, the restriction on some investigations of the functions belonging to suitable subsets of the family of Darboux functions permits to obtain analogous statements, as for continuous functions [6], [19], [16]. For example, a transitive map with two points of discontinuity does not have a dense orbit in general [18], however if we consider a special kind of Darboux functions, we can obtain statements analogous to results known for continuous functions [16].

This paper is a continuation of the considerations contained in [16] (in the case of functions mapping the real line into the real line). Consequently, we will use convenient tools connected with bi-topological spaces (which lead us to T-continuous functions) but we will not exhibit this topic.

We will consider functions $f : \mathbb{R} \longrightarrow \mathbb{R}$.

In the monograph [2] one can read that a point x_0 is said to be *a stable point of* f if for each open set $U \ni x_0$ there exists an open set $V \ni x_0$ such that

$$\gamma_f(x) = \left\{ f^n(x) : n \geq 0 \right\} \subset U, \qquad \text{for any} \quad x \in V.$$

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Consequently, we can deduce that (with no assumptions connected with considered functions):

- If x_0 is a stable point of f, then x_0 is a fixed point of f.
- If x_0 is a stable point of f, then x_0 is a continuity point of f.

For the sake of the second property, if we intend to consider discontinuous functions then it is necessary to modify the above definition. The main idea of this modification is to preserve the character such kinds points, and, on the other hand, to introduce of a generalization useful for Darboux functions.

1. Preliminaries

We will use standard definitions and notations mostly (see [1], [2], [4], [12]). In particular, by the letter $\mathbb{N}$ ($\mathbb{R}$) we will denote the set of *all natural* [1] (*real*) *numbers.*

The symbols $\overline{A}$ and Int(A) stand for the *closure* and *interior* of the set A, respectively. By the symbol $C_P(A)$ we shall denote the family of all components of A.

We will consider the behaviour of some real functions defined on the real line $\mathbb{R}$ and we will consider iterations of such kinds of functions: $f^0(x) = x$, and $f^n(x) = f\big(f^{n-1}(x)\big)$, if $n > 0$.

If $\mathcal{T}$ is a topology in $\mathbb{R}$, then we will use the notation $\mathcal{T}$-open set, $\mathcal{T}$-closed set, etc. to denote that these properties of sets are connected with the topology $\mathcal{T}$. The notations open set, closed set, etc., stand for properties in the natural topology of subsets of $\mathbb{R}$. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$. The notation g is *$\mathcal{T}$-continuous function* means that $g^{-1}((a, b)) \in \mathcal{T}$, for any open interval (a, b).

If we intend to consider Darboux functions, then we can require, the $\mathcal{T}$-continuous functions to have Darboux property ([21], [17]) and consequently, we have to exclude the topologies giving isolated points. Simultaneously, these topologies will be used for generalization of the notion of “stable points”. Endeavouring for this generalization we require, usual stable points to be obtained by putting the usual topology instead of $\mathcal{T}$ (so we have to assume that our topologies are finer than the natural topology of the real line). Consequently, the above considerations and results presented in [16] seem to suggest that in our case it is profitable to distinguish the following class of topologies.

Let $\tau_{\mathbb{R}}$ be the family of topologies finer than the natural topology of $\mathbb{R}$ such that if $\mathcal{T} \in \tau_{\mathbb{R}}$, then each $\mathcal{T}$-continuous function f has a Darboux property.

We make only one change in the basic definition of a stable point.

[1] That is *positive integer.*

Let $\mathcal{T} \in \tau_{\mathbb{R}}$. A point x_0 is said to be a $\mathcal{T}$-*stable point* of f if for each open set $U \ni x_0$ there exists a $\mathcal{T}$-open set $V \ni x_0$ such that

$$\gamma_f(x) = \{f^n(x) : n \geq 0\} \subset U, \qquad \text{for any} \quad x \in V.$$

Let us notice that in the above definition we use tools connected with bi-topological spaces, since they are only the tools, we will not consider this problem more widely.

We will denote by $\omega_f(x)$ the set of all accumulation points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. Moreover, we establish another notation: $\mathrm{Int}_f(A) = \{t \in A : f^n(t) \in \mathrm{Int} f^n(A),\ n = 0, 1, 2, \dots\}$. By the symbol $\mathrm{Fix}(f)$ we will denote the set of all fixed points of f, i.e., $\mathrm{Fix}(f) = \{x :\ f(x) = x\}$.

If A is a subset of the domain of f, then $f \restriction A$ denotes the restriction of f on A. We say that a set A is f-*invariant* if $f(A) \subset A$.

Let $\{A_n\}$ be a sequence of subsets of the real line. We will denote $A_n \searrow x_0$ ($A_n \nearrow x_0$) if for each $\epsilon > 0$ there exists a positive integer N such that

$$A_n \subset (x_0, x_0 + \epsilon)\big(A_n \subset (x_0 - \epsilon, x_0)\big), \quad \text{for any} \quad n \geq N.$$

An idea of the undermentioned notions derives from [7], [9], [10], [5], [12], [15]. Let J be a nondegenerate interval. By a J-*trajectory* we mean any sequence $\{d_n\} \subset \mathrm{Int}(J)$ of distinct points such that $\{d_n : n = 1, 2, \dots\}$ is a dense set in J. For a given nonempty J-open set[2] V, $r_{\{d_n\}}(V)$ (or shortly $r(V)$, if the J-trajectory is fixed) will be the first element of the sequence $\{d_n\}$ in V.

For $x \in J$ the *left first return path* to x based on $\{d_n\}$, $P_x^l = \{t_k : k = 1, 2, \dots\}$ is defined recursively via

$$t_1 = r((-\infty, x)), \qquad t_{k+1} = r((t_k, x)).$$

For $x \in J$ the *right first return path* to x based on $\{d_n\}$, $P_x^r = \{s_k : k = 1, 2, \dots\}$ is defined analogously

$$s_1 = r((x, +\infty)), \qquad s_{k+1} = r((x, s_k)).$$

Of course, if x is an endpoint of J, then we have only one-side first return path to x based on $\{d_n\}$.

A function $f : J \to \mathbb{R}$ is first return continuous from the left (right) at x with respect to the J-trajectory $\{d_n\}$ provided that

$$\lim_{\substack{t \to x \\ t \in P_x^l}} f(t) = f(x) \left(\lim_{\substack{t \to x \\ t \in P_x^r}} f(t) = f(x) \right).$$

[2] V is an open set in the subspace J of the space $\mathbb{R}$ such that $V \cap J \neq \emptyset$.

We say that $f : J \to \mathbb{R}$ is a *first return continuous function at x* with respect to the J-trajectory $\{d_n\}$ provided it is both left and right first return continuous at x with respect to the J-trajectory $\{d_n\}$.

Let $\{d_n\}$ be a fixed J-trajectory. A function $f : J \to \mathbb{R}$ is an $\big(J, \{d_n\}\big)$-first return continuous function (we denote $f \in \mathrm{FRC}\big(J, \{d_n\}\big)$) if it is first return continuous at each point $x \in (a, b) = \mathrm{Int}(J)$ (with respect to the J-trajectory $\{d_n\}$) and if $a \neq -\infty$ ($b \neq +\infty$) then f is first return continuous from the right (left) at a (b) with respect to the J-trajectory $\{d_n\}$.

We will say that $f \in \mathrm{FRC}(J)$ if there exists a J-trajectory $\{d_n\}$ such that $f \in \mathrm{FRC}\big(J, \{d_n\}\big)$.

Let $\mathcal{D}$ ($\mathcal{B}_1$) denote the class of all Darboux functions, i.e., functions having Darboux property or, in other words, *intermediate value property* ([3]) (functions in Baire class 1). If we wish to consider the intersection of $\mathcal{D}$ and $\mathcal{B}_1$, we shall write them next to each other, i.e., $\mathcal{D}B_1$ consists of all Darboux functions in Baire class 1.

Let H be an open and dense set (od-set for short) in $\mathbb{R}$. We will say that $f \in \mathrm{FRC}^*(H)$ if for each component $J = (a, b)$ of H $f \restriction \overline{J} \in \mathcal{D}B_1$ and there exist open intervals $I, K \subset J$ such that $\mathrm{Int}_f(I) \neq \emptyset \neq \mathrm{Int}_f(K)$ and $f^n(I) \searrow a$, $f^n(K) \nearrow b$. Of course, if $a = -\infty$ ($b = +\infty$) then (in the above definition) we have to omit the set I (K) and assumptions connected with this set.

2. Main results

In the further considerations, if we write "*x_0 is a $\mathcal{T}$-stable point*", then we always assume that $\mathcal{T} \in \tau_{\mathbb{R}}$.

We start with the observation that (similar as in the case of stable points): $\mathcal{T}$-stable points are fixed points of considered functions.

Proposition 2.1. *If x_0 is a $\mathcal{T}$-stable point of f, then x_0 is a fixed point of f.*

Proof. Suppose, on the contrary, that $f(x_0) \neq x_0$. Let $\epsilon = \frac{1}{2} \mid f(x_0) - x_0 \mid > 0$. Then $f(x_0) \notin (x_0 - \epsilon, x_0 + \epsilon)$. On the other hand, $\gamma_f(x_0) \subset (x_0 - \epsilon, x_0 + \epsilon)$, a contradiction. □

As we see, this very important property of stable points has been preserved in the case of $\mathcal{T}$-stable points. However, it can be shown, that $\mathcal{T}$-stable points need not be continuity points of f (in the usual topology).

Now, we have a base for the introduction of the notion of a $\mathcal{T}$-attractor.

Let $\mathcal{T} \in \tau_{\mathbb{R}}$. A nonempty closed set $A \subset \mathbb{R}$ is said to be a *$\mathcal{T}$-attractor* of f if a is a $\mathcal{T}$-stable point of f, for any $a \in A$ and there exists a $\mathcal{T}$-open set $V \supset A$ such that $\omega_f(x) \subset A$ for each $x \in V$.

Proposition 2.2. *Any $\mathcal{T}$-attractor of f consists of fixed points of f.*

As a result, if we intend to consider a set A to be a $\mathcal{T}$-attractor, then it is natural that $A \subset \operatorname{Fix}(f)$.

According to the results in [7] and [8] we have:

Lemma 2.3. *If J is a compact interval then a function $f : J \longrightarrow \mathbb{R}$ is a Darboux Baire one function iff $f \in \operatorname{FRC}(J)$.*

Now, we shall give some sufficient conditions for the fact that a fixed set A is a $\mathcal{T}$-attractor of f (for some topology $\mathcal{T} \in \tau_{\mathbb{R}}$).

Theorem 2.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $A \subset \mathbb{R}$ be a nowhere dense and closed set, $f \in \operatorname{FRC}^*(\mathbb{R} \setminus A)$ and $A \subset \operatorname{Fix}(f)$. Then f is a Darboux function and there exists a topology $\mathcal{T}_f \in \tau_{\mathbb{R}}$ such that A is a $\mathcal{T}_f$-attractor of f.*

In the further considerations, if we write $\mathcal{T}_f$, then we understand that it is topology constructed by means of the method described in this proof.

P r o o f. First, we will construct a suitable topology $\mathcal{T}_f \in \tau_{\mathbb{R}}$.

Let us denote $H_A = \mathbb{R} \setminus A$ and let $x_0 \in \mathbb{R}$.

Of course, H_A is an open and dense set.

If $x_0 \in H_A$, then there exists $n_{x_0} \in \mathbb{N}$ such that $\left(x_0 - \frac{1}{n_{x_0}}, x_0 + \frac{1}{n_{x_0}}\right) \subset H_A$. Let us put

$$\begin{aligned} \mathcal{B}_{\mathcal{T}_f}(x_0) = \Bigg\{ & U_m(x_0) = \left(x_0 - \frac{1}{m}, x_0 + \frac{1}{m}\right) \\ & \cap\ f^{-1}\left(\left(f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m}\right)\right) : m = n_{x_0}, n_{x_0} + 1, \dots \Bigg\}. \end{aligned} \tag{1}$$

Now, we will consider the case $x_0 \notin H_A$. First, we will define the "right-hand neighbourhood" of x_0. To do this it is convenient to consider two cases.

1) There exists $(a, b) \in C_P(H_A)$ $(a < b)$ such that $x_0 = a$. Let I_0 be an open interval such that $\operatorname{Int}_f(I_0) \neq \emptyset$ and $f^n(I_0) \searrow a$. Then there exists n_0 such that $f^n(I_0) \subset (a, b)$, for $n \geq n_0$.

   Let us denote $I_1 = f^{n_0}(I_0)$. Then $f^n(I_1) \subset (a, b)$ $(n = 1, 2, \dots)$, $\operatorname{Int}_f(I_1) \neq \emptyset$ and $f^n(I_1) \searrow a$. Let $x_1 \in I_1$ be a point such that $f^n(x_1) \in \operatorname{Int} f^n(I_1)$ $(n = 0, 1, 2, \dots)$. So, let us put

   $$x_{n+1} = f^n(x_1), \qquad \text{for} \quad n = 1, 2, \dots$$

   It is easy to see that $\{x_n : n = 0, 1, 2, \dots\} \subset (a, b) \subset H_A$ and $x_n \to a$.

   For each $n = 1, 2, \dots$ let k_n be a positive integer such that

   $$U_{k_n}(x_n) \in \mathcal{B}_{\mathcal{T}_f}(x_n) \quad \text{and} \quad U_{k_n}(x_n) \subset \operatorname{Int}\big(f^{n-1}(I_1)\big).$$

Let us put

$$\mathcal{B}^+_{\mathcal{T}_f}(x_0) = \left\{ U^+_m(x_0) = \{x_0\} \cup \bigcup_{n=m}^{\infty} U_{k_n}(x_n); \qquad m = 1, 2, \dots \right\}.$$

Note that in this case for every $\eta > 0$ there exists m^* such that

$$f\big(U^+_m(x_0)\big) \subset [x_0, x_0 + \eta), \qquad \text{for} \quad m \geq m^*. \tag{2}$$

Let m^* be a positive integer such that $f^m(I_1) \subset (x_0, x_0 + \eta)$, for $m \geq m^*$. So let

$$U^+_m(x_0) = \{x_0\} \cup \bigcup_{n=m}^{\infty} U_{k_n}(x_n) \in \mathcal{B}^+_{\mathcal{T}_f}(x_0) \quad (\text{for} \quad m \geq m^*).$$

Note that

$$f\big(U_{k_n}(x_n)\big) \subset (x_0, x_0 + \eta), \quad \text{for} \quad n = m^*, m^* + 1, \dots$$

This finishes the proof of (2).

Moreover, it is easy to see that if

$$(x_0, y) \in C_P(H_A), \quad \text{then} \quad f^k\big(U^+_m(x_0)\big) \subset [x_0, y), \tag{3}$$

for any k and m.

2) There is no $(a, b) \in C_P(H_A)$ such that $x_0 = a$. Let us denote by $R^r_m(x_0)$ the set of all points $t \in \big(x_0, x_0 + \frac{1}{m}\big)$ such that t is a left endpoint of some component of H_A. Since $A \subset \operatorname{Fix}(f)$, then $\lim_{t \to x_0, t \in R^r_m(x_0)} f(t) = f(x_0)$, for any m.

So let us define

$$B^+_{\mathcal{T}_f}(x_0) = \left\{ U^+_m(x_0) = \{x_0\} \cup \bigcup_{t \in R^r_m(x_0)} \big(U^+_m(t) \setminus \{t\}\big) : m = 1, 2, \dots \right\}.$$

Let us note that in this case for each $\eta > 0$ there exists m^* such that

$$f\big(U^+_m(x_0)\big) \subset [x_0, x_0 + \eta) = \big[f(x_0), f(x_0) + \eta\big), \qquad \text{for} \quad m \geq m^*. \tag{4}$$

Let m^* be a positive integer such that if $(t_1, t_2) \in C_P(H_A)$ and $t_1 < x_0 + \frac{1}{m^*}$ then $t_2 < x_0 + \eta$. Then for any $m \geq m^*$ and $t \in R^r_m(x_0)$ we have $U^+_m(t) \subset (x_0, x_0 + \eta)$ which, according to (3), finishes the proof of (4).

In a similar way we can define a "left-hand" base at x_0 :

$$B^-_{\mathcal{T}_f}(x_0) = \big\{U^-_m(x_0) : m = 1, 2, \dots\big\}.$$

Finally, let $B_{\mathcal{T}_f}(x_0) = \big\{U_m(x_0) : \ m = 1, 2, \dots\big\}$, where if $x_0 \in H_A$ then $U_m(x_0)$ have been defined in (1) and if $x_0 \notin H_A$, then $U_m(x_0) = U^+_m(x_0) \cup U^-_m(x_0)$, for $m = 1, 2, \dots$

It is not hard to verify that the family $\{B(x) : x \in \mathbb{R}\}$ fulfils the conditions (BP1), (BP2), (BP3) from [11]. Then we can consider topology $\mathcal{T}_f$ in $\mathbb{R}$ generated by the neighbourhood system $\{B(x) : x \in \mathbb{R}\}$ (Proposition 1.2.3, [11]).

In the next step of the proof, we will show that

$$\mathcal{T}_f \in \tau_{\mathbb{R}}. \tag{5}$$

Of course, $\mathcal{T}_f$ is a finer topology than the natural topology of $\mathbb{R}$. Now, let $C(\mathcal{T}_f)$ be a family of all $\mathcal{T}_f$-continuous functions. Let us establish a function $h \in C(\mathcal{T}_f)$. For the proof of (5) it is necessary to show that

$$h \quad \text{is a Darboux function.} \tag{6}$$

First, we remark that if $[p, q] \subset (a, b) \in C_P(H_A)$ is a compact interval, then

$$h \restriction [p, q] \in DB_1. \tag{7}$$

Let $f_* = f \restriction [p, q]$ and $h_* = h \restriction [p, q]$. Since $f_* \in \mathcal{D}B_1$ then (Lemma 2.3) there exists $[p, q]$-trajectory $\{d_n\}$ such that $f_* \in \mathrm{FRC}([p, q], \{d_n\})$. It is not hard to verify, that $h_* \in \mathrm{FRC}([p, q], \{d_n\})$. Consequently, according to Lemma 2.3, $h \restriction [p, q] \in DB_1$.

According to (7) we can deduce that if $(p, q) \subset (a, b) \in C_P(H_A)$, then

$$h \restriction [p, q] \in DB_1. \tag{8}$$

Now, we set about proving (6). On the contrary, suppose that h is not a Darboux function. This means that there exist real numbers $a, b \in \mathbb{R}$ (there is no loss of generality in assuming $a < b$) and $\alpha \in \mathbb{R}$ such that

$$h(a) < \alpha,\ h(b) > \alpha \quad \text{and} \quad h^{-1}(\alpha) \cap (a, b) = \emptyset.$$

On account of $\mathcal{T}_f$-continuity of h and the form of local bases $B(a)$ and $B(b)$ we infer that there exist $a_1, b_1 \in H_A \cap [a, b]$ such that $h(a_1) < \alpha$ and $h(b_1) > \alpha$. Let (p, q) be a component of H_A such that $a_1 \in (p, q)$. From (8), $h \restriction [p, q] \in DB_1$, which means that

$$a_0 = \sup\{x > a_1 : [a_1, x) \subset h^{-1}((-\infty, \alpha))\} > a_1.$$

It is obvious that $a_0 < b_1$. Moreover, let us notice that if

$$a_0 \notin H_A \quad \text{and} \quad C \in C_P(H_A), \tag{9}$$

then a_0 is not a left-hand endpoint of C. So, we can consider two cases:

1) $a_0 \in h^{-1}((\alpha, \infty))$. Thus, according to $\mathcal{T}_f$-continuity of h at a_0 and the form of $B(a)$, we deduce that there exists $a_2 \in (a_1, a_0)$ such that $h(a_2) > \alpha$, which contradicts the definition of a_0.
2) $a_0 \in h^{-1}((-\infty, \alpha))$. Let $U_{m'}(a_0) \in B(a_0)$ be a set such that $h\big(U_{m'}(a_0)\big) \subset (-\infty, \alpha)$ and let $a^* > a_0$ be a fixed element of $U_{m'}(a_0)$. Let us consider a family

$$T = \big\{(t_1, t_2) \in C_P(H_A) : \ a_0 < t_2 < a^*\big\} \neq \emptyset.$$

Then

$$(t_1, t_2) \cap U_{m'}(a_0) \neq \emptyset, \qquad \text{for any} \quad (t_1, t_2) \in T.$$

Since $h \restriction [t_1, t_2]$ is a Darboux function, then

$$\bigcup_{(t_1,t_2)\in T} [t_1, t_2] \subset h^{-1}((-\infty, \alpha)).$$

According to the $\mathcal{T}_f$-continuity of h, we can infer that

$$z \in h^{-1}((-\infty, \alpha)), \qquad \text{for every} \quad z \in [a_0, a^*],$$

which contradicts the definition of a_0.

In both cases we obtain a contradiction. This finishes the proof of (6) and, at the same time, the proof of (5) is finished, too.

For the proof of a Darboux property of the function f it is sufficient to show that

$$f \text{ is a } \mathcal{T}_f\text{-continuous function.} \tag{10}$$

We shall show that f is $\mathcal{T}_f$-continuous at any point $y_0 \in \mathbb{R}$. It is sufficient to restrict our considerations for the case $y_0 \notin H_A$. Thus $f(y_0) = y_0$. On account of (2) and (4) we infer that for every ϵ there exists m_ϵ such that

$$f\big(U^+_{m_\epsilon}(y_0)\big) \subset \big[f(y_0), f(y_0) + \epsilon\big).$$

In a similar way we can consider the "left-hand" $\mathcal{T}_f$-neighbourhood of y_0, that permits us to conclude that f is a $\mathcal{T}_f$-continuous function at y_0.

Now, we shall show that

$$A \text{ is a } \mathcal{T}_f\text{-attractor of } f. \tag{11}$$

Let us fix a point $w \in A$. First, we shall show that

$$w \text{ is a } \mathcal{T}_f\text{-stable point of } f. \tag{12}$$

Let U be an arbitrary open set containing w and let ϵ_w be a positive real number such that $(w - \epsilon_w, w + \epsilon_w) \subset U$.

Let us first examine a case when there exists a number $v > w$ such that $(w, v) \in C_P(H_A)$. Let us go back to the construction of the "right-hand" base

$B^+_{\mathcal{T}_f}$ at points being left endpoint of some component of H_A. We will use a notation similar as in the case of this construction, to that except for a and b (we replace a with w and b with $w+\epsilon_w$) and let m_w be a positive integer such that $f^m(I_1) \subset (w, w+\epsilon_w)$, for any $m \geq m_w$. Then $f^n(z) \in [w, w+\epsilon_w)$, for any $z \in U^+_m(w)$ and positive integer n and $m \geq m_w$.

Let $m \geq m_w$ and n be a fixed positive integer. Then

$$U^+_m(w) = \{w\} \cup \bigcup_{n=m}^{\infty} U_{k_n}(x_n) \subset \{w\} \cup \bigcup_{n=m}^{\infty} f^{n-1}(I_1).$$

Let $z \in U^+_m(w)$. We can omit easy considerations connected with the case $z = w$. So, we can assume that $z \in \bigcup\limits_{n=m}^{\infty} f^{n-1}(I_1)$. Let n' be a number such that $n' \geq m$ and $z \in f^{n'-1}(I_1)$. Thus $f^n(z) \in f^{n+n'-1}(I_1)$ and consequently $f^n(z) \in (w, w+\varepsilon_w)$.

Now, we will consider the case when there is no positive integer $v > w$ such that $(w, v) \in C_P(H_A)$. So, let m' be a natural number such that if $(t_1, t_2) \in C_P(H_A)$, $t_1 < t_2$ and if $t_1 < w + \frac{1}{m'}$, then $t_2 < w + \epsilon_w$.

Let m_0 be a fixed positive integer such that $m_0 > m'$ and let us consider

$$U^+_{m_0}(w) = \{w\} \cup \bigcup_{t \in R^r_{m_0}(w)} \left(U^+_{m_0}(t) \setminus \{t\}\right).$$

If n is any positive integer and $z \in U^+_{m_0}(w)$ then, according to (3), we infer that $f^n(z) \in [w, w+\epsilon_w)$.

We can give similar considerations for the case of a "left-hand" $\mathcal{T}_f$-neighbourhood of w.

The above considerations permit us to infer that there exists a $\mathcal{T}_f$-open set $V \ni x_0$ such that

$$\gamma_f(x) \subset (w-\epsilon_w, w+\epsilon_w) \subset U, \quad \text{for each } x \in V.$$

It remains to prove that there exists; a $\mathcal{T}_f$-open set V such that

$$V \supset A \quad \text{and} \quad \omega_f(x) \subset A, \qquad \text{for any } x \in V. \tag{13}$$

Let k be a fixed positive integer and let us put $V = \bigcup_{x \in A} U_k(x)$, where $U_k(x) \in B_{\mathcal{T}_f}(x)$. Then V is a $\mathcal{T}_f$-open set and $A \subset V$. Finally, let $z \in V$. If $z \in A$, then $f^n(z) = z \in A$ $(n = 1, 2, \dots)$. So, we will restrict our considerations for the case $z \notin A$. Then $z \in (a, b) \in C_P(H_A)$ and $z \in U_k(x)$, for some $x \in A$. Consequently, $z \in f^l(I_2)$, where I_2 is an interval such that $f^n(I_2) \searrow a$ and l is a positive integer. It is easy to see that $f^n(z) \to a \in A$. □

In the above theorem we have considered attractors which are outside of a suitable od-set. The next theorem is some kind of complement of these considerations. The notion of *path derivatives* ([3],[4]) will be very useful in it.

Let $x_0 \in \mathbb{R}$. A *path leading to* x_0 is a set $P_{x_0} \subset \mathbb{R}$ such that $x_0 \in P_{x_0}$ and x_0 is a point of accumulation of P_{x_0}. The collection $\{P_x : x \in \mathbb{R}\}$ is called the *path system*.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and let $P = \{P_x : x \in \mathbb{R}\}$ be any path system. If the

$$\lim_{y \to x, y \in P_x \setminus \{x\}} \frac{f(y) - f(x)}{y - x} = F(x)$$

exists and is finite, then we say that f is P-differentiable at x. We use the notation $F = f'_P$.

The next theorem is similar to the well-known theorem connected with the usual derivative (and usual attractors) ([2]). Of course, in account of our considerations, it is impossible to apply the usual derivative.

Theorem 2.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and let $A \subset \mathbb{R}$ be a closed set such that $A \subset \mathrm{Fix}(f)$. If $\mathcal{T} \in \tau_{\mathbb{R}}$ is a topology such that f is a $\mathcal{T}$-continuous function and for every $a \in A$ there exists a local base $B_{\mathcal{T}}(a)$ of $(\mathbb{R}, \mathcal{T})$ at a consisting of f-invariant sets such that*

$$f'_U(a) \in (-1, 1), \qquad \text{for some} \quad U \in B_{\mathcal{T}}(a),$$

then A is a $\mathcal{T}$-attractor of f.

Proof. Let us fix $a \in A$ and let W be an arbitrary open set containing a. Now, we can establish a set $U \in B_{\mathcal{T}}(a)$ such that $U \subset W$ and $f'_U(a) = \alpha_a \in (-1, 1)$. Let $\beta_a \in (0, 1)$ be a real number such that $| \alpha_a | < \beta_a$. This yields there exists $\delta > 0$ such that

$$\frac{f(x) - f(a)}{x - a} \in (-\beta_a, \beta_a), \qquad \text{for} \quad x \in (a - \delta, a + \delta) \cap U \setminus \{a\}. \tag{14}$$

Let $U_a \in B_{\mathcal{T}}(a)$ be an f-invariant set such that $U_a \subset (a - \delta, a + \delta) \cap U$. On account of (14), we have

$$\frac{| f(x) - f(a) |}{| x - a |} < \beta_a, \qquad \text{for} \quad x \in U_a \setminus \{a\}.$$

Consequently,

$$| f(x) - a | \leq \beta_a \cdot | x - a |, \qquad \text{for} \quad x \in U_a. \tag{15}$$

It is obvious that if $y \in U_a$, then

$$f^n(y) \in U_a, \qquad \text{for} \quad n = 1, 2, \ldots \tag{16}$$

Therefore

$$f^n(y) \in U \subset W, \qquad \text{for} \quad n = 1, 2, \ldots$$

This finishes the proof that a is a $\mathcal{T}$-stable point of f.

Now, let us put $U_A = \bigcup_{a \in A} U_a$ and let $z \in U_A$. Finally, let $t \in \omega_f(z)$, i.e., there exists an increasing sequence $\{n_k\}$ such that $f^{n_k}(z) \to t$. Let us remark that there exists $a_z \in A$ such that $z \in U_{a_z}$. So, according to (15) and (16), we have

$$\mid f^n(z) - a_z \mid \leq \beta_{a_z} \cdot \mid f^{n-1}(z) - a_z \mid \leq \cdots \leq \beta_{a_z}^n \cdot \mid z - a_z \mid .$$

Consequently, $f^n(z) \to a_z \in A$, which gives $t = a_z \in A$. $\square$

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