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ON GENERALIZATIONS OF THE NOTION OF CONTINUITY OF MULTIFUNCTIONS IN BITOPOLOGICAL SPACES

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ABSTRACT. The relations between inclusion quasi-continuity (upper inclusion quasi-continuity, lower inclusion quasi-continuity) and somewhat continuity (upper somewhat continuity, lower somewhat continuity) of the multifunction $F : X \to Y$, where X is a bitopological space are considered.

1. Introduction

The concept of bitopological spaces was introduced by J. Kelly in [KE]. Quasi-continuous multivalued functions in topological spaces have been studied by various authors, e.g., T. Neubrunn, J. Ewert, T. Lipski (see [EL], [EN], [NE1] and [NE2]). Quasi-continuous multivalued functions in bitopological spaces have been studied by T. Lipski (see [LI1], [LI2]).

The principal purpose of this article is to generalize the quasi-continuity notion of multifunctions in bitopological spaces in such a way that some properties of this notion can be transferred from topological spaces to bitopological spaces.

2. Preliminaries and basic considerations

Let $F : X \to Y$ first be a multifunction, where X and Y are topological spaces. We know that a multifunction $F : X \to Y$ is said to be upper (lower) quasi-continuous multifunctions at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \subset V$, $\big(F(x_0) \cap V \neq \emptyset\big)$ and for any open neighborhood U of x_0 there exists a non-empty open subset W of U such that $F(x) \subset V$, $\big(F(x) \cap V \neq \emptyset\big)$ for any $x \in W$.

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A multifunction $F : X \to Y$ is said to be quasi-continuous at $x_0 \in X$ if for any open sets $V_1 \subset Y$, $V_2 \subset Y$, $F(x_0) \subset V_1$, $F(x_0) \cap V_2 \neq \emptyset$ and for any open neighborhood U of x_0 there exists a non-empty open subset W of U such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$ (see [EL], [NE1], [PO1], [PO2]).

A multifunction F is upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction if it is upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction at any point of its domain.

Let us consider a bitopological case.

The space X on which two (arbitrary) topologies τ_1, τ_2 are defined is called a *bitopological space* and is denoted by (X, τ_1, τ_2) (see [KE]).

Let X be a bitopological space equipped with two topologies τ_1, τ_2, and Y be a topological space with topology τ. There are two ways of generalizing the notion of quasi-continuity of a multifunction.

I. A multifunction $F : X \to Y$ is said to be τ_1 *upper intersection quasi-continuous with respect to* τ_2 ($\cap uQC^{\tau_1}_{\tau_2}$ for short) at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \subset V$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \subset V$ for any $x \in W$. A multifunction $F : X \to Y$ is said to be τ_1 *lower intersection quasi-continuous with respect to* τ_2 ($\cap lQC^{\tau_1}_{\tau_2}$ for short) at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \cap V \neq \emptyset$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \cap V \neq \emptyset$ for any $x \in W$. A multifunction $F : X \to Y$ is said to be τ_1 *intersection quasi-continuous with respect to* τ_2 ($\cap QC^{\tau_1}_{\tau_2}$ for short) at $x_0 \in X$ if for any open sets $V_1 \subset Y$, $V_2 \subset Y$, $F(x_0) \subset V_1$, $F(x_0) \cap V_2 \neq \emptyset$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$.

A multifunction F is τ_1 upper intersection quasi-continuous, (lower intersection quasi-continuous, intersection quasi-continuous) with respect to τ_2 if it is τ_1 upper intersection quasi-continuous (lower intersection quasi-continuous, intersection quasi-continuous) multifunction with respect to τ_2 at any point of its domain.

A lot of results connected with intersection generalization are contained in Lipski's papers (see [LI1], [LI2]). This kind of generalization is interesting, however some results known for ordinary topological space are not valid in bitopological space. One of these is the following Neubrunn's result (see [NE3]).

A multifunction $F : X \to Y$ is upper (lower) quasi-continuous if and only if its restriction $F|_B$ is upper (lower) somewhat continuous for any set B belonging to a base β of open sets in X.

So, it seems to be a good idea to consider the other way of generalizing the quasi-continuty.

II. A multifunction $F : X \to Y$ is said to be τ_1 *upper inclusion quasi-continuous with respect to* τ_2 ($\subset uQC^{\tau_1}_{\tau_2}$ for short) at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \subset V$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $F(x) \subset V$ for any $x \in W$. A multifunction $F : X \to Y$ is said to be τ_1 *lower inclusion quasi-continuous with respect to* τ_2 ($\subset lQC^{\tau_1}_{\tau_2}$ for short) at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \cap V \neq \emptyset$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $F(x) \cap V \neq \emptyset$ for any $x \in W$. A multifunction $F : X \to Y$ is said to be τ_1 *inclusion quasi-continuous with respect to* τ_2 ($\subset QC^{\tau_1}_{\tau_2}$ for short) at $x_0 \in X$ if for any open sets $V_1 \subset Y$, $V_2 \subset Y$, $F(x_0) \subset V_1$, $F(x_0) \cap V_2 \neq \emptyset$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$.

A multifunction F is τ_1 upper inclusion quasi-continuous (lower inclusion quasi-continuous, inclusion quasi-continuous) with respect to τ_2 if it is τ_1 upper inclusion quasi-continuous (lower inclusion quasi-continuous, inclusion quasi-continuous) multifunction with respect to τ_2 at any point of its domain.

Note that the inclusion versions of quasi continuity have been introduced by M. Matejdes in more general setting in such a way that the condition "there exists a non-empty open set $W \in \tau_1, W \subset U$" is replaced by the condition "there exists a non-empty set $W \in \mathcal{E}, W \subset U$ where $\mathcal{E}$ is not topology but a non-empty family of non-empty subset of X" ([MA1], [MA2]).

Let us mention that not any τ_1 intersection quasi-continuous with respect to τ_2 is τ_1 inclusion quasi-continuous with respect to τ_2 multifunction. Let us consider the following example.

EXAMPLE 1. Let $F : (X, \tau_1, \tau_2) \to \mathbf{R}$, where $\mathbf{R}$ denotes a space of reals with natural topology, be a multifunction defined by the following formula

$$F(x) = \begin{cases} [0, x] & \text{if} \quad x > 0, \\ \{0\} & \text{if} \quad x = 0, \\ [x, 0] & \text{if} \quad x < 0, \end{cases}$$

and let τ_1 denotes a natural topology of the real line $X = \mathbf{R}$, τ_2 denotes a discrete topology. It is easy to verify that F is τ_1 intersection upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction with respect to τ_2, however it is not τ_1 inclusion upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction with respect to τ_2.

We can observe that if X is an ordinary topological space then the above definitions and definitions of upper quasi-continuity, lower quasi-continuity and quasi-continuity are the same.

If $F : X \to Y$, then for $A \subset Y$ we denote

$$F^{+1}(A) = \{x : F(x) \subset A\} \quad \text{and} \quad F^{-1}(A) = \{x : F(x) \cap A \neq \emptyset\}.$$

Let (X, τ_0) be a topological space. A multifunction $F : X \to Y$ is said to be τ_0 upper (lower) somewhat continuous if for any open subset V of topological space Y such that $F^{+1}(V) \neq \emptyset$ $\big(F^{-1}(V) \neq \emptyset\big)$ we have $\mathrm{int}_{\tau_0}\big(F^{+1}(V)\big) \neq \emptyset$ $\big(\mathrm{int}_{\tau_0}(F^{-1}(V)) \neq \emptyset\big)$ (see [NE1]). A multifunction $F : X \to Y$ is said to be τ_0 somewhat continuous if for any open subsets V_1, V_2 of topological space Y such that $F^{+1}(V_1) \cap F^{-1}(V_2) \neq \emptyset$ we have $\mathrm{int}_{\tau_0}\big(F^{+1}(V_1) \cap F^{-1}(V_2)\big) \neq \emptyset$ (see [EN]).

Let A be a non-empty topological subspace of (X, τ_0). It is clear that a restricted multifunction $F|_A : A \to Y$ is τ_0^A upper (lower) somewhat continuous (where τ_0^A denotes the induced topology) if for any open set $V \subset Y$ such that $(F|_A)^{+1}(V) \neq \emptyset$ $\big((F|_A)^{-1}(V) \neq \emptyset\big)$ we have $\mathrm{int}_{\tau_0^A}\big((F|_A)^{+1}(V)\big) \neq \emptyset$ $\big(\mathrm{int}_{\tau_0^A}((F|_A)^{-1}(V)) \neq \emptyset\big)$. In the case of a restricted multifunction $F|_A$ we can also consider upper (lower) somewhat continuity with respect to topology τ_0. A restricted multifunction $F|_A$ is said to be τ_0 upper (lower) somewhat continuous if $\mathrm{int}_{\tau_0}\big(F^{+1}(V) \cap A\big) \neq \emptyset$ $\big(\mathrm{int}_{\tau_0}(F^{-1}(V) \cap A) \neq \emptyset\big)$ for any open set $V \subset Y$ such that $(F|_A)^{+1}(V) \neq \emptyset$ $\big((F|_A)^{+1}(V) \neq \emptyset\big)$. Of course, if $F|_A$ is τ_0 upper (lower) somewhat continuous multifunction then $F|_A$ is τ_0^A upper (lower) somewhat continuous multifunction. If $A \in \tau_0$ then the converse theorem is also true. Analogous considerations can be provided for the case of somewhat continuity (not upper or lower). A restricted multifunction $F|_A$ is said to be τ_0 somewhat continuous if $\mathrm{int}_{\tau_0}\big(F^{+1}(V_1) \cap F^{-1}(V_2) \cap A\big) \neq \emptyset$ for any open sets $V_1, V_2 \subset Y$ such that $F^{+1}(V_1) \cap F^{-1}(V_2) \neq \emptyset$. Note that in case of a single valued map the upper somewhat continuity, lower somewhat continuity and somewhat continuity become the somewhat continuity for a single valued map introduced in [GH]. It is easy to show the following two theorems which are generalizations of Neubrunn's results.

3. Main results

Theorem 1. *Let (X, τ_1, τ_2) be a bitopological space, and Y—a topological space. IF $F : X \to Y$ is a multifunction then the following statements are equivalent:*

(a) *Multifunction F is $\subset uQC^{\tau_1}_{\tau_2}$ $\big(\subset lQC^{\tau_1}_{\tau_2}\big)$ multifunction.*

(b) *Multifunction $F|_U$ is τ_1 upper (lower) somewhat continuous multifunction for any non-empty $U \in \tau_2$.*

(c) *There exists a basis β_2 of topology τ_2 such that $F|_U$ is τ_1 upper (lower) somewhat continuous multifunction for any non-empty $U \in \beta_2$.*

We will prove only the "upper case" of this theorem.

Proof.

(a) $\Rightarrow$ (b). Let F be $\subset uQC^{\tau_1}_{\tau_2}$ multifunction and let $U \in \tau_2$. Let $V \subset Y$ be an open set such that $(F|_U)^{+1}(V) \neq \emptyset$. Therefore, there exists $x_0 \in U$ such that $(F|_U)(x_0) \subset V$. It follows that there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $F(x) \subset V$ for any $x \in W$. It means that $W \subset F^{+1}(V)$. Therefore, $W \subset U \cap F^{+1}(V)$. It implies that $\operatorname{int}_{\tau_1}\big(F^{+1}(V) \cap U\big) \neq \emptyset$. Since V was arbitrary chosen, the multifunction $F|_U$ is τ_1 upper somewhat continuous multifunction.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a). Now, let $x_0 \in X$ and let G be an open subset of Y such that $F(x_0) \subset G$. Let $V \in \tau_2$, $x_0 \in V$. There exists $U \in \beta_2$ such that $x_0 \in U \subset V$. Because of $F(x_0) \subset G$ and $x_0 \in U$ then $F|_U(x_0) \subset G$ and, consequently, $(F|_U)^{+1}(G) \neq \emptyset$. By the assumptions $\operatorname{int}_{\tau_1}\big(F^{+1}(G) \cap U\big) \neq \emptyset$ and therefore there exists a non-empty set $W \in \tau_1$ such that $W \subset F^{+1}(G) \cap U$. So, there exists a non-empty set $W \in \tau_1$ $W \subset V$ such that $F^{+1}(x) \subset G$ for any $x \in W$. We have shown that F is $\subset uQC^{\tau_1}_{\tau_2}$ multifunction at x_0. Since x_0 was arbitrary chosen, the multifunction F is $\subset uQC^{\tau_1}_{\tau_2}$ multifunction. □

Theorem 2. *Let (X, τ_1, τ_2) be a bitopological space, and Y—a topological space, $F : X \to Y$—a multifunction. The following statements are equivalent:*

(a) *Multifunction F is $\subset QC^{\tau_1}_{\tau_2}$ multifunction.*

(b) *Multifunction $F|_U$ is τ_1 somewhat continuous multifunction for any non-empty $U \in \tau_2$.*

(c) *There exists a basis β_2 of topology τ_2 such that $F|_U$ is τ_1 somewhat continuous multifunction for any non-empty $U \in \beta_2$.*

Proof.

(a) $\Rightarrow$ (b). Let F be $\subset QC^{\tau_1}_{\tau_2}$ multifunction and let $U \in \tau_2$. Let V_1, V_2 be open subsets of the space Y such that $(F|_U)^{+1}(V_1) \cap (F|_U)^{-1}(V_2) \neq \emptyset$. Therefore there exists $x_0 \in U$ such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$. Since F is $\subset QC^{\tau_1}_{\tau_2}$ multifunction, it follows that there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$. It means that $W \subset F^{+1}(V_1) \cap F^{-1}(V_2)$. As a result $\operatorname{int}_{\tau_1}\big(F^{+1}(V_1) \cap F^{-1}(V_2)\big) \neq \emptyset$. Since V_1 and V_2 were arbitrary chosen, the multifunction $F|_U$ is τ_1 somewhat continuous multifunction.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a). Let β_2 be a basis of topology τ_2 such that $F|_U$ is τ_1 somewhat continuous multifunction for any $U \in \beta_2$. Let $x_0 \in X$ and let G_1, G_2 be open subsets of Y such that $F(x_0) \subset G_1$ and $F(x_0) \cap G_2 \neq \emptyset$. Let $V \in \tau_2$, $x_0 \in V$. There exists $U \in \beta_2$ such that $x_0 \in U \subset V$. Therefore $(F|_U)^{+1}(G_1) \cap (F|_U)^{-1}(G_2) \neq \emptyset$. The assumptions imply $\mathrm{int}_{\tau_1}\big((F|_U)^{+1}(G_1) \cap (F|_U)^{-1}(G_2) \cap U\big) \neq \emptyset$ and that there exists a non-empty set $W \in \tau_1$ such that $W \subset F^{+1}(G_1) \cap F^{-1}(G_2) \cap U$. So, there exists a non-empty set $W \in \tau_1$ $W \subset V$ such that $F^{+1}(x) \subset G_1$ and $F^{-1}(x) \cap G_2 \neq \emptyset$ for any $x \in W$. We have shown that F is $\subset QC^{\tau_1}_{\tau_2}$ multifunction at x_0. Since x_0 was arbitrary chosen, F is $\subset QC^{\tau_1}_{\tau_2}$ multifunction. □

Remark 1. If we replace the symbol τ_1 with the symbol τ_1^U in Theorem 1 (b), (c) (2 (b), (c)), then implication (c) ⇒ (a) does not hold.

Example 2. Let $F : X \to \mathbf{R}$, where $X = \{1, 2, 3\}$ and $\mathbf{R}$ denotes the space of reals with natural topology, and a multifunction F be defined by the following formula

$$F(x) = \begin{cases} [-1, 0] \cup \{1\} & \text{if } x \in \{1, 2\}, \\ [-2, 0] & \text{if } x = 3. \end{cases}$$

Let τ_1, τ_2 be topologies in X defined as follows.

$$\tau_1 = \big\{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\big\},$$
$$\tau_2 = \big\{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

We can show that $F|_U$ is τ_1^U somewhat continuous (and lower and upper somewhat continuous at the same time) for any non-empty $U \in \tau_2$. This multifunction is not τ_1 somewhat continuous and neither τ_1 lower nor τ_1 upper somewhat continuous multifunction. Indeed, let $U = \{2, 3\}$ and $V_1 = (-\frac{3}{2}, \frac{3}{2})$. Then the interior of the set $(F|_U)^{+1}(V_1) = \{2\}$ is not empty in induced topology τ_1^U but empty in topology τ_1. Let now $V_2 = \big(\frac{1}{2}, \frac{3}{2}\big)$. Then $(F|_U)^{-1}(V_2) = \{2\}$ is again not empty in induced topology τ_1^U but empty in topology τ_1. Let us observe that $(F|_U)^{+1}(V_1) \cap (F|_U)^{-1}(V_2) = \{2\}$. Now, let $x_0 = 2$, then $F(x_0) = [-1, 0] \cup \{1\}$. It implies that $x_0 \in (F|_U)^{+1}(V_1)$, but there does not exist a non-empty set $W \in \tau_1$, $W \subset U$ such that $W \subset (F|_U)^{+1}(V_1)$. A multifunction F is not $\subset uQC^{\tau_1}_{\tau_2}$ at the point x_0. In a similar way we can show that F is not $\subset lQC^{\tau_1}_{\tau_2}$ and is not $\subset QC^{\tau_1}_{\tau_2}$ at the point x_0.

In the case of "intersection" we can prove the two following theorems.

Theorem 3. *Let (X, τ_1, τ_2) be a bitopological space, and Y—a topological space. If a multifunction $F : X \to Y$ is $\cap uQC^{\tau_1}_{\tau_2}$ ($\cap lQC^{\tau_1}_{\tau_2}$) multifunction then $F|_U$ is τ_1^U upper (lower) somewhat continuous multifunction for any non-empty $U \in \tau_2$.*

We will only prove the "lower case" of this theorem.

Proof. Let F be $\cap lQC_{\tau_2}^{\tau_1}$ multifunction and let $U \in \tau_2$. Let $V \subset Y$ be an open set such that $(F|_U)^{-1}(V) \neq \emptyset$. Therefore there exists $x_0 \in U$ such that $F(x_0) \cap V \neq \emptyset$. It follows that there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \cap V \neq \emptyset$ for any $x \in W$. It means that $W \subset F^{-1}(V)$. Therefore, $W \cap U \subset (F|_U)^{-1}(V)$. It implies that $\operatorname{int}_{\tau_1^U}\big((F|U)^{-1}(V)\big) \neq \emptyset$. Since V was arbitrary chosen, the multifunction $F|_U$ is τ_1^U upper somewhat continuous multifunction. □

Theorem 4. *Let (X, τ_1, τ_2) be a bitopological space, and Y—a topological space. If a multifunction $F : X \to Y$ is $\cap QC_{\tau_2}^{\tau_1}$ multifunction then $F|_U$ is τ_1^U somewhat continuous multifunction for any non-empty $U \in \tau_2$.*

Proof. Let F be $\cap QC_{\tau_2}^{\tau_1}$ multifunction and $U \in \tau_2$. Let $V_1, V_2 \subset Y$ be open sets such that $(F|_U)^{+1} \cap (F|_U)^{-1}(V_2) \neq \emptyset$. Therefore there exists $x_0 \in U$ such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$. It follows that there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$. It means that $W \subset F^{+1}(V_1) \cap F^{-1}(V_2)$. Therefore $W \cap U \subset F^{+1}(V_1) \cap F^{-1}(V_2) \cap U$. It implies that $\operatorname{int}_{\tau_1^U}\big((F|U)^{+1}(V_1)\big) \cap \big((F|U)^{-1}(V_2)\big) \neq \emptyset$. Since V_1, V_2 were arbitrary chosen, the multifunction $F|_U$ is τ_1^U somewhat continuous multifunction. □

We cannot replace the symbol τ_1^U with the symbol τ_1 in Theorems 3 and 4. Indeed, a multifunction F from Example 2 is $\cap uQC_{\tau_2}^{\tau_1}$, $\cap lQC_{\tau_2}^{\tau_1}$ and $\cap QC_{\tau_2}^{\tau_1}$ but neither τ_1 upper somewhat continuous nor lower somewhat continuous, and is not somewhat continuous multifunction.

If $F : X \to Y$ is a multifunction such that $F|_U$ is τ_1 somewhat continuous (upper somewhat continuous, lower somewhat continuous) multifunction for any non-empty $U \in \tau_2$, then by Theorem 1 (Theorem 2) a multifunction F is $\subset QC_{\tau_2}^{\tau_1}\big(\subset uQC_{\tau_2}^{\tau_1}, \subset lQC_{\tau_2}^{\tau_1}\big)$ multifunction. It implies that F is $\cap QC_{\tau_2}^{\tau_1}\big(\cap uQC_{\tau_2}^{\tau_1}$, $\cap lQC_{\tau_2}^{\tau_1}\big)$ multifunction. It is wondering whether τ_1^U somewhat continuity (upper somewhat continuity, lower somewhat continuity) of $F|_U$ for any non-empty $U \in \tau_2$ implies $\cap QC_{\tau_2}^{\tau_1}$ continuity ($\cap uQC_{\tau_2}^{\tau_1}$ continuity, $\cap lQC_{\tau_2}^{\tau_1}$ continuity) of F. An example below gives an negative answer. Consequently, the converse theorems of Theorems 3 and 4 are not valid.

Example 3. Let $F : X \to \mathbf{R}$, where $X = \{1, 2\}$ and $\mathbf{R}$ denotes the space of reals with natural topology and a multifunction F be defined by the following formula

$$F(x) = \begin{cases} [-2, 0] & \text{if } \ x = 1, \\ [-1, 1] & \text{if } \ x = 2. \end{cases}$$

Let τ_1, τ_2 be topologies in X defined as follows.

$$\tau_1 = \big\{\emptyset, \{1, 2\}\big\} \quad \text{and} \quad \tau_2 = \big\{\emptyset, \{2\}, \{1, 2\}\big\}.$$

We can show that $F|_U$ is τ_1^U somewhat continuous (and lower and upper somewhat continuous at the same time) for any non-empty $U \in \tau_2$ but neither $\cap uQC_{\tau_2}^{\tau_1}$ nor $\cap lQC_{\tau_2}^{\tau_1}$ (and, of course, not $\cap QC_{\tau_2}^{\tau_1}$). Indeed, let $x_0 = 2$ and $U = \{2\}$. Then $F(x_0) = [-1, 1]$. Let $V_1 = \left(-\frac{3}{2}, \frac{3}{2}\right)$. We can see that there does not exist a set $W \in \tau_1$ such that $F(x) \subset V_1$ for any $x \in W$. Now, let $V_2 = \left(\frac{1}{2}, \frac{3}{2}\right)$. Then there does not exist a set $W \in \tau_1$ such that $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$.

Let us recall the notions of u-density, l-density and density of a collection of subsets.

A collection α of non-empty subsets of Y is said to be u-dense (l-dense) in collection β of non-empty subsets of Y if for any $U \in \beta$ and any open set $V \subset Y$ such that $U \subset V$ ($U \cap V \neq \emptyset$) there exists a set $W \in \alpha$ such that $W \subset V$ ($W \cap V \neq \emptyset$) (see [NE2]).

A collection α of non-empty subsets of Y is said to be dense in collection β of non-empty subsets of Y if for any $U \in \beta$ and any open sets $V_1, V_2 \subset Y$ such that $U \subset V_1$ and $U \cap V_2 \neq \emptyset$ there exists a set $W \in \alpha$ such that $W \subset V_1$ and $W \cap V_2 \neq \emptyset$ (see [EN]).

We know that a multivalued map $F : X \to Y$ is upper (lower) somewhat continuous if and only if for any dense set $S \subset X$ the collection $\{F(s) : s \in S\}$ is u-dense (l-dense) in the collection $\{F(x) : x \in X\}$ (see [NE3]). We also know that a multivalued map $F : X \to Y$ is somewhat continuous if and only if for any dense set $S \subset X$ the collection $\{F(s) : s \in S\}$ is dense in the collection $\{F(x) : x \in X\}$ (see [EN]). We can generalize these results.

Theorem 5. *Let (X, τ_1, τ_2) be a bitopological space, and Y—a topological space. A multifunction $F : X \to Y$ is $\subset uQC_{\tau_2}^{\tau_1}$ $\left(\subset lQC_{\tau_2}^{\tau_1}\right)$ multifunction if and only if for any dense set S in topology τ_1 and for any non-empty set $U \in \tau_2$ the collection $\{F(s) : s \in U \cap S\}$ is u-dense (l-dense) in the collection $\{F(x) : x \in U\}$.*

We will only prove the "lower case" of this theorem.

Proof.

$\Rightarrow$: Let F be $\subset lQC_{\tau_2}^{\tau_1}$ multifunction and let S be a dense subset of X, V an open subset of Y, and let $\emptyset \neq U \in \tau_2$. Let $F(x_0)$ be a set such that $F(x_0) \cap V \neq \emptyset$, where $x_0 \in U$. By the assumptions that there exists a non-empty set $W \in \tau_1$, $W \subset U$ such that $W \subset F^{-1}(V)$. Because of τ_1 density of the set S and relation $W \in \tau_1$, we have $S \cap W = S \cap W \cap U \neq \emptyset$. Let $x_1 \in S \cap W \cap U$. It implies that $F(x_1) \cap V \neq \emptyset$. Moreover, $x_1 \in S \cap U$, then $F(x_1) \in \{F(x) : x \in S \cap U\}$. The l-density of the family $\{F(x) : x \in S \cap U\}$ in the family $\{F(x) : x \in U\}$ was shown.

$\Leftarrow$: Let us assume that for any dense set S in topology τ_1 and for any non-empty set $U \in \tau_2$ the collection $\{F(s) : s \in U \cap S\}$ is l-dense in the collection $\{F(x) : x \in U\}$. Suppose that F is not $\subset lQC^{\tau_1}_{\tau_2}$ at some point x_0. Then there exists an open set $V \subset Y$ such that $F(x_0) \cap V \neq \emptyset$, and there exists a non-empty set $U \in \tau_2$ such that for any non-empty subset $W \in \tau_1$ of U we have $F(x_W) \cap V = \emptyset$ for some element $x_W \in W$. Let $A = (X \setminus U) \cup \{x_W \in W : \emptyset \neq W \in \tau_1 \wedge W \subset U\}$. We will show τ_1-density of the set A. Let $G \in \tau_1$. If $G \cap (X \setminus U) \neq \emptyset$ then, of course, $A \cap G \neq \emptyset$. Now, let $G \subset U$. Then there exists $x_G \in G$ such that $F(x_G) \cap V = \emptyset$. It follows that $x_G \in A$ and $x_G \in G$, and again, $A \cap G \neq \emptyset$. Let us observe that $F(x_0) \in \{F(x) : x \in U\}$ and $F(x_0) \cap V \neq \emptyset$ but $F(x) \cap V = \emptyset$ for any $x \in A \cap U$. □

Theorem 6. *Let (X, τ_1, τ_2) be a bitopological space, and Y—a topological space. Then $F : X \to Y$ is $\subset QC^{\tau_1}_{\tau_2}$ multifunction if and only if for any dense set S in topology τ_1 and for any non-empty set $U \in \tau_2$ the collection $\{F(s) : s \in U \cap S\}$ is dense in the collection $\{F(x) : x \in U\}$.*

Proof.

$\Rightarrow$: Let F be $\subset QC^{\tau_1}_{\tau_2}$ multifunction and let S be a dense subset of X, let V_1, V_2 be an open subsets of Y and let $\emptyset \neq U \in \tau_2$. Let $F(x_0)$ be such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$, where $x_0 \in U$. By the assumptions that there exists a non-empty set $W \in \tau_2$, $W \subset U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$. Because of τ_1 density of the set S and relation $W \in \tau_1$, we have $S \cap W = S \cap W \cap U \neq \emptyset$. Let $x_1 \in S \cap W \cap U$. It implies that $F(x_1) \subset V_1$ and $F(x_1) \cap V_2 \neq \emptyset$. Moreover, $x_1 \in S \cap U$ then $F(x_1) \in \{F(x) : x \in S \cap U\}$. The density of the family $\{F(x) : x \in S \cap U\}$ in the family $\{F(x) : x \in U\}$ was shown.

$\Leftarrow$: Let us assume that for any dense set S in topology τ_1 and for any non-empty set $U \in \tau_2$ the collection $\{F(s) : s \in U \cap S\}$ is dense in the collection $\{F(x) : x \in U\}$. Suppose that F is not $\subset QC^{\tau_1}_{\tau_2}$ at some point x_0. Then there exist open subsets V_1, V_2 of the space Y such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$, and there exists a non-empty set $U \in \tau_2$ such that for any non-empty subset $W \in \tau_1$ of U we have $\neg F(x_W) \subset V_1$ or $F(x_W) \cap V = \emptyset$ for some element $x_W \in W$. Let $A = (X \setminus U) \cup \{x_W \in W : \emptyset \neq W \in \tau_1 \wedge W \subset U\}$. We will show τ_1-density of the set A. Let $G \in \tau_1$. If $G \cap (X \setminus U) \neq \emptyset$ then, of course, $A \cap G \neq \emptyset$. Now, let $G \subset U$. Then there exists $x_G \in G$ such that $\neg F(x_G) \subset V_1$ or $F(x_G) \cap V_2 = \emptyset$. It follows that $x_G \in A$ and $x_G \in G$ and again $A \cap G \neq \emptyset$. Let us observe that $F(x_0) \in \{F(x) : x \in U\}$ and $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$ but $\neg F(x) \subset V_1$ or $F(x) \cap V_2 = \emptyset$ for any $x \in A \cap U$. □

Let us consider "intersection" case.

Let $F : X \to Y$ be a multifunction. By Theorem 6 (Theorem 5) the condition "the collection $\{F(s) : s \in U \cap S\}$ is dense (u-dense, l-dense) in the collection $\{F(x) : x \in U\}$" implies that "F is $\subset QC^{\tau_1}_{\tau_2}$ $\left(\subset uQC^{\tau_1}_{\tau_2}, \subset lQC^{\tau_1}_{\tau_2}\right)$ multifunction" and, consequently, it implies that F is $\cap QC^{\tau_1}_{\tau_2}$ $\left(\cap uQC^{\tau_1}_{\tau_2}, \cap lQC^{\tau_1}_{\tau_2}\right)$ multifunction. But the last condition does not imply that the collection $\{F(s) : s \in U \cap S\}$ is dense (u-dense, l-dense) in the collection $\{F(x) : x \in U\}$. Let us consider the multifunction F from Example 2. We can see that $2^X \setminus \tau_1 = \{\{1,2,3\},\{2,3\},\{1,3\},\{2\},\emptyset\}\}$. Let $S = \{1,2\}$. It is obvious that $\mathrm{cl}_{\tau_1}(\{1,2\}) = \{1,2,3\} = X$. Therefore S is a τ_1-dense subset of X. Let $U = \{2,3\}$. Then $\{F(x) : x \in U\} = \{F(2), F(3)\} = \{[-1,0] \cup \{1\}, [-2,0]\}$ and $\{F(s) : s \in U \cap S\} = \{F(2)\} = \{[-1,0] \cup \{1\}\}$. Let $V_1 = \left(-\frac{5}{2}, \frac{1}{2}\right)$. Then $F(3) \subset V_1$ but $\neg F(2) \subset V_1$. It was shown that the collection $\{F(s) : s \in U \cap S\}$ is not u-dense in the collection $\{F(x) : x \in U\}$. Let $V_2 = \left(-\frac{5}{2}, -\frac{3}{2}\right)$. Then $F(3) \cap V_2 \neq \emptyset$ but $F(2) \cap V_2 = \emptyset$. It was shown that the collection $\{F(s) : s \in U \cap S\}$ is not l-dense in the collection $\{F(x) : x \in U\}$. Of course, the collection $\{F(s) : s \in U \cap S\}$ is not dense in the collection $\{F(x) : x \in U\}$.

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