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INTERSECTIONS OF RANGES OF ADDITIVE GENERATORS OF ASSOCIATIVE FUNCTIONS

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ABSTRACT. The structure of the set $\mathcal{B}$ of all ranges of additive generators of associative functions is studied here. Sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$, where $A_n \in \mathcal{B}$ for all $n \in N$, are introduced. Examples of elements of $\mathcal{B}$ which are nowhere dense in $[0, \infty]$ and contain 0 as an accumulation point are presented.

1. Introduction

Non-continuous additive generators of associative functions are investigated. The associativity of a generated function depends only on properties of the range of its additive generator. In this paper the intersections of ranges of additive generators of associative functions are studied.

The idea of representing the special associative functions by means of functions of one variable goes back to Abel [1]. Many results concerning the representation of associative functions appeared later in the framework of the semigroup theory and the theory of functional equations. In the context of triangular norms (triangular conorms) the representation theorems were introduced by Ling [4] and Schweizer and Sklar [5], [6]. Klement, Mesiar and Pap [3] studied additive generators of triangular norms (triangular conorms) whose ranges are relatively closed under the usual addition. Many results concerning the additive generators of associative functions whose ranges are not relatively closed under the usual addition were introduced in [8]–[12].

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2. Preliminaries

Each strictly monotone function $f : [0,1] \to [0,\infty]$ yields the function $F : [0,1]^2 \to [0,1]$ via the formula

$$F(x,y) = f^{(-1)}\big(f(x) + f(y)\big) \qquad \text{for all} \quad x,y \in [0,1], \tag{1}$$

where $f^{(-1)} : [0,\infty] \to [0,1]$ is the pseudo-inverse of f, i.e.,

$$f^{(-1)}(y) = \begin{cases} \sup\big(x \in [0,1] \mid f(x) > y\big) & \text{if} \quad f \text{ is strictly decreasing,} \\ \sup\big(x \in [0,1] \mid f(x) < y\big) & \text{if} \quad f \text{ is strictly increasing,} \end{cases}$$

where $\sup \emptyset = 0$. The function f is said to be an *additive generator* of F. In general, the function F need not be associative. The associativity of F depends only on the properties of $\mathrm{Ran}(f) = \big\{x \in [0,\infty] \mid \exists\, t \in [0,1], f(t) = x\big\}$ of f.

Write

$$\mathcal{F} = \big\{f : [0,1] \to [0,\infty] \mid f \text{ is a strictly monotone function }\big\},$$
$$\mathcal{G} = \big\{f : [0,1] \to [0,\infty] \mid f \in \mathcal{F} \text{ generates via (1) an associative function}\big\},$$

and

$$\mathcal{A} = \big\{A \subseteq [0,\infty] \mid \exists\, f \in \mathcal{F},\, \mathrm{Ran}(f) = A\big\},$$
$$\mathcal{B} = \big\{A \subseteq [0,\infty] \mid \exists\, f \in \mathcal{G}, \mathrm{Ran}(f) = A\big\}.$$

Obviously,

$$\mathcal{G} \subseteq \mathcal{F} \quad \text{and} \quad \mathcal{B} \subseteq \mathcal{A}.$$

Let $A \in \mathcal{B}$. If $f \in \mathcal{F}$ with $\mathrm{Ran}(f) = A$ then $f \in \mathcal{G}$. Let $A \in \mathcal{A}$. If A is relatively closed under the usual addition ($\forall\, x,y \in A$, $x + y \in A \cup [s,\infty]$, where $s = \sup\big(A \setminus \{\max(A)\}\big)$), then $A \in \mathcal{B}$ (see, [3]).

The main problem is the following one: What are the sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$, where $A_n \in \mathcal{B}$ for all $n \in N$, ($N = \{1, 2, \dots\}$)?

In Section 3 some sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$, where $A_n \in \mathcal{B}$ for all $n \in N$, are introduced. The first example of a set $A \in \mathcal{B}$ which is nowhere dense in $[0,\infty]$ and contains 0 as an accumulation point was introduced in [9] and can be found in [12]. In this paper some other examples of the sets $A \in \mathcal{B}$ which are nowhere dense in $[0,\infty]$ and contain 0 as an accumulation point are presented.

2.1. The range of $f \in \mathcal{F}$

We will use some properties of the range of a strictly monotone function [11]: *Let $A \in \mathcal{A}$ and $A \neq [0,\infty]$. Then there exist the uniquely determined non-empty countable system $\mathcal{S} = \big\{[b_k, d_k] \subseteq [0,\infty] \mid k \in K\big\}$ of pairwise disjoint* *intervals of a positive length and the uniquely determined non-empty countable set* $C = \{c_k \in [0,\infty] \mid k \in K\}$ *such that* $[b_k, d_k] \cap C = \{c_k\}$ *for all* $k \in K$*, and*

$$A = \big([0,\infty] \setminus (\cup_{k\in K}[b_k, d_k])\big) \cup \{c_k \in [0,\infty] \mid k \in K\}. \tag{2}$$

In fact, if $f : [0,1] \to [0,\infty]$ is a strictly increasing function with $\mathrm{Ran}(f) = A \neq [0,\infty]$, the sets $\mathcal{S} = \{[f(x_-), f(x_+)] \mid x \in [0,1],\ f(x_-) < f(x_+)\}$, $C = \{f(x) \mid x \in [0,1],\ f(x_-) < f(x_+)\}$ (where $f(x_-) = \lim_{t\to x^-} f(t)$ for all $x \in (0,1]$, $f(0_-) = 0$, and $f(x_+) = \lim_{t\to x^+} f(t)$ for all $x \in [0,1)$, $f(1_+) = \infty$) have all the required properties as shown in [11].

This pair $(\mathcal{S}, C)$ is said to be *associated with* $A \in \mathcal{A}$, $A \neq [0,\infty]$. The pair $(\mathcal{S}, C)$ is said to be *associated with* $A = [0,\infty]$ if $\mathcal{S} = \{[\infty,\infty]\}$ and $C = \{\infty\}$. We will write $(\mathcal{S}, C) = \big(\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in K\}\big)$.

Let $A \subseteq [0,\infty]$. The set $[0,\infty] \setminus A$ will be denoted by A^c in this paper. Observe that $(A \setminus C)^c = \cup_{k\in K}[b_k, d_k]$. Moreover, a set C is always non-empty, and $[b_k, d_k] \cap A = [b_k, d_k] \cap C = \{c_k\}$ for all $k \in K$. For all $I, J \in \mathcal{S}$, if $I \cap J \neq \emptyset$ then $I = J$, and if $I \neq J$ then $I \cap J = \emptyset$.

2.2. The addition on $\mathrm{Ran}(f)$

First of all, we will deal with the following operation [11]: $\oplus : \mathrm{Ran}(f)^2 \to \mathrm{Ran}(f)$ which is given by

$$x \oplus y = f\Big(F\big(f^{-1}(x), f^{-1}(y)\big)\Big) \qquad \text{for all} \quad x, y \in \mathrm{Ran}(f),$$

where f is an additive generator of F and $f^{-1} : \mathrm{Ran}(f) \to [0,1]$ is the (standard) inverse of f. Clearly, the operation $\oplus$ is associative if and only if F is associative.

Denote $\mathrm{Ran}(f)$ by A. Substituting (1) into the last equation it yields

$$x \oplus y = f\big(f^{(-1)}(x+y)\big) \qquad \text{for all} \quad x, y \in A.$$

Let $(\mathcal{S}, C) = \big(\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in K\}\big)$ be associated with $A \in \mathcal{A}$. It is a matter of straightforward verification that

$$f\big(f^{(-1)}(x)\big) = \begin{cases} x & \text{if} \quad x \in A, \\ c_k & \text{if} \quad x \in [b_k, d_k] \setminus \{c_k\} \qquad \text{for some} \quad k \in K, \end{cases}$$

which leads to the following definition:

Definition 1 (Section 4, [11]). Let $(\mathcal{S}, C) = \big(\{[b_k, d_k] \mid k \in K\}, \{c_k \mid k \in K\}\big)$ be associated with $A \in \mathcal{A}$. A function $F_A : [0,\infty] \to [0,1]$ given by

$$F_A(x) = \begin{cases} x & \text{if} \quad x \in A, \\ c_k & \text{if} \quad x \in [b_k, d_k] \setminus \{c_k\} \qquad \text{for some} \quad k \in K, \end{cases}$$

is said to be the *function determined* by A. An operation $\oplus : A^2 \to A$ given by

$$x \oplus y = F_A(x+y) \tag{3}$$

is called the *addition* on A.

The function F_A determined by $A \in \mathcal{A}$ is always non-decreasing on $[0,\infty]$ and strictly increasing on A. Moreover, for all $x, y \in A$,

$$x \oplus y = \begin{cases} x+y & \text{if} \quad x+y \in A, \\ c_k & \text{if} \quad x+y \in [b_k, d_k] \setminus \{c_k\} \qquad \text{for some} \quad k \in K. \end{cases}$$

The addition $\oplus$ on A is always commutative, non-decreasing (for all $x, y, u, v \in A$, if $x \leq u$ and $y \leq v$, then $x \oplus y \leq u \oplus v$), $\max A$ is its annihilator and $\max(x, y) \leq x \oplus y$ for all $x, y \in A$. Assuming that $\oplus$ is the addition on $A \in \mathcal{A}$, $A \in \mathcal{B}$ if and only if the addition $\oplus$ is associative on A, i.e., for all $x, y, z \in A$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

Let $A \subseteq [0, \infty]$. Write

$$\mathrm{Acc}_-(A) = \big\{x \in [0,\infty] \mid \forall\, \epsilon > 0, (x-\epsilon, x) \cap A \neq \emptyset\big\},$$
$$\mathrm{Acc}_+(A) = \big\{x \in [0,\infty] \mid \forall\, \epsilon > 0, (x, x+\epsilon) \cap A \neq \emptyset\big\}.$$

In proofs we will often use the following properties: Let $A \in \mathcal{A}$, $\oplus$ be the addition on A and $x, y \in A$. Then

(i) If $x + y \leq a$ ($a \leq x + y$) and $a \in A$, then $x \oplus y \leq a$ ($a \leq x \oplus y$).
(ii) If $x + y < a$ and $a \in \mathrm{Acc}_-(A)$, then $x \oplus y < a$.
(iii) If $a < x + y$ and $a \in \mathrm{Acc}_+(A)$, then $a < x \oplus y$.

In particular, we will use (ii) with $a = b_k > 0$ for some $k \in K$ and (iii) with $a = d_k < \infty$ for some $k \in K$.

2.3. Examples of elements of $\mathcal{B}$

Let $A, B \subseteq [0,\infty]$. Denote the set $\big\{x \in [0,\infty] \mid \exists\, a \in A, \exists\, b \in B, a + b = x\big\}$ by $A + B$. Instead of $A + \{c\}$ ($c \in [0,\infty]$) we will write $A + c$.

We can use the following results [11] for constructing elements of $\mathcal{B}$.
Let $A \in \mathcal{B}$, $a = \min A$ and $b = \max A$. Then

(S1) $A \cap [u, v] \in \mathcal{B}$ where $u, v \in A$, $u < v$.
(S2) $cA = \big\{cx \in [0,\infty] \mid x \in A\big\} \in \mathcal{B}$, where $c \in (0,\infty)$.
(S3) $A \cup \big(A \setminus \{a\} + c\big) \in \mathcal{B}$ where $2b \leq c < \infty$.
(S4) $\{a\} \cup \big(\cup_{n=0}^{\infty}(A \setminus \{a\} + nc)\big) \cup \{\infty\} \in \mathcal{B}$ where $2b \leq c < \infty$.

For instance, obviously $[0,\infty] \in \mathcal{B}$, then $[0,2] \in \mathcal{B}$ by (S1), $[0,1] \in \mathcal{B}$ by (S2), $[0,1] \cup (2,3] \in \mathcal{B}$ by (S3), and $\{0\} \cup \big(\cup_{n=0}^{\infty}((6n, 6n+1] \cup (6n+2, 6n+3]\big) \cup \{\infty\}$ by (S4).

The next result is an immediate consequence of Theorem 5.5 in [10]:

Let

$$A = \{0\} \cup \Big(\cup_{n\in I}\big((a_n, b_n) \cup \{c_n\}\big)\Big), \quad I = \{1, \ldots, k\},\ (k \in N),$$

or

$$A = \{0\} \cup \Big(\cup_{n\in I}\big((a_n, b_n) \cup \{c_n\}\big)\Big) \cup \{\infty\}, \quad I = N,$$

where $0 \le a_n < b_n \le c_n$ for all $n \in I$, and $b_n < a_{n+1}$, $c_n \le a_{n+1}$ for all $n, n+1 \in I$.

(S5) If $b_n - a_n \le b_1$ for all $n \in I$ and $2c_n \le a_{n+1}$ for all $n, n+1 \in I$, then $A \in \mathcal{B}$.

3. Intersections of elements of $\mathcal{B}$

In general, the intersection of the sets of $\mathcal{B}$ need not be an element of $\mathcal{A}$. For instance, if $A_n = [0, 2] \cup (11 - 1/n, 12]$ for all $n \in N$, then $A_n \in \mathcal{B}$ for all $n \in N$ by (S5), but $\cap_{n=1}^{\infty} A_n = [0, 2] \cup [11, 12] \notin \mathcal{A}$.

Further, if the intersection of the sets of $\mathcal{B}$ is an element of $\mathcal{A}$, it need not be an element of $\mathcal{B}$. For instance, if $A = [0, 4) \cup \{4 + 1/n\} \cup (11 - 1/n, 12]$ for all $n \in N$, then $A_n \in \mathcal{B}$ for all $n \in N$ by (S5), $\cap_{n=1}^{\infty} A_n = [0, 4) \cup [11, 12] \in \mathcal{A}$ but $\cap_{n=1}^{\infty} A_n \notin \mathcal{B}$ since $(1 \oplus 2) \oplus 3 = 11 < 12 = 1 \oplus (2 \oplus 3)$.

The sets $A_n = [0, 1] \cup \big(\cup_{k=1}^{n} (10^k, 10^k + 1/k]\big) \cup \big(\cup_{k=n+1}^{\infty} (10^k, 10^k + 1/n]\big) \cup \{\infty\}$ for all $n \in N$ are elements of $\mathcal{B}$ by (S5). The set $\cap_{n=1}^{\infty} A_n = [0, 1] \cup \big(\cup_{k=1}^{\infty} (10^k, 10^k + 1/k]\big) \cup \{\infty\}$ is obviously an element of $\mathcal{A}$, and by (S5), it is an element of $\mathcal{B}$.

In this section we introduce several sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$ where $A_n \in \mathcal{B}$ for all $n \in N$.

3.1. Sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$

In this subsection we will suppose that $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$.

Theorem 1. *Let $\oplus_n$ be the addition on $A_n \in \mathcal{B}$ for all $n \in N$, and let $\oplus$ be the addition on $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$. Write*

(P1) *For all $x, y \in \cap_{n=1}^{\infty} A_n$, if $x + y \notin \cap_{n=1}^{\infty} A_n$ then there exists $m \in N$ such that $x \oplus y = x \oplus_n y$ for all $n \ge m$, $n \in N$.*

If condition (P1) is satisfied then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$.

P r o o f. Denote $\cap_{n=1}^{\infty} A_n$ by A. Observe that for all $x, y \in A$, if $x + y \in A$ then $(x, y, x + y \in A_n)$, $x \oplus y = x + y = x \oplus_n y$ for all $n \in N$. With respect to this observation condition (P1) is satisfied if and only if the following condition

(P) For all $x, y \in \cap_{n=1}^{\infty} A_n$, there exists $m \in N$ such that $x \oplus y = x \oplus_n y$ for all $n \geq m$, $n \in N$

is satisfied. Suppose that condition (P1) is satisfied. Fix $x, y, z \in A$. By (P), for the pairs $x \oplus y, z \in A$, $x, y \in A$, $x, y \oplus z \in A$ and $y, z \in A$, there exist numbers $m_1, m_2, m_3, m_4 \in N$, respectively, such that for all $n \geq m = \max(m_1, m_2, m_3, m_4)$, $(x \oplus y) \oplus z = (x \oplus y) \oplus_n z = (x \oplus_n y) \oplus_n z$ and $x \oplus (y \oplus z) = x \oplus_n (y \oplus z) = x \oplus_n (y \oplus_n z)$. Choose $n \geq m$, $n \in N$. Since the addition $\oplus_n$ is associative on A_n, we obtain $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ which completes the proof. $\square$

LEMMA 1. *Let $(\mathcal{S}_A, C_A)$ be associated with $A \in \mathcal{A}$ and $(\mathcal{S}_B, C_B)$ be associated with $B \in \mathcal{A}$. If $A \supseteq B$, $A \neq [0, \infty]$, then for all $I \in \mathcal{S}_A$ there exists an interval $J \in \mathcal{S}_B$ such that $I \subseteq J$.*

Proof. Suppose that $A \supseteq B$, $A \neq [0, \infty]$. Fix $I \in \mathcal{S}_A$, $I = [a, b]$. Since $A \neq [0, \infty]$, $a < b$. Choose $x \in [a, b]$, $x \notin A$. Since $A \supseteq B$, $x \notin B$. There exists $J \in \mathcal{S}_B$, $J = [c, d]$ such that $x \in J$. Obviously, $c \leq b$. We will prove that $c \leq a$. If $c = 0$ then $c \leq a$. If $c > 0$ then $c \in \mathrm{Acc}_-(B)$, and since $A \supseteq B$, $c \in \mathrm{Acc}_-(A)$. Since the set $[a, b]$ contains only one element of A, $c \notin (a, b]$ implying $c \leq a$. Similarly, we can prove that $b \leq d$. Hence, $I \subseteq J$ which completes the proof. $\square$

Assuming $A_n \supseteq A_{n+1}$ for all $n \in N$, we obtain the following corollary.

COROLLARY 1. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{B}$ for all $n \in N$ and $\oplus$ be the addition on $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$. Write*

(P2) *For all $x, y \in \cap_{n=1}^{\infty} A_n$, if $x \oplus y \neq x + y$ then there exist $m \in N$ and $I \in \mathcal{S}_m$ such that $x \oplus y, x + y \in I$.*

If condition (P2) *is satisfied and $A_n \supseteq A_{n+1}$ for all $n \in N$, then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$.*

Proof. Denote $\cap_{n=1}^{\infty} A_n$ by A. If $A = [0, \infty]$ ($A_n = [0, \infty]$ for all $n \in N$), then condition (P2) is satisfied, $A_n \supseteq A_{n+1}$ for all $n \in N$ and $A \in \mathcal{B}$.

Let $A \neq [0, \infty]$. Suppose that condition (P2) is satisfied and $A_n \supseteq A_{n+1}$ for all $n \in N$. With respect to Theorem 1 it is sufficient to prove that condition (P1) is satisfied. Let $(\mathcal{S}, C)$ be associated with A. Fix $x, y \in A$ such that $x + y \notin A$. Then there exists an interval $K \in \mathcal{S}$ such that $x + y \in K$ and $x \oplus y = c$, where $K \cap A = \{c\}$. Obviously, $c \neq x + y$. By (P2), there exist $m \in N$ and $J_m \in \mathcal{S}_m$ such that $c, x + y \in J_m$. By Lemma 1, there exists a sequence $\{J_n\}_{n=m}^{\infty}$, $J_n \in \mathcal{S}_n$ such that $J_n \subseteq J_{n+1}$ for all $n \geq m$, $n \in N$. For all $n \geq m$, $n \in N$, obviously $c \in J_n \cap A_n$, and since $J_n \cap A_n$ contains only one element, $J_n \cap A_n = \{c\}$, and therefore $x \oplus_n y = c$. We have proved that $x \oplus y = c = x \oplus_n y$ for all $n \geq m$, $n \in N$ which completes the proof. $\square$

In general, condition (P1) implies condition (P2). In the proof of Corollary 1 we have shown that assuming $A_n \supseteq A_{n+1}$ for all $n \in N$, condition (P1) is equivalent to (P2).

Lemma 2. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{A}$ for all $n \in N$. If $A_n \supseteq A_{n+1}$ and $C_n \subseteq C_{n+1}$ for all $n \in N$, then $\cup_{n=1}^{\infty} C_n \subseteq \cap_{n=1}^{\infty} A_n$.*

P r o o f. Fix $m \in N$. Since $C_m \subseteq C_n \subseteq A_n$ for all $n \geq m$, $n \in N$, and since $A_n \supseteq A_m \supseteq C_m$ for all $n \leq m$, $n \in N$, we have that $C_m \subseteq \cap_{n=1}^{\infty} A_n$ which completes the proof. □

Assuming that $C_n \subseteq C_{n+1}$ for all $n \in N$, we obtain the following result.

Corollary 2. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{B}$ for all $n \in N$, and let $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$. If $A_n \supseteq A_{n+1}$ and $C_n \subseteq C_{n+1}$ for all $n \in N$, then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$.*

P r o o f. Denote $\cap_{n=1}^{\infty} A_n$ by A. Let $\oplus_n$ be the addition on A_n for all $n \in N$, $\oplus$ be the addition on A, and let $(\mathcal{S}, C)$ be associated with A. Suppose that $A_n \supseteq A_{n+1}$ and $C_n \subseteq C_{n+1}$ for all $n \in N$. With respect to Corollary 1 it is sufficient to show that condition (P2) is satistied. Fix $x, y \in A$ such that $x \oplus y \neq x + y$. Then $x + y \notin A$, and consequently there exists $K \in \mathcal{S}$ such that $x + y \in K$ and $x \oplus y = c$, where $K \cap A = \{c\}$. Further, $x + y \notin A_m$ for some $m \in N$, and consequently, there exists $J_m \in \mathcal{S}_m$ such that $x + y \in J_m$ and $x \oplus_m y = c_m$, where $J_m \cap A_m = \{c_m\}$. It remains to prove that $c \in J_m$. By Lemma 2, $c_m \in A$. By Lemma 1, there exists $L \in \mathcal{S}$ such that $J_m \subseteq L$. Obviously, $x + y \in K \cap L$, and since the intervals of $\mathcal{S}$ are pairwise disjoint, $L = K$. Hence, $c_m \in K \cap A$, and since $K \cap A = \{c\}$, we have $c_m = c$ implying $c \in J_m$ which completes the proof. □

In the proof of Corollary 2 we have showed that assuming $A_n \supseteq A_{n+1}$ for all $n \in N$, the condition $C_n \subseteq C_{n+1}$ for all $n \in N$ implies condition (P2).

Lemma 3. *Let $(\mathcal{S}_A, C_A)$ be associated with $A \in \mathcal{A}$, $(\mathcal{S}_B, C_B)$ be associated with $B \in \mathcal{A}$, and let $C_A \subseteq C_B$. Then $A \supseteq B$ if and only if for all $I \in \mathcal{S}_A$ there exists an interval $J \in \mathcal{S}_B$ such that $I \subseteq J$.*

P r o o f. $(\Rightarrow)$ Suppose that $A \supseteq B$. If $A = [0, \infty]$ then the assertion is obviously true. If $A \neq [0, \infty]$ the assertion is true by Lemma 1.
$(\Leftarrow)$ Suppose that for all $I \in \mathcal{S}_A$, there exists $J \in \mathcal{S}_B$ such that $I \subseteq J$. Then $\cup \mathcal{S}_A \subseteq \cup \mathcal{S}_B$, and consequently $A \supseteq A \setminus C_A = (\cup \mathcal{S}_A)^c \supseteq (\cup \mathcal{S}_B)^c = B \setminus C_B$. By assumptions $C_A \subseteq C_B$. Clearly, $C_B = C_A \cup (C_B \setminus C_A)$ and $C_A \subseteq A$. It remains to prove that $C_B \setminus C_A \subseteq A$. Fix $b \in C_B \setminus C_A$. Then $b \in K$ for some $K \in \mathcal{S}_B$ and $K \cap B = \{b\}$. We will prove that

$$K \cap I = \emptyset \qquad \text{for all} \quad I \in \mathcal{S}_A. \tag{4}$$

The proof is by contradiction. Suppose that $K \cap I \neq \emptyset$ for some $I \in \mathcal{S}_A$. Then $I \subseteq J$ for some $J \in \mathcal{S}_B$. Hence, $J \cap K \neq \emptyset$, and since intervals of $\mathcal{S}_B$ are pairwise disjoint, $J = K$. Moreover, I contains just one element a of C_A, and since $C_A \subseteq C_B$, $a \in B$. It follows that $a \in K \cap B = \{b\}$, and so $a = b$ implying $b \in C_A$ contrary to $b \notin C_A$. We have proven (4). Finally, from (4) it follows immediately that $K \subseteq (\cup \mathcal{S}_A)^c = A \setminus C_A$, and consequently $b \in A$ which completes the proof. $\square$

The following corollary is an equivalent formulation of Corollary 2.

Corollary 3. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{B}$ for all $n \in N$, and let $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$. Write*

(P3) *For all $n \in N$, if $I \in \mathcal{S}_n$ then there exists $J \in \mathcal{S}_{n+1}$ such that $I \subseteq J$.*

If condition (P3) *is satisfied and $C_n \subseteq C_{n+1}$ for all $n \in N$, then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$.*

Proof. Assuming that $C_n \subseteq C_{n+1}$ for all $n \in N$, by Lemma 3, condition (P3) is equivalent to $A_n \supseteq A_{n+1}$ for all $n \in N$. The rest follows immediately from Corollary 2. $\square$

We conclude this subsection with one consequence of Corollary 2.

Corollary 4. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{B}$ for all $n \in N$, and let $A_n \supseteq A_{n+1}$ and $C_n \subseteq C_{n+1}$ for all $n \in N$. Then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$ if and only if $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$.*

3.2. Sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$

In this subsection we will assume that $A_n \supseteq A_{n+1}$ and $C_n \subseteq C_{n+1}$ for all $n \in N$, and with respect to Corollary 4 we will try to find sufficient conditions for $\cap_{n=1}^{\infty} A_n \in \mathcal{A}$.

First, we will study the structure of an element $A \in \mathcal{A}$. The following result gives a topological characterization of the range of a function $f \in \mathcal{F}$.

Lemma 4. *Let $A \subseteq [0, \infty]$ and $A \neq [0, \infty]$. If there exist a non-empty countable system $\mathcal{S} = \big\{[b_k, d_k] \subseteq [0, \infty] \mid k \in K\big\} \neq \{[0, \infty]\}$ of pairwise disjoint intervals of a positive length and a non-empty countable set $C = \big\{c_k \in [0, \infty] \mid k \in K\big\}$ such that $[b_k, d_k] \cap C = \{c_k\}$ for all $k \in K$, and*

$$A = \big([0, \infty] \setminus (\cup_{k \in K} [b_k, d_k])\big) \cup \big\{c_k \in [0, \infty] \mid k \in K\big\},$$

then

$$A \in \mathcal{A}.$$

Proof. We will construct a function $f \in \mathcal{F}$ with $\mathrm{Ran}(f) = A$. We only give the main ideas of the proof.

First of all, there exist $\min A = a$, $\max A = b$, and $a < b$. Further, for all $x \in (A \setminus C) \cap (a,b)$,

$$x \in \mathrm{Acc}_-(C) \Leftrightarrow x \in \mathrm{Acc}_-(A^c), \tag{5}$$

$$x \in \mathrm{Acc}_+(C) \Leftrightarrow x \in \mathrm{Acc}_+(A^c). \tag{6}$$

Write $(A \setminus C) \cap (a,b) = B$. Clearly, $B \subseteq A$. Using (5) and (6) yields

$$B = A_{00} \cup A_{01} \cup A_{10} \cup A_{11},$$

where the sets
$A_{00} = \big\{x \in B \mid \exists\, \epsilon > 0, (x-\epsilon, x) \subseteq A,\ (x, x+\epsilon) \subseteq A\big\}$,
$A_{01} = \big\{x \in B \mid \exists\, \epsilon > 0, (x-\epsilon, x) \subseteq A,\ \forall\, \delta > 0 (x, x+\delta) \cap C \neq \emptyset\big\}$,
$A_{10} = \big\{x \in B \mid \forall\, \delta > 0, (x-\delta, x) \cap C \neq \emptyset,\ \exists\, \epsilon > 0, (x, x+\epsilon) \subseteq A\big\}$,
$A_{11} = \big\{x \in B \mid \forall\, \delta > 0, (x-\delta, x) \cap C \neq \emptyset,\ (x, x+\delta) \cap C \neq \emptyset\big\}$
are pairwise disjoint.

Define $g(a) = 0$ and $g(b) = 1$. The set $\big(C \cap (a,b)\big) \cup A_{01} \cup A_{10}$ is countable, and if it is non-empty, we can write $\big(C \cap (a,b)\big) \cup A_{01} \cup A_{10} = \{a_n \mid n \in I\}$, where $I = \{1, \ldots, k\}$ or $I = N$, and $a_i \neq a_j$ for all $i, j \in I$. Define $g(a_1) = 1/2$. Suppose that we have defined $g(a_j)$ for all $j \in \{1, \ldots, n\}$. If $n+1 \in I$, we define $g(a_{n+1})$ as

$$\frac{1}{2}\Big(\sup\big(g(a_j) \mid j \in \{1, \ldots, n\}, a_j < a_{n+1}\big) + \inf\big(g(a_j) \mid j \in \{1, \ldots, n\}, a_{n+1} < a_j\big)\Big),$$

where $\sup(\emptyset) = 0$ and $\inf(\emptyset) = 1$. Denote the set $\{a, b\} \cup C \cup A_{01} \cup A_{10}$ by M. The function $g : M \to [0,1]$ is strictly increasing and possesses the following two properties:

(i) For all $c \in M$, $0 < c$, if $s = \sup\big([0,c) \cap M\big) \notin M$, $0 < s$, then $\sup\big(g(x) \mid x \in [0,c) \cap M\big) = g(c)$.

(ii) For all $c \in M$, $c < \infty$, if $i = \inf\big(M \cap (c, \infty]\big) \notin M$, $i < \infty$, then $\inf\big(g(x) \mid x \in M \cap (c, \infty]\big) = g(c)$,

($\sup\big([0,c) \cap M\big) = 0$, if $[0,c) \cap M = \emptyset$, and $\inf\big(M \cap (c,\infty]\big) = \infty$, if $M \cap (c, \infty] = \emptyset$).

Define the function $h : A_{11} \to [0,1]$ by

$$h(x) = \sup\big(g(t) \mid t \in [0,x) \cap M\big),$$

where $\sup(\emptyset) = 0$, and then put $h(x) = g(x)$ for all $x \in M$. The function $h : A \setminus A_{00} \to [0,1]$ is strictly increasing.

Since the set A_{00} is open in $(0,\infty)$ we can write that $A_{00} = \cup_{l\in L}(a'_l, a''_l)$, where $(a'_i, a''_i) \cap (a'_j, a''_j) = \emptyset$ for all $i, j \in L$. For an arbitrary $l \in L$, we have

$$\begin{aligned} s_l &= \sup\bigl([0,a'_l] \cap (A \setminus A_{00})\bigr) = \max\bigl([0,a'_l] \cap (A \setminus A_{00})\bigr),\\ i_l &= \inf\bigl([a''_l,\infty] \cap (A \setminus A_{00})\bigr) = \min\bigl([a''_l,\infty] \cap (A \setminus A_{00})\bigr),\end{aligned}$$

and

$$h(s_l) < h(i_l).$$

For all $l \in L$, choose a strictly increasing bijection $h_l : (a'_l, a''_l) \to \bigl(h(s_l), h(i_l)\bigr)$, and define $h(x) = h_l(x)$ for all $x \in (a'_l, a''_l)$ and $l \in L$. The function $h : A \to [0,1]$ is strictly increasing.

Finally, define $h(x) = h(c_k)$ for all $x \in [b_k, d_k] \setminus \{c_k\}$ and $k \in K$. The function $h : [0,\infty] \to [0,1]$ is non-decreasing and $h(0) = 0$, $h(1) = 1$. Moreover, the function h is continuous on $[0,\infty]$. Hence, $h\bigl([0,\infty]\bigr) = [0,1]$. Obviously, $h\bigl([0,\infty]\bigr) = h(A)$. It follows that the function $h : A \to [0,1]$ is a strictly increasing bijection, and its inverse $f : [0,1] \to A$ is an element of $\mathcal{F}$ with $\mathrm{Ran}(f) = A$ which completes the proof. □

Theorem 2. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{B}$ for all $n \in N$, and let $A_n \supseteq A_{n+1}$ and $C_n \subseteq C_{n+1}$ for all $n \in N$. Write*

(P4) *For all $\{J_n\}_{n=m}^{\infty}$, $J_n \in \mathcal{S}_n$, if $J_n \subseteq J_{n+1}$ for all $n \ge m$, $n \in N$ then there exists $k \ge m$, $k \in N$ such that $J_k = J_n$ for all $n \ge k$, $n \in N$.*

Then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$ if and only if condition (P4) *is satisfied.*

Proof. Denote $\cap_{n=1}^{\infty} A_n$ by A. With respect to Corollary 4 it is sufficient to prove that $A \in \mathcal{A}$ if and only if condition (P4) is satisfied.
($\Rightarrow$) Suppose that $A \in \mathcal{A}$. Let $(\mathcal{S}, C)$ be associated with A. Fix $\{J_n\}_{n=m}^{\infty}$, $J_n \in \mathcal{S}_n$, $J_n = [a_n, b_n]$ such that $J_n \subseteq J_{n+1}$ for all $n \ge m$, $n \in N$. By Lemma 3, for all J_n, $n \ge m$, there exists $K_n \in \mathcal{S}$ such that $J_n \subseteq K_n$. Since intervals of $\mathcal{S}$ are pairwise disjoint, $K_n = K = [a,b]$ for all $n \ge m$. Obviously,

$$a \le a_{n+1} \le a_n \le b_n \le b_{n+1} \le b.$$

Now, we will prove that there exists $k_1 \ge m$, $k_1 \in N$ such that $b_{k_1} = b$. The proof is by contradiction. Suppose that $b_n < b$ for all $n \ge m$, $n \in N$. The interval $J_m \subseteq K$ contains just one element c of C_m. By Lemma 2, $c \in A$. Hence, $c \in K \cap A$, and since the interval K contains just one element of A, we have $K \cap A = \{c\}$. Obviously, $c \le b_m < b$, and so $b \notin A$. Since $A_n \supseteq A_{n+1}$ for all $n \in N$, there exists $l \ge m$, $l \in N$ such that $b \notin A_l$, and consequently, there exists $I_l \in \mathcal{S}_l$ such that $b \in I_l$. Clearly, $I_l \ne J_l$, and since the intervals of $\mathcal{S}_l$ are pairwise disjoint, $I_l \cap J_l = \emptyset$. By Lemma 3, $I_l \subseteq L$ for some $L \in \mathcal{S}$. Clearly, $L \cap K \ne \emptyset$, and since the intervals of $\mathcal{S}$ are pairwise disjoint, $L = K$. The interval $I_l \subseteq K$ contains just one element d of C_l. By Lemma 2, $d \in A$. Hence, $d \in K \cap A$. Thus,

$c, d \in K \cap A$, $c \neq d$ contrary to $K \cap A = \{c\}$. Similarly, we can prove that there exists $k_2 \geq m$, $k_2 \in N$ such that $a_n = a$. It follows that $a_n = a$ and $b_n = b$ for all $n \geq k$, $n \in N$, where $k = \max(k_1, k_2) \geq m$.
$(\Leftarrow)$ If $A = [0, \infty]$, then condition (P4) is satisfied and $A \in \mathcal{A}$. Let $A \neq [0, \infty]$. Suppose that (P4) is satisfied. First, we will prove that for all $x \notin A$, there exists the uniquely determined interval $[a, b]$, $a < b$ containing x with the following property: there exists $k \in N$ such that $[a, b] \in \mathcal{S}_n$ for all $n \geq k$, $n \in N$. In order to show it, fix $x \notin A$. Then $x \notin A_m$ for some $m \in N$, and consequently, $x \in J_m$ for some $J_m \in \mathcal{S}_m$. By Lemma 3, there exists a sequence $\{J_n\}_{n=m}^{\infty}$ of intervals $J_n \in \mathcal{S}_n$, $J_n = [a_n, b_n]$ such that $J_n \subseteq J_{n+1}$ for all $n \geq m$, $n \in N$. By (P4), there exists $k \geq m$, $k \in N$ such that $J_k = J_n = [a, b]$ for all $n \geq k$, $n \in N$. Obviously, $a < b$ and $x \in [a, b]$.

Now, we will prove that the interval $[a, b]$ is uniquely determined. Suppose that there is an interval $[c, d]$, $c < d$ containing x with the property: there exists $l \in N$ such that $[c, d] \in \mathcal{S}_n$ for all $n \geq l$, $n \in N$. Then $[a, b]$, $[c, d] \in \mathcal{S}_n$ for all $n \geq \max(k, l)$, $n \in N$ implying that $[a, b] = [c, d]$.

Finally, we will prove that the interval $[a, b]$ contains just one element of A. The interval $J_k = [a, b]$ contains just one element e of C_k. By Lemma 2, $e \in A$, and consequently, $e \in [a, b] \cap A$. Further, $[a, b] \cap A \subseteq [a, b] \cap A_k = [a, b] \cap C_k = \{e\}$. Hence, $[a, b] \cap A = \{e\}$.

For all $x \notin A$, denote the interval $[a, b]$ described above by I_x. Write $\mathcal{S} = \big\{I \subseteq [0, \infty] \mid \exists\, x \in [0, \infty] \setminus A, I_x = I\big\}$ and $C = \big\{c \in [0, \infty] \mid \exists\, I \in \mathcal{S}, I \cap A = \{c\}\big\}$. The system $\mathcal{S}$ is non-empty and contains intervals of a positive length. Moreover, for all $I \in \mathcal{S}$, there exists $m \in N$ such that $I \in \mathcal{S}_n$ for all $n \geq m$, $n \in N$. It follows that intervals of $\mathcal{S}$ are pairwise disjoint, and $\mathcal{S} \neq \big\{[0, \infty]\big\}$. Further, for all $I \in \mathcal{S}$, the set $I \cap C$ contains only one element, and $A = \big([0, \infty] \setminus (\cup \mathcal{S})\big) \cup C$. By Lemma 4, $A \in \mathcal{A}$ which completes the proof. $\square$

We conclude this subsection with one simple and useful consequence of Theorem 2.

Corollary 5. *Let $(\mathcal{S}_n, C_n)$ be associated with $A_n \in \mathcal{B}$ for all $n \in N$. If $C_n \subseteq C_{n+1}$ and $\mathcal{S}_n \subseteq S_{n+1}$ for all $n \in N$, then $\cap_{n=1}^{\infty} A_n \in \mathcal{B}$.*

3.3. Examples

The results presented in this subsection extend the construction of the nowhere dense set $A \in \mathcal{B}$ with $0 \in \mathrm{Acc}_+(A) = \big\{x \in [0, \infty] \mid \forall \epsilon > 0, (x, x+\epsilon) \cap A \neq \emptyset\big\}$ introduced in [12].

Corollary 6. *Let $A \in \mathcal{B}$, $\min A = 0$, $\max A = 1$ and let $p \in [1/3, 1)$, $q = (1 - p)/2$ and $r = (1 + p)/2$. Then $q\big(A \cup (A \setminus \{0\} + r/q)\big) \in \mathcal{B}$.*

Proof. It follows from (S3) with $c = r/q$ ($r/q = (1+p)/(1-p) \geq 2$ for all $p \in [1/3, 1)$) and (S2) with $c = q$. $\square$

Starting with $A = [0,1]$ Corollary 6 yields the following sequence of elements of $\mathcal{B}$.

EXAMPLE 1. Let $p \in [1/3, 1)$. Write $q = (1-p)/2$, $r = (1+p)/2$, define the function $f : \{0,1,2\} \to \{0,q,r\}$ by $f(0) = 0$, $f(1) = q$, $f(2) = r$ and the function $g : \{0,1,2\} \to \{q,p\}$ by $g(0) = g(2) = q$ and $g(1) = p$. Write

$$I_{i_1,\dots,i_n} = \left\{ x \in [0,1] \mid \sum_{j=1}^{n} f(i_j)q^{j-1} < x \leq \sum_{j=1}^{n} f(i_j)q^{j-1} + g(i_n)q^{n-1} \right\}, \quad (7)$$

where $i_1, \dots, i_{n-1} \in \{0,2\}$, $i_n \in \{0,1,2\}$, $n \in N$.

For all $n \in N$, define the set $A_{pn} \in \mathcal{A}$ such that A_{pn} consists of 0 and 2^n intervals $I_{i_1,\dots,i_n}$ given by (7), where $i_1, \dots, i_n \in \{0,2\}$. We will prove that $A_{pn} \in \mathcal{B}$ for all $n \in N$ by induction. Obviously, $\min A_{pn} = 0$, $\max A_{pn} = \sum_{j=1}^{n} rq^{j-1} + q^n = 1$ for all $n \in N$. The set $q\big([0,1] \cup ([0,1] \setminus \{0\} + r/q)\big) \in \mathcal{B}$ by Corollary 6. Further, $q\big([0,1] \cup ([0,1] \setminus \{0\} + r/q)\big) = \{0\} \cup I_0 \cup I_2 = A_{p1}$. Hence, $A_{p1} \in \mathcal{B}$.

Suppose that $A_{pn} \in \mathcal{B}$. Then $q\big(A_{pn} \cup (A_{pn} \setminus \{0\} + r/q)\big) \in \mathcal{B}$ by Corollary 6. It remains to show that $q\big(A_{pn} \cup (A_{pn} \setminus \{0\} + r/q)\big) = A_{pn+1}$.

Observe that $x \in A_{pn} \setminus \{0\}$ if and only if $x \in I_{i_1,\dots,i_n}$ for some $i_1, \dots, i_n \in \{0,2\}$, i.e.,

$$\sum_{j=1}^{n} f(i_j)q^{j-1} < x \leq \sum_{j=1}^{n} f(i_j)q^{j-1} + q^n.$$

Further, $y \in q\big(A_{pn} \cup (A_{pn} \setminus \{0\} + r/q)\big) \setminus \{0\}$ if and only if $y = qx$ or $y = q(x + r/q)$ for some $x \in A_{pn} \setminus \{0\}$, i.e.,

$$\sum_{j=1}^{n} f(i_j)q^{j} < qx \leq \sum_{j=1}^{n} f(i_j)q^{j} + q^{n+1}$$

or

$$r + \sum_{j=1}^{n} f(i_j)q^{j} < q(x + r/q) \leq r + \sum_{j=1}^{n} f(i_j)q^{j} + q^{n+1},$$

which is equivalent to $y \in I_{0,i_1,\dots,i_n}$ or $y \in I_{2,i_1,\dots,i_n}$ for some $i_1, \dots, i_n \in \{0,2\}$, i.e., $y \in A_{pn+1} \setminus \{0\}$. Hence, $A_{pn+1} \in \mathcal{B}$.

EXAMPLE 2. Let $p \in [1/3, 1)$, and $\{A_{pn}\}_{n=1}^{\infty}$, be a sequence of the sets defined in Example 1. Write $\mathcal{T}_{p0} = \{[1,\infty]\}$, $\mathcal{T}_{pn} = \{\bar{I}_{i_1,\dots,i_n} \mid i_1, \dots, i_{n-1} \in \{0,2\}, i_n = 1\}$, where $\bar{I}_{i_1,\dots,i_n}$ is the closure of $I_{i_1,\dots,i_n}$, $D_{p0} = \{1\}$, $D_{pn} = \{l(I) \mid I \in \mathcal{T}_{pn}\}$, where $l(I)$ is the left-hand end point of the interval I, and $\mathcal{S}_{pn} = \cup_{j=0}^{n}\mathcal{T}_{pj}$, $C_{pn} = \cup_{j=0}^{n} D_{pj}$.

The pair $(\mathcal{S}_{pn}, C_{pn})$ is associated with $A_{pn} \in \mathcal{A}$ for all $n \in N$. The set $A_{pn} \in \mathcal{B}$ for all $n \in N$ by Example 1. Further, $\mathcal{S}_{pn} \subseteq \mathcal{S}_{pn+1}$ and $C_{pn} \subseteq C_{pn+1}$ for all $n \in N$. Hence, $A_p = \cap_{n=1}^{\infty} A_{pn} \in \mathcal{B}$ by Corollary 5.

Observe that the set $A_p \in \mathcal{B}$ consists of 0 and of all $x \in [0,1]$ such that $x = \sum_{j=1}^{\infty} f(i_j) q^{j-1}$, $\{i_j\}_{j=1}^{\infty}$, $i_j \in \{0,2\}$ with infinite many $i_j \neq 0$. If $p = 1/3$ then the set $A_{1/3}$ is a proper subset of the *Cantor set*. The set $A_{1/3}$ was introduced in [9] for the first time and can be found in [12].

Corollary 7. *Let $A \in \mathcal{B}$, $\min A = 0$, $\max A = 1$, $k \in N$. Then $\frac{1}{2k+1}\big(\{0\} \cup (\cup_{j=0}^{k}(A \setminus \{0\} + 2j))\big) \in \mathcal{B}$.*

Proof. It follows from (S4) with $c = 2$, (S1) with $u = 0$, $v = 2k+1$, and (S2) with $c = 1/(2k+1)$. □

Starting with $A = [0,1]$ Corollary 7 yields the following sequence of elements of $\mathcal{B}$:

Example 3. Let $k \in N$. Write

$$I_{i_1,\dots,i_n} = \Big\{x \in [0,1] \mid \sum_{j=1}^{n} i_j/(2k+1)^j < x \leq \sum_{j=1}^{n} i_j/(2k+1)^j + 1/(2k+1)^n\Big\}, \quad (8)$$

where $i_1, \dots, i_{n-1} \in \{0, 2, \dots, 2k\}$ are even, $i_n \in \{0, 1, \dots, 2k\}$, $n \in N$.

For all $n \in N$, define the set $A_{kn} \in \mathcal{A}$ such that A_{kn} consists of 0 and $(k+1)^n$ intervals $I_{i_1,\dots,i_n}$ given by (8) where $i_1, \dots, i_n \in \{0, 2, \dots, 2k\}$ are even. We will prove that $A_{kn} \in \mathcal{B}$ for all $n \in N$ by induction. Obviously, $\min A_{kn} = 0$ and $\max A_{kn} = \sum_{j=1}^{n} \frac{2k}{(2k+1)^j} + \frac{1}{(2k+1)^n} = 1$ for all $n \in N$. The set $\frac{1}{2k+1}\big(\{0\} \cup (\cup_{j=0}^{k}([0,1] \setminus \{0\} + 2j))\big) \in \mathcal{B}$ by Corollary 7. Further, $\frac{1}{2k+1}\big(\{0\} \cup (\cup_{j=0}^{k}([0,1] \setminus \{0\} + 2j))\big) = \{0\} \cup I_0 \cup I_2 \cup \dots \cup I_{2k} = A_{k1}$. Hence, $A_{k1} \in \mathcal{B}$.

Suppose that $A_{kn} \in \mathcal{B}$. Then $\frac{1}{2k+1}\big(\{0\} \cup (\cup_{j=0}^{k}(A_{kn} \setminus \{0\} + 2j))\big) \in \mathcal{B}$ by Corollary 7. It remains to show that $\frac{1}{2k+1}\big(\{0\} \cup (\cup_{j=0}^{k}(A_{kn} \setminus \{0\} + 2j))\big) = A_{kn+1}$.

Observe that $x \in A_{kn} \setminus \{0\}$ if and only if $x \in I_{i_1,\dots,i_n}$ for some $i_1, \dots, i_n \in \{0, 2, \dots, 2k\}$, i.e.,

$$\sum_{j=1}^{n} \frac{i_j}{(2k+1)^j} < x \leq \sum_{j=1}^{n} \frac{i_j}{(2k+1)^j} + \frac{1}{(2k+1)^n}.$$

Hence, $y \in \frac{1}{2k+1}\big(\{0\} \cup (\cup_{j=0}^{k}(A_{kn} \setminus \{0\} + 2j))\big) \setminus \{0\}$ if and only if $y = (x+2j)/(2k+1)$ for some $x \in A_{kn} \setminus \{0\}$ and $j \in \{0,1,\dots,k\}$, i.e.,

$$\frac{2j}{2k+1} + \sum_{j=1}^{n} \frac{i_j}{(2k+1)^{j+1}} < \frac{x+2j}{2k+1} \le \frac{2j}{(2k+1)} + \sum_{j=1}^{n} \frac{i_j}{(2k+1)^{j+1}} + \frac{1}{(2k+1)^{n+1}}$$

which is equivalent to $y \in I_{2j,i_1,\dots,i_n}$ for some $i_1,\dots,i_n \in \{0,2\dots,2k\}$ and $j \in \{0,1,\dots,k\}$, i.e., $y \in A_{kn+1} \setminus \{0\}$. Hence, $A_{kn+1} \in \mathcal{B}$.

Example 4. Let $k \in N$, and $\{A_{kn}\}_{n=1}^{\infty}$ be a sequence of the sets defined in Example 3. Write

$$\mathcal{T}_{k0} = \big\{[1,\infty]\big\},$$
$$\mathcal{T}_{kn} = \big\{\bar{I}_{i_1,\dots,i_n} \mid i_1,\dots,i_{n-1} \in \{0,2,\dots,2k\}, i_n \in \{1,3,\dots,2k-1\}\big\},$$

where $\bar{I}_{i_1,\dots,i_n}$ is the closure of $I_{i_1,\dots,i_n}$, $D_{k0} = \{1\}$, $D_{kn} = \big\{l(I) \mid I \in \mathcal{T}_{kn}\big\}$, where $l(I)$ is the left-hand end point of the interval I, and $\mathcal{S}_{kn} = \cup_{j=0}^{n}\mathcal{T}_{kj}$, $C_{kn} = \cup_{j=0}^{n} D_{kj}$.

The pair $(\mathcal{S}_{kn}, C_{kn})$ is associated with $A_{kn} \in \mathcal{A}$ for all $n \in N$. The set $A_{kn} \in \mathcal{B}$ for all $n \in N$ by Example 3. Further, $\mathcal{S}_{kn} \subseteq \mathcal{S}_{kn+1}$ and $C_{kn} \subseteq C_{kn+1}$ for all $n \in N$. Hence, $A_k = \cap_{n=1}^{\infty} A_{kn} \in \mathcal{B}$ by Corollary 5.

Observe that the set $A_k \in \mathcal{B}$ which consists of 0 and of all $x \in [0,1]$ such that $x = \sum_{j=1}^{\infty} i_j/(2k+1)^j$, $\{i_j\}_{j=1}^{\infty}$, $i_j \in \{0,2,\dots,2k\}$ with infinite many $i_j \ne 0$. If $k = 1$ and $p = 1/3$, then the set A_1 coincides with the set $A_{1/3}$. Observe the sets $A_p \setminus \{0\}$, $p \in [1/3,1)$ and $A_k \setminus \{0\}$, $k \in N$ are fractals.

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