ALGEBRAIC PROPERTIES OF THE CLASSES $\mathcal{M} \cap \mathcal{B}_\alpha$

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ABSTRACT. The sums and the maximums of functions from the class $\mathcal{M} \cap \mathcal{B}_\alpha$, for each countable ordinal $\alpha$ are characterized. Moreover, the maximal additive class and the maximal class with respect to maximums for these classes of functions are characterized as well.

1. Preliminaries

The letters $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ stand for the set of positive integers, the set of all integers, and the real line, respectively. If $A \subseteq \mathbb{R}$, then $\text{int} A$, $\text{bd} A$, and $\chi_A$ denote the (Euclidean) interior, the boundary, and the characteristic function of the set $A$, respectively. The word function denotes a real function defined on $\mathbb{R}$. The symbol $\text{sgn}$ denotes the sign function; i.e., $\text{sgn} x = |x|/x$ if $x \neq 0$, and $\text{sgn} 0 = 0$.

For each countable ordinal $\alpha$, let $\mathcal{B}_\alpha$ denote the $\alpha$th Baire class; i.e., $\mathcal{B}_0$ stands for the family of all continuous functions, and for each $\alpha > 0$,

$$\mathcal{B}_\alpha \overset{\text{df}}{=} \left\{ f \in \mathbb{R}^\mathbb{R} : (\exists (f_n) \subseteq \bigcup_{\beta < \alpha} \mathcal{B}_\beta) f = \lim_{n \to \infty} f_n \right\}.$$

(We require the above sequence to be pointwise convergent.)

Let $f : \mathbb{R} \to \mathbb{R}$. For each $a \in \mathbb{R}$ we define $[f = a] \overset{\text{df}}{=} \{ x \in \mathbb{R} : f(x) = a \}$. Similarly we define the sets $[f > a]$, $[f < a]$, etc. We say that $f$ is Darboux, if it has the intermediate value property; i.e.,

$$\forall a < b \forall y \in \mathbb{R} \left( (f(a) - y)(f(b) - y) < 0 \Rightarrow (a, b) \cap [f = y] \neq \emptyset \right).$$

We will denote the class of all Darboux functions by $\mathcal{D}$.

Following J. G. Ceder [3], we define $\mathcal{M}$ as the family of all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\forall a < b \ (f(a)f(b) < 0 \Rightarrow (a, b) \cap [f = 0] \neq \emptyset).$$
The goal of this paper is to examine basic algebraic properties of the class \( \mathcal{M} \cap \mathcal{B}_\alpha \), for each countable ordinal \( \alpha \).

2. Addition

**Theorem 2.1.** Let \( f : \mathbb{R} \to \mathbb{R} \). For each ordinal \( \alpha > 1 \), the following are equivalent:

a) there are functions \( g, h \in \mathcal{M} \cap \mathcal{B}_\alpha \) such that \( f = g + h \),

b) there are functions \( g \in \mathcal{D} \cap \mathcal{B}_\alpha \) and \( h \in \mathcal{D} \cap \mathcal{B}_2 \) such that \( f = g + h \),

c) \( f \in \mathcal{B}_\alpha \).

**Proof.** The implication c) \( \Rightarrow \) b) follows by [4, Theorem II.1.2], and the implications b) \( \Rightarrow \) a) and a) \( \Rightarrow \) c) are obvious. \( \square \)

The proof of the next theorem is analogous. (We use [2, Theorem B] instead of [4, Theorem II.1.2].)

**Theorem 2.2.** Let \( f : \mathbb{R} \to \mathbb{R} \). The following are equivalent:

a) there are functions \( g, h \in \mathcal{M} \cap \mathcal{B}_1 \) such that \( f = g + h \),

b) \( f \in \mathcal{B}_1 \).

Recall that the maximal additive class for a family \( \mathcal{F} \subset \mathbb{R}^\mathbb{R} \) is defined as follows:

\[
\mathcal{M}_\alpha(\mathcal{F}) \overset{\text{def}}{=} \left\{ f \in \mathbb{R}^\mathbb{R} : (\forall g \in \mathcal{F}) f + g \in \mathcal{F} \right\}.
\]

It turns out that the maximal additive class for each family \( \mathcal{M} \cap \mathcal{B}_\alpha \) is the smallest possible one.

**Theorem 2.3.** For each ordinal \( \alpha > 0 \), we have \( \mathcal{M}_\alpha(\mathcal{M} \cap \mathcal{B}_\alpha) = \{ \chi_\emptyset \} \).

**Proof.** For each \( g \in \mathcal{M} \cap \mathcal{B}_\alpha \), we have \( g + \chi_\emptyset = g \in \mathcal{M} \cap \mathcal{B}_\alpha \). So, \( \chi_\emptyset \in \mathcal{M}_\alpha(\mathcal{M} \cap \mathcal{B}_\alpha) \).

Now, let \( f \neq \chi_\emptyset \). Then \( f(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R} \). We consider three cases.

**Case 1.** If \( x_0 \) is a point of continuity of \( f \), then there is a \( \delta > 0 \) such that

\[
|f(x) - f(x_0)| < |f(x_0)| \quad \text{whenever} \quad |x - x_0| \leq \delta.
\]

Define \( g(x) \overset{\text{def}}{=} 2|f(x_0)| \cdot \text{sgn}(x - x_0) \). Then clearly \( g \in \mathcal{M} \cap \mathcal{B}_1 \subset \mathcal{M} \cap \mathcal{B}_\alpha \). Moreover,

\[
(f + g)(x_0 - \delta) < (f(x_0) + |f(x_0)|) - 2|f(x_0)| \leq 0
\]

\[
\leq (f(x_0) - |f(x_0)|) + 2|f(x_0)| < (f + g)(x_0 + \delta)
\]

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and \( f + g \neq 0 \) on \([x-\delta, x+\delta]\). Consequently, \( f + g \notin \mathcal{M} \) and \( f \notin \mathcal{M}_a(\mathcal{M} \cap \mathcal{B}_\alpha) \).

**Case 2.** If \( f \notin \mathcal{D} \), then there exist \( a < b \) and \( y \in \mathbb{R} \) such that
\[
(f(a) - y)(f(b) - y) < 0 \quad \text{and} \quad (a, b) \cap \{f = y\} = \emptyset.
\]
Define \( g \overset{\text{df}}{=} -y\chi_{\mathbb{R}} \). Then \( g \) is continuous (so \( g \in \mathcal{M} \)), \((f + g)(a)(f + g)(b) < 0\) and \((a, b) \cap \{f + g = 0\} = \emptyset \). So, \( f + g \notin \mathcal{M} \) and \( f \notin \mathcal{M}_a(\mathcal{M} \cap \mathcal{B}_\alpha) \).

**Case 3.** Finally, assume that \( f \) is Darboux and discontinuous at \( x_0 \). Using [6] (if \( \alpha > 1 \)) or [1, Theorem 3.2, p. 14] (if \( \alpha = 1 \)) we can find a function \( g \in \mathcal{D} \cap \mathcal{B}_\alpha \) such that \( f + g \notin \mathcal{D} \). Proceeding as in Case 2., we can find a \( y \in \mathbb{R} \) such that \( f + g - y\chi_{\mathbb{R}} \notin \mathcal{M} \). Clearly \( g - y\chi_{\mathbb{R}} \in \mathcal{D} \cap \mathcal{B}_\alpha \subset \mathcal{M} \cap \mathcal{B}_\alpha \). It follows that \( f \notin \mathcal{M}_a(\mathcal{M} \cap \mathcal{B}_\alpha) \).

\[ \square \]

3. Multiplication

**Theorem 3.1.** For each ordinal \( \alpha \), the class \( \mathcal{M} \cap \mathcal{B}_\alpha \) is closed with respect to multiplication.

**Proof.** Indeed, let \( f, g \in \mathcal{M} \cap \mathcal{B}_\alpha \). Then clearly \( fg \in \mathcal{B}_\alpha \). J. G. Ceder proved that \( \mathcal{M} \) is the class of all finite products of Darboux functions [3]. Hence \( fg \) is also a finite product of Darboux functions, and consequently, \( fg \in \mathcal{M} \). \[ \square \]

From the above theorem we can easily obtain the following corollaries.

**Corollary 3.2.** Let \( f : \mathbb{R} \to \mathbb{R} \). For each ordinal \( \alpha \), the following are equivalent:

a) there are functions \( g, h \in \mathcal{M} \cap \mathcal{B}_\alpha \) such that \( f = gh \),

b) \( f \in \mathcal{M} \cap \mathcal{B}_\alpha \).

Recall that the **maximal multiplicative class** for a family \( \mathcal{F} \subset \mathbb{R}^{\mathbb{R}} \) is defined as follows:

\[
\mathcal{M}_m(\mathcal{F}) \overset{\text{df}}{=} \left\{ f \in \mathbb{R}^{\mathbb{R}} : (\forall g \in \mathcal{F}) \; fg \in \mathcal{F} \right\}.
\]

**Corollary 3.3.** For each ordinal \( \alpha > 0 \), we have \( \mathcal{M}_m(\mathcal{M} \cap \mathcal{B}_\alpha) = \mathcal{M} \cap \mathcal{B}_\alpha \).

**Proof.** Indeed, if \( f \in \mathcal{M}_m(\mathcal{M} \cap \mathcal{B}_\alpha) \), then \( f \chi_{\mathbb{R}} \in \mathcal{M} \cap \mathcal{B}_\alpha \).

On the other hand, if \( f \in \mathcal{M} \cap \mathcal{B}_\alpha \), then by Theorem 3.1, \( fg \in \mathcal{M} \cap \mathcal{B}_\alpha \) for each \( g \in \mathcal{M} \cap \mathcal{B}_\alpha \). It follows that \( f \in \mathcal{M}_m(\mathcal{M} \cap \mathcal{B}_\alpha) \). \[ \square \]
4. Maximums

**Theorem 4.1.** Let \( f : \mathbb{R} \to \mathbb{R} \). For each ordinal \( \alpha > 0 \), the following are equivalent:

a) there are functions \( g, h \in \mathcal{M} \cap \mathcal{B}_\alpha \) such that \( f = \max\{g, h\} \),

b) \( f \in \mathcal{B}_\alpha \) and

\[
(\forall a, b \in \mathbb{R}) \left( f \neq f \implies a, b \in [f < 0] \right).
\]

**Proof.** a) \( \Rightarrow \) b). Let \( g, h \in \mathcal{M} \cap \mathcal{B}_\alpha \) be such that \( f = \max\{g, h\} \). Then clearly \( f \in \mathcal{B}_\alpha \). Take \( a < b \) such that \( f < 0 \) on \( (a, b) \). Since \( g \in \mathcal{M} \) and \( g < 0 \) on \( (a, b) \), we conclude that \( g \leq 0 \) on \( (a, b) \). Analogously, \( h \leq 0 \) on \( (a, b) \). Consequently, \( f = \max\{g, h\} \leq 0 \) on \( (a, b) \).

b) \( \Rightarrow \) a). Now assume that \( f \in \mathcal{B}_\alpha \) and \( f \) fulfills condition (1). The proof is different in cases \( \alpha > 1 \) and \( \alpha = 1 \).

Case 1. \( \alpha > 1 \). Let \( \mathcal{I} \) be the family of all open intervals \( I \) with rational end points for which the intersection \( I \cap [f > 0] \) is infinite. Enumerate all the elements of \( \mathcal{I} \) as \( \{I_n : n < N\} \), where \( N \in \mathbb{N} \cup \{\infty\} \). For each \( n < N \) choose two distinct points

\[
a_n, b_n \in I_n \cap [f > 0] \setminus \{(a_k : k < n) \cup \{b_k : k < n\}\}.
\]

Put

\[
A \overset{\text{df}}{=} \{a_n : n < N\}, \quad B \overset{\text{df}}{=} \{b_n : n < N\}.
\]

Define

\[
g \overset{\text{df}}{=} f - f \chi_A, \quad h \overset{\text{df}}{=} f - f \chi_B.
\]

Then clearly \( f = \max\{g, h\} \) on \( \mathbb{R} \). Moreover, since \( A \) and \( B \) are countable, we have \( \chi_A, \chi_B \in \mathcal{B}_2 \) and \( g, h \in \mathcal{B}_\alpha \). To complete the proof we will show that \( g, h \in \mathcal{M} \).

Let \( a < b \) be such that \( g(a)g(b) < 0 \). Let, e.g., \( g(a) < 0 \). (The opposite case is analogous.) Then by definition, \( f(a) = g(a) < 0 < g(b) = f(b) \). If there are \( x \in (a, b) \cap [f > 0] \) and \( \delta > 0 \) such that \((x - \delta, x) \cap [f > 0] = \emptyset \), then by (1), there is an \( x_0 \in (a, b) \cap (x - \delta, x) \cap [f = 0] \). Since \([g = 0] = [f = 0] \cup A \), we conclude that \( x_0 \in (a, b) \cap [g = 0] \neq \emptyset \).

So, assume the opposite case. Then, in particular, \((a, b) \cap [f > 0] \neq \emptyset \), and there is an \( n < N \) such that \( I_n \subset (a, b) \). Consequently, \( a_n \in (a, b) \cap [g = 0] \neq \emptyset \).

We have proved that \( g \in \mathcal{M} \). Analogously we can prove that \( h \in \mathcal{M} \).

Case 2. \( \alpha = 1 \). First we construct two disjoint isolated sets \( A, B \subset \text{int}[f > 0] \) such that if

\[
\text{int}((a, b) \cap [f > 0]) \neq \emptyset \quad \text{and} \quad [a, b] \not\subset [f > 0], \quad \text{for all} \quad a, b \in \mathbb{R},
\]

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then
\[(a, b) \cap A \neq \emptyset \quad \text{and} \quad (a, b) \cap B \neq \emptyset.\]

Let \(\{I_n: n < N\}\), where \(N \in \mathbb{N} \cup \{\infty\}\), be the family of all connected components of \(\text{int}[f > 0]\). For each \(n < N\) let \((a_{n,z}, z \in \mathbb{Z})\) be a strictly increasing sequence with limit points \(\inf I_n\) and \(\sup I_n\). Define
\[A \overset{\text{def}}{=} \{a_{n,2z}: n < N, z \in \mathbb{Z}\}, \quad B \overset{\text{def}}{=} \{a_{n,2z+1}: n < N, z \in \mathbb{Z}\}.
\]
One can easily verify that the sets \(A\) and \(B\) fulfill condition (2).

Define
\[g \overset{\text{def}}{=} f - f\chi_A, \quad h \overset{\text{def}}{=} f - f\chi_B.\]
Then clearly \(f = \max\{g, h\}\) on \(\mathbb{R}\). Moreover, since both \(A\) and \(B\) are \(F_\sigma\) and \(G_\delta\) sets, we have \(\chi_A, \chi_B \in \mathcal{B}_1\) and \(g, h \in \mathcal{B}_1\). To complete the proof we will show that \(g, h \in \mathcal{M}\).

Let \(a < b\) be such that \(g(a)g(b) < 0\). Let, e.g., \(g(a) < 0\). (The other case is analogous.) Then \(f(a) = g(a) < 0 < g(b) = f(b)\). If \(\text{int}((a, b) \cap [f > 0]) \neq \emptyset\), then by (2),
\[(a, b) \cap [g = 0] \supset (a, b) \cap A \neq \emptyset.\]
So, assume that
\[\text{int}((a, b) \cap [f > 0]) = \emptyset.\] (3)

Put
\[P \overset{\text{def}}{=} \text{bd}([a, b] \cap [f < 0]) \neq \emptyset,\]
and let \(x_0 \in P\) be a continuity point of \(f\mid P\). Since \((a, b) \not\subset [f > 0]\), we can assume that \(x_0 > a\).

If \(f(x_0) > 0\), then there is an open interval \(I \ni x_0\) such that \(f > 0\) on \(I \cap P\). Since \(f(a) < 0\), using (3) we can find an \(x_1 \in [a, b] \cap I \cap P\) which is isolated from the left in \([a, b]\). But then \(x_1 \in [f > 0]\) and \(x_1\) is an end point of some connected component of \(\text{int}[f < 0]\), contrary to (1).

If \(f(x_0) < 0\), then there is an open interval \(I \ni x\) such that \(f < 0\) on \(I \cap P\). By (3), we conclude that \(x_0 \in (a, b) \cap I \subset \text{int}[f < 0]\) and \(x_0 \notin \text{bd} I\), a contradiction.

It follows that \(f(x_0) = 0\). Since \([g = 0] = [f = 0] \cup A\), we conclude that \(x_0 \in (a, b) \cap [g = 0] \neq \emptyset\).

We have proved that \(g \in \mathcal{M}\). Analogously we can prove that \(h \in \mathcal{M}\). \(\square\)

For any family \(\mathcal{F} \subset \mathbb{R}\), we define \(\mathcal{L}(\mathcal{F})\) to be the smallest lattice of functions (i.e., a family of functions which is closed both with respect to maximums and with respect to minimums) containing \(\mathcal{F}\).

**Theorem 4.2.** For each ordinal \(\alpha\), we have \(\mathcal{L}(\mathcal{M} \cap \mathcal{B}_\alpha) = \mathcal{B}_\alpha\).
Proof. By [5, Theorem 1], we have $\mathcal{L}(\mathcal{D} \cap \mathcal{B}_\alpha) = \mathcal{B}_\alpha$. Since we clearly have

$$\mathcal{L}(\mathcal{D} \cap \mathcal{B}_\alpha) \subset \mathcal{L}(\mathcal{M} \cap \mathcal{B}_\alpha) \subset \mathcal{L}(\mathcal{B}_\alpha) = \mathcal{B}_\alpha,$$

our theorem follows. $\Box$

Recall that the maximal class with respect to maximums for a family $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$

is defined as follows:

$$\mathcal{M}_{\text{max}}(\mathcal{F}) \overset{\text{df}}{=} \left\{ f \in \mathbb{R}^\mathbb{R} : (\forall g \in \mathcal{F}) \, \max\{ f, g \} \in \mathcal{F} \right\}.$$

**Theorem 4.3.** For each ordinal $\alpha$, $\mathcal{M}_{\text{max}}(\mathcal{M} \cap \mathcal{B}_\alpha)$ is the family of all functions $f \in \mathcal{M} \cap \mathcal{B}_\alpha$ such that

$$[f < 0] \subset \text{int}[f \leq 0].$$

**Proof.** ‘$\subset$’. First, assume that $f \in \mathcal{M}_{\text{max}}(\mathcal{M} \cap \mathcal{B}_\alpha)$. Then $\max\{ f, y \chi_{\mathbb{R}} \} \in \mathcal{B}_\alpha$ for each $y \in \mathbb{R}$. Consequently, since for each $y \in \mathbb{R}$, we have

$$[f > y] = \left[ \max\{ f, y \chi_{\mathbb{R}} \} > y \right],$$

$$[f < y] = \left[ \max\{ f, (y - 1) \chi_{\mathbb{R}} \} < y \right],$$

we can conclude that $f \in \mathcal{B}_\alpha$.

Now, let $a < b$ be such that $f(a)f(b) < 0$. Let, e.g., $f(a) < 0$. Then since

$$\max\{ f, f(a)\chi_{\mathbb{R}} \} \in \mathcal{M},$$

there is an $x_0 \in (a, b) \cap \left[ \max\{ f, f(a)\chi_{\mathbb{R}} \} > 0 \right]$. Hence $f(x_0) = 0$ and $f \in \mathcal{M}$.

Suppose that there is a sequence $(x_n) \subset [f > 0]$ such that $x_n \searrow x_0 \in [f < 0]$. (The case $x_n \nearrow x_0$ is analogous.) Define

$$g(x) \overset{\text{df}}{=} \begin{cases} f(x_0) & \text{if } x \leq x_0, \\ 0 & \text{if } x = x_n, \quad n \in \mathbb{N}, \\ 1 & \text{otherwise}. \end{cases}$$

Then clearly $g \in \mathcal{M} \cap \mathcal{B}_1 \subset \mathcal{M} \cap \mathcal{B}_\alpha$. On the other hand, $\max\{ f, g \}(x_0) = f(x_0) < 0$ and $\max\{ f, g \} > 0$ on $(x_0, \infty)$. Hence $\max\{ f, g \} \notin \mathcal{M}$, contrary to the assumption $f \in \mathcal{M}_{\text{max}}(\mathcal{M} \cap \mathcal{B}_\alpha)$.

‘$\supset$’. Now, assume that $f \in \mathcal{M} \cap \mathcal{B}_\alpha$ fulfills condition (4). Let $g \in \mathcal{M} \cap \mathcal{B}_\alpha$.

Clearly $h \overset{\text{df}}{=} \max\{ f, g \} \in \mathcal{B}_\alpha$. We will show that $h \in \mathcal{M}$.

Let $a, b \in \mathbb{R}$ be such that $a < b$ and $h(a)h(b) < 0$. Let, e.g., $h(a) < 0$. Then $f(a) < 0$ and $g(a) < 0$. We consider the following cases.

Case 1. If $[a, b] \subset [f \leq 0]$, then choose an $x_0 \in [g = 0] \cap (a, b)$ (we use the assumption $g \in \mathcal{M}$ and the relation $g(b) = h(b) > 0$) and observe that $h(x_0) = 0$.

Case 2. Put $c \overset{\text{df}}{=} \inf([a, b] \cap [f > 0])$ otherwise. By (4), we have $c > a$ and $f(c) \geq 0$. 68
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Case 2.a) If $[a, c] \subset [g \leq 0]$, then choose an $x_0 \in [f = 0] \cap (a, c)$ (we use the assumption $f \in \mathcal{M}$ and the inequality $f(b) \geq 0$) and observe that $h(x_0) = 0$.

Case 2.b) If $g(d) > 0$ for some $d \in [a, c]$, then we repeat the argumentation of Case 1 for the interval $[a, d]$.

In any case we conclude that $h \in \mathcal{M}$. It follows that $f \in \mathcal{M}_{\text{max}}(\mathcal{M} \cap \mathcal{B}_\alpha)$.

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