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NOTE ON DECOMPOSABLE SETS

Jolanta Olko

ABSTRACT. We consider the notion of decomposable hull of a subset of the space of vector measures and give some elementary properties. The relation between decomposability of the set of measure selections of a multimeasure and additivity of multimeasure is studied. We give an example of multimeasure with nonconvex and nondecomposable set of measure selections.

Let (T, Σ) be a measurable space and let X be a real normed space. In the sequel, $\mathcal{M}$ stands for the space of all vector measures defined on Σ with values in X.

Definition 1. We say that $D \subset \mathcal{M}$ is decomposable iff for every $A \in \Sigma$ and $m, n \in D$

$$m\chi_A + n\chi_{T\setminus A} \in D,$$

where the measure $m\chi_A$ is given by $m\chi_A(B) = m(B \cap A)$.

Lemma 1. *Let (T, Σ) be a measurable space, X a normed space. Then*

(1) *$\mathcal{M}$ is decomposable;*

(2) *the intersection of any family of decomposable sets in $\mathcal{M}$ is decomposable; if $\{S_n : n \in \mathbb{N}\}$ is an increasing sequence of decomposable sets in $\mathcal{M}$, then $\bigcup_{n\in\mathbb{N}} S_n$ is decomposable;*

(3) *if $S_1, S_2 \subset \mathcal{M}$ are decomposable sets, then $S_1 + S_2$ is decomposable.*

The above lemma allows us to define a decomposable hull of a set $S \subset \mathcal{M}$ in the following way

$$\operatorname{dec} S = \bigcap \{D : S \subset D, \quad D \text{ is decomposable}\}.$$

It is easily seen that the decomposable hull of S is the smallest decomposable set containing S.

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We say that sets $A_1, \dots, A_k$ form Σ-partition of the space T iff they are mutually disjoint elements of Σ such that $\bigcup_{i=1}^k A_i = T$. The following lemma yields information about elements of the decomposable hull of a set.

Lemma 2. *Let $S \subset \mathcal{M}$. Then*

$$\operatorname{dec} S = \left\{ \sum_{i=1}^k m_i \chi_{A_i} \,:\, k \in \mathbb{N},\ m_i \in S,\ \{A_1, \dots, A_k\} \text{ is } \Sigma\text{-partition of } T \right\}.$$

Proof. Throughout the proof R denotes the set on the right-hand side of the above equality. We first show that $\operatorname{dec} S \subset R$. It is clear that $S \subset R$, so it suffices to prove that R is decomposable.

Fix $m, n \in R$ and $C \in \Sigma$. Therefore $m = \sum_{i=1}^k m_i \chi_{A_i}$, $n = \sum_{i=1}^l n_i \chi_{B_i}$, where $m_i, n_i \in S$, $A_1, \dots, A_k$, $B_1, \dots, B_l$ are Σ-partitions of T. Therefore

$$\begin{aligned} m\chi_C + n\chi_{T \setminus C} &= \left(\sum_{i=1}^k m_i \chi_{A_i} \right) \chi_C + \left(\sum_{i=1}^l n_i \chi_{B_i} \right) \chi_{T \setminus C} \\ &= \sum_{i=1}^k m_i \chi_{A_i} \chi_C + \sum_{i=1}^l n_i \chi_{B_i} \chi_{T \setminus C} \\ &= \sum_{i=1}^k m_i \chi_{A_i \cap C} + \sum_{i=1}^l n_i \chi_{B_i \cap (T \setminus C)}. \end{aligned}$$

Observe that the sets $A_1 \cap C, \dots, A_k \cap C$, $B_1 \cap (T \setminus C), \dots, B_l \cap (T \setminus C)$ form Σ-partition of T. Thus the measure $m\chi_C + n\chi_{T \setminus C} \in R$.

Now, we will show that for every $k \in \mathbb{N}$

$$\sum_{i=1}^k m_i \chi_{A_i} \in \operatorname{dec} S \quad \text{for } m_1, \dots, m_k \in S, \quad A_1, \dots, A_k \ \Sigma\text{-partition of } T. \tag{1}$$

The proof is by induction on k. Obviously $m\chi_T \in S \subset \operatorname{dec} S$ for $m \in S$, hence (1) holds for $k = 1$.

Let $k = 2$, $m_1, m_2 \in S$, $A_1, A_2 \in \Sigma$ be such that $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = T$. Therefore by definition the decomposable hull of S the measure $m_1\chi_{A_1} + m_2\chi_{A_2}$ is an element of $\operatorname{dec} S$.

Assuming (1) holds for k, we will prove it for $k+1$. Let $A_1, \dots, A_{k+1}$ be Σ-partition of T and let $m_1, \dots, m_{k+1} \in S$.

Define

$$m = \sum_{i=1}^{k-1} m_i \chi_{A_i} + m_k \chi_{A_k \cup A_{k+1}}.$$

By the induction hypothesis $m \in \operatorname{dec} S$. Since $m_{k+1} \in S \subset \operatorname{dec} S$ and $\operatorname{dec} S$ is decomposable, $m\chi_{A_1 \cup \ldots \cup A_k} + m_{k+1}\chi_{A_{k+1}}$ belongs to $\operatorname{dec} S$. But

$$\sum_{i=1}^{k+1} m_i \chi_{A_i} = \sum_{i=1}^{k} m_i \chi_{A_i} + m_{k+1}\chi_{A_{k+1}} = m\chi_{A_1 \cup \ldots \cup A_k} + m_{k+1}\chi_{A_{k+1}},$$

which completes the proof. □

Here we give some elementary properties of the decomposable hull of a set.

Lemma 3. *Let $S \subset \mathcal{M}$. Then*

(1) $S \subset \operatorname{dec} S$*;*

(2) *S is decomposable if and only if $S = \operatorname{dec} S$.*

(3) *If S is a convex set, then $\operatorname{dec} S$ is convex.*

(4) *If S is decomposable, then $\operatorname{conv} S$ (convex hull of S) is decomposable.*

P r o o f. The first two properties are obvious. To prove the third one, take $m = \sum_{i=1}^{k} m_i \chi_{A_i}$, $n = \sum_{i=1}^{l} n_i \chi_{B_i}$ from $\operatorname{dec} S$ and $\lambda \in [0,1]$. Define sets $C_{ij} = A_i \cap B_j$, $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$, which form Σ-partition of T. Observe that

$$m = \sum_{i=1}^{k} \sum_{j=1}^{l} m_i \chi_{C_{ij}} \quad \text{and} \quad n = \sum_{i=1}^{k} \sum_{j=1}^{l} n_j \chi_{C_{ij}}.$$

Therefore

$$\begin{aligned} \lambda m + (1-\lambda) n &= \lambda \left(\sum_{i=1}^{k} \sum_{j=1}^{l} m_i \chi_{C_{ij}} \right) + (1-\lambda) \sum_{i=1}^{k} \sum_{j=1}^{l} n_j \chi_{C_{ij}} \\ &= \sum_{i=1}^{k} \sum_{j=1}^{l} \big(\lambda m_i + (1-\lambda) n_j \big) \chi_{C_{ij}}. \end{aligned}$$

Since S is convex, $\lambda m_i + (1-\lambda) n_j \in S$ for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$ and finally $\lambda m + (1-\lambda) n \in \operatorname{dec} S$.

Now, assume that S is decomposable. Take $A \in \Sigma$. Let us first prove that

$$m \in S, \quad n \in \operatorname{conv} S \Longrightarrow m\chi_A + n\chi_{T \setminus A} \in \operatorname{conv} S. \tag{2}$$

To do this, consider $m \in S$ and $n = \sum_{i=1}^{k} \lambda_i n_i$, a convex combination of elements of S. Then we have

$$m\chi_A + n\chi_{T\setminus A} = \left(\sum_{i=1}^{k} \lambda_i m \right) \chi_A + \left(\sum_{i=1}^{k} \lambda_i n_i \right) \chi_{T\setminus A} = \sum_{i=1}^{k} \lambda_i \big(m\chi_A + n_i \chi_{T\setminus A} \big).$$

Since $m\chi_A + n_i\chi_{T\setminus A} \in S$, $i \in \{1, \ldots, k\}$, it folows that $m\chi_A + n\chi_{T\setminus A} \in \operatorname{conv} S$.

Now, we will show decomposability of $\operatorname{conv} S$. Fix $A \in \Sigma$. We prove that for every $k \in I\!N$

$$m = \sum_{i=1}^{k} \lambda_i m_i, \quad n = \sum_{i=1}^{k} \mu_i n_i \in \operatorname{conv} S \Longrightarrow m\chi_A + n\chi_{T\setminus A} \in \operatorname{conv} S. \tag{3}$$

The proof runs by induction on k. For $k = 1$ (3) is evident, by decomposability of S. Assuming (3) for k, take $m = \sum_{i=1}^{k+1} \lambda_i m_i, n = \sum_{i=1}^{k+1} \mu_i n_i \in \operatorname{conv} S$. Hence we can write

$$m = \lambda\overline{m} + (1-\lambda)m_{k+1} \quad \text{and} \quad n = \mu\overline{n} + (1-\mu)n_{k+1},$$

where

$$\lambda := \sum_{i=1}^{k} \lambda_i, \quad \overline{m} := \sum_{i=1}^{k} \frac{\lambda_i}{\lambda} m_i \in \operatorname{conv} S,$$

and

$$\mu := \sum_{i=1}^{k} \mu_i, \quad \overline{n} := \sum_{i=1}^{k} \frac{\mu_i}{\mu} n_i \in \operatorname{conv} S.$$

Without loss of generality we can assume that $\lambda \le \mu$. Consequently,

$$\begin{aligned}
m\chi_A + n\chi_{T\setminus A} &= \big(\lambda\overline{m} + (1-\lambda)m_{k+1}\big)\chi_A + \big(\mu\overline{n} + (1-\mu)n_{k+1}\big)\chi_{T\setminus A} \\
&= \lambda\big(\overline{m}\chi_A + \overline{n}\chi_{T\setminus A}\big) + (\mu-\lambda)\overline{n}\chi_{T\setminus A} \\
&\quad + (1-\mu)\big(m_{k+1}\chi_A + n_{k+1}\chi_{T\setminus A}\big) + (\mu-\lambda)m_{k+1}\chi_A \\
&= \lambda\big(\overline{m}\chi_A + \overline{n}\chi_{T\setminus A}\big) + (\mu-\lambda)\big(m_{k+1}\chi_A + \overline{n}\chi_{T\setminus A}\big) \\
&\quad + (1-\mu)\big(m_{k+1}\chi_A + n_{k+1}\chi_{T\setminus A}\big).
\end{aligned}$$

By the induction hypothesis, $\overline{m}\chi_A + \overline{n}\chi_{T\setminus A} \in \operatorname{conv} S$. According to (2), $m_{k+1}\chi_A + \overline{n}\chi_{T\setminus A} \in \operatorname{conv} S$. By the decomposability of S, $m_{k+1}\chi_A + n_{k+1}\chi_{T\setminus A} \in \operatorname{conv} S$. Thus $m\chi_A + n\chi_{T\setminus A}$ is the convex combination of elements of $\operatorname{conv} S$ and the proof is complete. □

The notion of decomposability is related to multimeasures. We recall some usefull definitions. Let $P_f(X)$ be a family of all nonempty closed subsets of X.

Among three different definitions of a multimeasure given in [2] (see also [1]), the most popular and general is the next one.

Definition 2. A multifunction $M : \Sigma \to P_f(X)$ is said to be a weak multimeasure iff for every $x^* \in X^*$ the function $A \mapsto \sigma\big(x^*, M(A)\big) = \sup\big\{x^*(x) : x \in M(A)\big\}$ is an $\mathbb{R} \cup \{+\infty\}$-valued signed measure.

A weak multimeasure will be called a multimeasure for abbreviation.

We say that a vector measure $m \colon \Sigma \to X$ is a measure selection of multimeasure M iff $m(A) \in M(A)$ for every $A \in \Sigma$. The set of all measure selections of M will be denoted by S_M.

EXAMPLE 1. Let $T = [0,1] \subset \mathbb{R}$, Σ be σ-algebra of Lebesgue measurable subsets of T. Let m be Lebesgue measure on Σ. Then the multifunction given by $M(A) = \bigl\{m(A), 2m(A)\bigr\}$, $A \in \Sigma$ is the multimeasure, $S_M = \{m, 2m\}$, which is neither convex nor decomposable. Moreover, $\operatorname{conv} S_M$ is not decomposable as well.

Proof. It is clear that M is multimeasure. We will show that the set S_M consists of two elements. Obviously $m, 2m \in S_M$. Suppose, contrary to our claim, that there exist $n \in S_M$ and $A \in \Sigma$ such that

$$n(T) = m(T) \quad \text{and} \quad n(A) = 2m(A) \neq m(A).$$

Hence

$$\begin{aligned} n(T \setminus A) &= n(T) - n(A) = m(T) - 2m(A) \\ &= m(A) + m(T \setminus A) - 2m(A) \\ &= m(T \setminus A) - m(A) \in \bigl\{m(T \setminus A), 2m(T \setminus A)\bigr\}. \end{aligned}$$

If

$$n(T \setminus A) = m(T \setminus A),$$

then

$$m(A) = 0 = 2m(A), \quad \text{which contradicts our assumption.}$$

If

$$n(T \setminus A) = 2m(T \setminus A), \quad \text{then} \quad m(T) = 0, \quad \text{a contradiction.}$$

The similar arguments are applied to the case $n(T) = 2m(T)$ and $n(A) = m(A)$.

It is easily seen that S_M is not convex. To prove that it is not decomposable, take

$$A = \left[0, \frac{1}{2}\right] \in \Sigma$$

We will show that $\widehat{m} := \chi_A + 2m\chi_{T \setminus A}$ is not an element of S_M. Let

$$B = \left[\tfrac{1}{4}, \tfrac{3}{4}\right].$$

Therefore

$$\begin{aligned} \widehat{m}(B) &= m(B \cap A) + 2m\bigl(B \cap (T \setminus A)\bigr) \\ &= m\left(\left[\tfrac{1}{4}, \tfrac{1}{2}\right]\right) + 2m\left(\left[\tfrac{1}{2}, \tfrac{3}{4}\right]\right) = \tfrac{3}{4}. \end{aligned}$$

Therefore $\widehat{m}(B) \notin M(B) = \bigl\{\tfrac{1}{2}, 1\bigr\}$ and consequently $\widehat{m} \notin S_M$.

Observe that

$$S_M \subset \operatorname{conv} S_M = \big\{\lambda m + (1-\lambda)2m :\ \lambda \in [0,1]\big\} = \big\{(2-\lambda)m :\ \lambda \in [0,1]\big\}.$$

If there existed $\lambda \in [0,1]$ such that $\widehat{m} = (2-\lambda)m$, we would have

$$\tfrac{3}{2} = \widehat{m}\big([0,1]\big) = (2-\lambda)m\big([0,1]\big) = 2-\lambda$$

and

$$\tfrac{1}{2} = \widehat{m}(A) = (2-\lambda)m(A) = \tfrac{1}{2}(2-\lambda),$$

a contradiction.

Convexity of values of a multimeasure implies convexity of the set S_M. Decomposability is related to additivity of M. We say that M is an additive set-multifunction iff $M(A\cup B) = \operatorname{cl}\big(M(A)+M(B)\big)$ for disjoint sets $A, B \in \Sigma$. $\square$

Lemma 4. *Let the multimeasure $M : \Sigma \to P_f(X)$ be an additive multifunction, then S_M is decomposable.*

The proof is straightforward.

According to the above lemma, the multimeasure from Example 1 is not additive set-multifunction.

Definition 3. The multimeasure $M : \Sigma \to P_f(X)$ is said to be rich iff $M(A) = \operatorname{cl}\big\{m(A) : m \in S_M\big\}$ for every $A \in \Sigma$.

If the multimeasure M has closed convex values, then it is additive set-multifunction and consequently S_M is decomposable. Here we give a partial converse of Lemma 4.

Lemma 5. *Let $M : \Sigma \to P_f(X)$ be a rich multimeasure. If S_M is decomposable, then M is additive set-multifunction.*

Proof. Let $A, B \in \Sigma$ be disjoint and let $\varepsilon > 0$.

If $x \in M(A\cup B)$, then there exists $m \in S_M$, such that

$$\|x - m(A\cup B)\| < \varepsilon.$$

Since $m(A\cup B) = m(A) + m(B)$, then

$$\big\|x - \big(m(A)+m(B)\big)\big\| < \varepsilon$$

and consequently $x \in \operatorname{cl}\big(M(A)+M(B)\big)$.

To prove the converse inclusion, take $x \in \operatorname{cl}\big(M(A)+M(B)\big)$. Then there exist

$$m_1, m_2 \in S_M$$

such that

$$\big\|x - \big(m_1(A)+m_2(B)\big)\big\| < \varepsilon.$$

Therefore, by decomposability of S_M, we have

$$\begin{aligned} m_1(A)+m_2(B) &= m_1\big((A\cup B)\cap A\big)+m_2\big((A\cup B)\cap (T\setminus A)\big) \\ &= m_1\chi_A(A\cup B)+m_2\chi_{T\setminus A}(A\cup B) \\ &= \big(m_1\chi_A+m_2\chi_{T\setminus A}\big)(A\cup B)\in M(A\cup B). \end{aligned}$$

Therefore

$$x\in \mathrm{cl}M(A\cup B)=M(A\cup B)$$

and the proof is complete. □

Consider the space $\mathcal{M}$ with the weak pointwise convergence topology $\widehat{\omega}$, which is the weak topology on $\mathcal{M}$ given by functionals of the from

$$u(\cdot)=\sum_{k=1}^{n}\chi_{A_k}(\cdot)x_k^*,$$

where $x_1^*,\dots,x_n^*\in X^*$ and $A_1,\dots,A_n$ is Σ-partition of T. If $m\in\mathcal{M}$, then

$$(m,u)=\sum_{k=1}^{n}\big(x_k^*,m(A_k)\big)$$

(see [2]). Namely a net $m_\alpha \xrightarrow{\widehat{\omega}} m$ iff $(x^*,m_\alpha(A))\to\big(x^*,m(A)\big)$ for every $A\in\Sigma$ and $x^*\in X^*$.

Remark 1. If $S\subset\mathcal{M}$ is decomposable, then $\mathrm{cl}_{\widehat{\omega}}S$ is decomposable as well.

Let M be the multimeasure from Example 1. Define the multimeasure $\mathrm{clconv}\, M$ by

$$(\mathrm{clconv}\, M)(A):=\mathrm{clconv}\, M(A),\quad A\in\Sigma.$$

Therefore $S_{\mathrm{clconv}\, M}$ is convex and decomposable. On the other hand the set $\mathrm{cl}_{\widehat{\omega}}\mathrm{conv}\, S_M=\mathrm{conv}\, S_M$ is not decomposable (see Example 1). Hence Proposition 4.30 in [2] is false. The proof of this proposition strongly depends on the assumptions that M is rich and S_M is decomposable. If M has closed convex values, then M is rich and S_M is decomposable (cf. [2, Theorem 4.17]), however in this case the proposition is obvious.

When using an analogous proof, we have the following version without assumption of convexity of values of multimeasure.

Theorem 1. *Let $M:\Sigma\to P_f(X)$ be a rich multimeasure. Then*

$$S_{\mathrm{clconv}\, M}=\mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M.$$

P r o o f. Obviously $S_{\mathrm{clconv}\,M}$ is $\widehat{\omega}$-closed, decomposable and convex and therefore $\mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M \subset S_{\mathrm{clconv}\,M}$. Suppose that the inclusion is strict. Then there exists $\widehat{m} \in S_{\mathrm{clconv}\,M}$ such that $\widehat{m} \notin \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M$. According to the separation theorem, there exists a functional $\sum_{k=1}^{n} \chi_{A_k} x_k^*$ such that

$$\sup\left\{ \sum_{k=1}^{n} \big(x_k^*,\, m(A_k)\big) : m \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M \right\} < \sum_{k=1}^{n} \big(x_k^*,\, \widehat{m}(A_k)\big).$$

Since

$$\sum_{k=1}^{n} \big(x_k^*,\, \widehat{m}(A_k)\big) \le \sum_{k=1}^{n} \sigma\big(x_k^*,\, \mathrm{clconv}\, M(A_k)\big) = \sum_{k=1}^{n} \sigma\big(x_k^*,\, M(A_k)\big),$$

thus

$$\sup\left\{ \sum_{k=1}^{n} \big(x_k^*,\, m(A_k)\big) : m \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M \right\} < \sum_{k=1}^{n} \sigma\big(x_k^*,\, M(A_k)\big). \tag{4}$$

On the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} \sigma\big(x_k^*, M(A_k)\big) &= \sum_{k=1}^{n} \sup\big\{(x_k^*, x) :\; x \in M(A_k)\big\} \\
&= \sum_{k=1}^{n} \sup\Big\{\big(x_k^*, m(A_k)\big) :\; m \in S_M\Big\} \\
&\le \sum_{k=1}^{n} \sup\Big\{\big(x_k^*, m(A_k)\big) :\; m \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M\Big\} \\
&= \sum_{k=1}^{n} \sup\Big\{\big(x_k^*, m_k(A_k)\big) :\; m_k \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M\Big\} \\
&= \sup\left\{\sum_{k=1}^{n} \big(x_k^*, m_k(A_k)\big) :\; m_k \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M\right\} \\
&= \sup\left\{\sum_{k=1}^{n} \big(x_k^*, m(A_k)\big) : m \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M\right\}
\end{aligned}$$

which contradicts (4).

Note that the last equality relies on the decomposability of $\mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M$. Indeed, for $m_k \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M$

$$\sum_{k=1}^{n} m_k(A_k) = \sum_{k=1}^{n} m(A_k),$$

where $m = \sum_{k=1}^{n} m_k \chi_{A_k} \in \mathrm{cl}_{\widehat{\omega}}\mathrm{decconv}S_M$. □

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Institute of Mathematics
Pedagogical University
Podchorążych 2
PL–30-084 Kraków
POLAND
E-mail: jolko@ap.krakow.pl