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CONVERGENCE FIELDS OF REGULAR MATRIX TRANSFORMATIONS 2

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ABSTRACT. Let $A = (a_{nk})$ be a regular matrix and $S(A)$ be the set of all sequences $x = (x_k)$ such that the series $\sum_{k=1}^{\infty} a_{nk}x_k$ converges for each $n = 1, 2, \dots$ It is shown that the convergence field $F(A)$ of the matrix A is a σ-porous set in the linear metric space $S(A)$ endowed with the Fréchet metric. This result improves the result of the paper [Ko].

1. Introduction

Let (a_{nk}) $(n, k = 1, 2, \dots)$ be an infinite matrix of real numbers. A sequence $(x_k)_1^{\infty}$ of reals is said to be A-limitable (limitable by method (A)) to a real number t, if $\lim_{n\to\infty} t_n = t$, where

$$t_n = t_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k, \qquad n = 1, 2, \dots$$

If $x = (x_k)$ is A-limitable to the number t, we write $A\text{-}\lim x_k = t$. The symbol $F(A)$ denotes the set of all A-limitable sequences. The set $F(A)$ is called the convergence field of the method (A) or of the matrix transformation A (see [Pe]). The method (A) defined by the matrix A is said to be regular provided $F(A)$ contains all convergent sequences and $\lim_{k\to\infty} x_k = A - \lim x_k$. If the method (A) is regular then the matrix A is called a regular matrix. It is well-known that the method (A) is regular if and only if the matrix A fulfils the following three conditions ([Co, p. 79]; [Pe, p. 8]):

(i) There exists $c > 0$ such that for each $n = 1, 2, \dots$ we have $\sum_{k=1}^{\infty} |a_{nk}| \le c$;

(ii) $\lim_{n\to\infty} a_{nk} = 0$ holds for each fixed positive integer k;

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

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Further, we will deal with the set s of all real sequences and with some of its subsets endowed with the Fréchet metric

$$\varrho(x,y)=\sum_{k=1}^{\infty}2^{-k}|x_k-y_k|(1+|x_k-y_k|)^{-1},\quad x=(x_k)\,,\quad y=(y_k).$$

We will consider them to be a linear metric space. Operations of addition and multiplication by a real number are defined in the natural way.

The notion of porosity has been introduced by L. Zajíček [Za]. It is a suitable tool to describe the small sets in a metric space. Let (Y,σ) be a metric space, $Z\subset Y$. Let $y\in Y$, $\delta>0$ and $B(y,\delta)=\{x\in Y:\sigma(x,y)<\delta\}$. We put $\gamma(y,\delta,Z)=\sup\bigl\{t>0:\text{there is } z\in B(y,\delta) \text{ such that } B(z,t)\subset B(y,\delta) \text{ and } B(z,t)\cap Z=\emptyset\bigr\}$. If such $t>0$ does not exist, we put $\gamma(y,\delta,Z)=0$. The numbers $\underline{p}(y,Z)=\liminf_{\delta\to 0+}\gamma(y,\delta,Z)/\delta$ and $\overline{p}(y,Z)=\limsup_{\delta\to 0+}\gamma(y,\delta,Z)/\delta$ are called lower and upper porosity of the set Z at y, respectively. If we have $\overline{p}(y,Z)>0$ for all $y\in Y$, then Z is said to be porous in Y. Obviously, every set porous in Y is nowhere dense in Y. The set W is said to be σ-porous in Y, if $W=\bigcup_{n=1}^{\infty}Z_n$ and each Z_n is porous in Y.

In [Ko] it is shown that in any space (S,ϱ), $l_\infty\subset S$ (l_∞-bounded sequences), the set $F(A)$ is of the first Baire category in S for every regular matrix A. The aim of the present paper is to show that the above result can be formulated in a stronger form.

Notice that in a linear metric space (S,ϱ) of sequences, i.e. $S\subset s$, the functions $f_k:S\to\mathbb{R}$, $f_k(x)=x_k$, $k=1,2,\dots$, are continuous. It is easy to verify that in (S,ϱ) this holds if and only if the convergence in the sense of the metric ϱ implies the pointwise convergence, i.e., if $x=(x_k)$, $x^{(r)}=\bigl(x_k^{(r)}\bigr)\in S$ and $\varrho\left(x^{(r)},x\right)\to 0$ as $r\to\infty$, then $x_k^{(r)}\to x_k$ as $r\to\infty$ for each $k=1,2,\dots$

2. Results

To each regular matrix $A=(a_{nk})$ we can assigne the set $S(A)\subset s$,

$$S(A)=\left\{x\in s:\exists\, t=(t_n)\,\forall\, n:t_n=\sum_{k=1}^{\infty}a_{nk}x_k\right\},$$

i.e., for every $x\in S(A)$ the series $\sum_{k=1}^{\infty}a_{nk}x_k$ is convergent for each n. Obviously, $\bigl(S(A),\varrho\bigr)$ is a linear metric space for each regular matrix A and $l_\infty\subset S(A)$. In general, $S(A)\neq l_\infty$, and l_∞ is not closed in $S(A)$. The following example shows this.

Example. Let $C = (c_{nk})$ be the Césaro matrix, i.e., $c_{nk} = \frac{1}{n}$ for $k \le n$ and $c_{nk} = 0$ for $k > n$. Obviously, $S(C) = s$. Let $x^{(m)} = \big(x_k^{(m)}\big)$, where $x_k^{(m)} = k$ for $k \le m$ and $x_k^{(m)} = 0$ for $k > m$. Then $x^{(m)} \in l_\infty$ for each m, and $\varrho(x^{(m)}, x) \to 0$ as $m \to \infty$, where $x = (x_k)$, $x_k = k$. Hence $x \in S(C) \setminus l_\infty$.

Further, we will use the following auxiliary result.

Lemma. *Let $A = (a_{nk})$ be a regular matrix. Then for each positive integer Q there exist $M > Q$, $u > Q$ and $v > Q$, such that $a_{uM} \neq a_{vM}$.*

Proof. Suppose that there is Q such that $a_{uM} = a_{vM}$ for each $M > Q$, $u > Q$ and $v > Q$. It follows from property (ii) of a regular matrix, that $a_{nk} = 0$ for every $n > Q$ and $k > Q$. The property (ii) implies that for each sufficiently large n we have

$$\sum_{k=1}^{\infty} a_{nk} \le \sum_{k=1}^{Q} |a_{nk}| < \frac{1}{2}.$$

This leads to a contradiction with (iii). □

Theorem. *Let A be a regular matrix. Then the convergence field $F(A)$ of the matrix A is a σ-porous set in $S(A)$.*

Proof. From the definition of $F(A)$ we have

$$\begin{aligned} F(A) &= \Big\{ x \in S(A) : \text{there exists } \lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} x_k \Big\} \\ &= \Big\{ x \in S(A) : \forall (p \ge 1)\, \exists\, (q \ge 1)\, \forall\, (m \ge q)\, \forall\, (n \ge q)\, |t_m - t_n| < \frac{1}{p} \Big\} \\ &= \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{m \ge q} \bigcap_{n \ge q} F_{pqmn}\,, \end{aligned}$$

where

$$F_{pqmn} = \Big\{ x \in S(A) : \Big| \sum_{k=1}^{\infty} (a_{mk} - a_{nk}) x_k \Big| < \frac{1}{p} \Big\}.$$

Put

$$F_{pqmn}(K) = \Big\{ x \in S(A) : \Big| \sum_{k=1}^{K} (a_{mk} - a_{nk}) x_k \Big| \le \frac{1}{p} \Big\}.$$

Then

$$F_{pqmn} \subset \bigcup_{L=1}^{\infty} \bigcap_{K \ge L} F_{pqmn}(K).$$

Due to the σ-porosity of $F(A)$ in $S(A)$ it suffices to show that for a given q there exist $u > q$ and $v > q$ such that $H(L) = \bigcap_{K \geq L} F_{pquv}(K)$ is porous in $S(A)$. Then $F_{pquv} \subset \bigcup_{L=1}^{\infty} H(L)$ is σ-porous in $S(A)$ and also $F(A) = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} F_{pq}$, where $F_{pq} = \bigcap_{m \geq q} \bigcap_{n \geq q} F_{pqmn}$ is σ-porous in $S(A)$.

We choose $y \in S(A)$ and prove $\overline{p}\big(y, H(L)\big) > 0$. First, suppose

$$y = (y_k) \in H(L).$$

Let the positive integers Q, M, u and v have the meaning introduced in Lemma and let $Q > L$. Put $\delta = 1/2^{M-2}$ and define $z = (z_k)$ such that $z_k = y_k$ for $k \neq M$, and $z_M = A + B$, where

$$A = |y_M| + 1 + \left(1 + \frac{1}{p}\right) |a_{uM} - a_{vM}|^{-1} \quad \text{and} \quad B = 2c\, |a_{uM} - a_{vM}|^{-1}$$

($c > 0$ is introduced in property (i) of a regular matrix). Obviously, $z_M > 0$. Then

$$\varrho(z, y) = 2^{-M} |y_M - z_M| (1 + |y_M - z_M|)^{-1} < 2^{-M} < \frac{\delta}{2} \qquad \text{and} \qquad z \in B(y, \delta).$$

Using $|z_M| > A$ we have

$$\begin{aligned} \left| \sum_{k=1}^{M} (a_{uk} - a_{vk}) z_k \right| &\geq |a_{uM} - a_{vM}|\, |z_M| - \left| \sum_{k=1}^{M-1} (a_{uk} - a_{vk}) y_k \right| \\ &> |a_{uM} - a_{vM}|\, (|y_M| + 1) + 1 + \frac{1}{p} - \frac{1}{p} > 1 \end{aligned}$$

and $z \notin F_{pquv}(M)$. Since $M > L$ we have $z \notin H(L)$. Put $\eta = 1/2^{M+1}$. The inclusion $B(z, \eta) \subset B(y, \delta)$ follows from the facts that $\eta < \delta/2$ and $\varrho(y, z) < \delta/2$. We show $B(z, \eta) \cap H(L) = \emptyset$. Choose $x \in B(z, \eta)$ and put

$$I_P = \left| \sum_{k=1}^{P} (a_{uk} - a_{vk}) x_k \right|, \qquad P = 1, 2, \dots$$

Then

$$\begin{aligned} I_M &\geq |a_{uM} - a_{vM}|\, |x_M| - I_{M-1} \\ &\geq |a_{uM} - a_{vM}|\, |z_M| - |a_{uM} - a_{vM}|\, |x_M - z_M| - I_{M-1}. \end{aligned} \tag{$*$}$$

Since $y_k = z_k$ for $k, 1 \le k \le M-1$,

$$\begin{aligned} I_{M-1} &= \left| \sum_{k=1}^{M-1} (a_{uk} - a_{vk}) x_k \right| \\ &\le \left| \sum_{k=1}^{M-1} (a_{uk} - a_{vk}) y_k \right| + \sum_{k=1}^{M-1} \bigl(|a_{uk}| + |a_{vk}| \bigr) \max_{1 \le k \le M-1} \bigl\{ |x_k - z_k| \bigr\} \\ &\le \frac{1}{p} + 2c \max_{1 \le k \le M-1} \bigl\{ |x_k - z_k| \bigr\}. \end{aligned}$$

For each $k = 1, 2, \dots, M$ we have

$$2^{-k} |x_k - z_k| \bigl(1 + |x_k - z_k|\bigr)^{-1} \le \varrho(x, z) < 2^{-M-1} \qquad \text{and} \qquad |x_k - z_k| < 1.$$

Hence $I_{M-1} < \frac{1}{p} + 2c$. Consequently, (see(*))

$$I_M > |a_{uM} - a_{vM}| \bigl(|y_M| + 1 \bigr) + 1 + \frac{1}{p} + 2c - |a_{uM} - a_{vM}| - \frac{1}{p} - 2c > 1.$$

Hence $x \notin F_{pquv}(M)$ and $B(z, \eta) \cap H(L) = \emptyset$.
Then

$$\gamma\bigl(y, \delta, H(L)\bigr) \ge \eta = \frac{1}{2^{M+1}}, \qquad \frac{\gamma\bigl(y, \delta, H(L)\bigr)}{\delta} \ge \frac{2^{M-2}}{2^{M+1}} = \frac{1}{8} > 0$$

and $$\overline{p}\bigl(y, H(L)\bigr) = \limsup_{\delta \to 0+} \frac{\gamma\bigl(y, \delta, H(L)\bigr)}{\delta} > 0.$$

Suppose $y \notin H(L)$. Since $H(L)$ is a closed set, $B(y, \delta) \cap H(L) = \emptyset$ holds for all sufficiently small $\delta > 0$, and $\overline{p}\bigl(y, H(L)\bigr) = 1$ in this case.

Consequently $H(L)$ is porous and $F(A)$ is σ-porous in $S(A)$. □

REFERENCES

[Co] COOKE, R. G.: *Infinite Matrices and Sequence Spaces*, Moscow, 1960. (In Russian)

[Ko] KOSTYRKO, P.: *Convergence fields of regular matrix transformations*, Tatra Mt. Math. Publ. **28** (2004), 153–157.

[Pe] PETERSEN, G. M.: *Regular Matrix Transformations*, McGraw-Hill, New York, 1966.

[Za] ZAJÍČEK, L.: *Porosity and σ-porosity*, Real Anal. Exange **13** (1987-88), 314–350.

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