ON GENERALIZATIONS OF THE NOTION OF CONTINUITY OF MULTIFUNCTIONS IN BITOPOLOGICAL SPACES

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ABSTRACT. The relations between inclusion quasi-continuity (upper inclusion quasi-continuity, lower inclusion quasi-continuity) and somewhat continuity (upper somewhat continuity, lower somewhat continuity) of the multifunction $F: X \rightarrow Y$, where $X$ is a bitopological space are considered.

1. Introduction

The concept of bitopological spaces was introduced by J. C. Kelly in [KE]. Quasi-continuous multivalued functions in topological spaces have been studied by various authors, e.g., T. Neubrunn, J. Ewert, T. Lipski (see [EL], [EN], [NE1] and [NE2]). Quasi-continuous multivalued functions in bitopological spaces have been studied by T. Lipski (see [LI1], [LI2]).

The principal purpose of this article is to generalize the quasi-continuity notion of multifunctions in bitopological spaces in such a way that some properties of this notion can be transferred from topological spaces to bitopological spaces.

2. Preliminaries and basic considerations

Let $F: X \rightarrow Y$ first be a multifunction, where $X$ and $Y$ are topological spaces. We know that a multifunction $F: X \rightarrow Y$ is said to be upper (lower) quasi-continuous multifunctions at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \subset V$, $(F(x_0) \cap V \neq \emptyset)$ and for any open neighborhood $U$ of $x_0$ there exists a non-empty open subset $W$ of $U$ such that $F(x) \subset V$, $(F(x) \cap V \neq \emptyset)$ for any $x \in W$.

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A multifunction $F : X \to Y$ is said to be quasi-continuous at $x_0 \in X$ if for any open sets $V_1 \subset Y$, $V_2 \subset Y$, $F(x_0) \subset V_1$, $F(x_0) \cap V_2 \neq \emptyset$ and for any open neighborhood $U$ of $x_0$ there exists a non-empty open subset $W$ of $U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$ (see [EL], [NE1], [PO1], [PO2]).

A multifunction $F$ is upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction if it is upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction at any point of its domain.

Let us consider a bitopological case.

The space $X$ on which two (arbitrary) topologies $\tau_1$, $\tau_2$ are defined is called a bitopological space and is denoted by $(X, \tau_1, \tau_2)$ (see [KE]).

Let $X$ be a bitopological space equipped with two topologies $\tau_1$, $\tau_2$, and $Y$ be a topological space with topology $\tau$. There are two ways of generalizing the notion of quasi-continuity of a multifunction.

I. A multifunction $F : X \to Y$ is said to be $\tau_1$ upper intersection quasi-continuous with respect to $\tau_2$ (\(\cap uQC^{\tau_2}_{\tau_1}\) for short) at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \subset V$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \subset V$ for any $x \in W$. A multifunction $F : X \to Y$ is said to be $\tau_1$ lower intersection quasi-continuous with respect to $\tau_2$ (\(\cap lQC^{\tau_1}_{\tau_2}\) for short) at $x_0 \in X$ if for any open set $V \subset Y$, $F(x_0) \cap V \neq \emptyset$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \cap V \neq \emptyset$ for any $x \in W$. A multifunction $F : X \to Y$ is said to be $\tau_1$ intersection quasi-continuous with respect to $\tau_2$ (\(\cap QC^{\tau_1}_{\tau_2}\)) for short) at $x_0 \in X$ if for any open sets $V_1 \subset Y$, $V_2 \subset Y$, $F(x_0) \subset V_1$, $F(x_0) \cap V_2 \neq \emptyset$ and for any set $U \in \tau_2$, $x_0 \in U$ there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$.

A multifunction $F$ is $\tau_1$ upper intersection quasi-continuous, (lower intersection quasi-continuous, intersection quasi-continuous) with respect to $\tau_2$ if it is $\tau_1$ upper intersection quasi-continuous (lower intersection quasi-continuous, intersection quasi-continuous) multifunction with respect to $\tau_2$ at any point of its domain.

A lot of results connected with intersection generalization are contained in Lipski’s papers (see [LI1], [LI2]). This kind of generalization is interesting, however some results known for ordinary topological space are not valid in bitopological space. One of these is the following Neubrunn’s result (see [NE3]).

A multifunction $F : X \to Y$ is upper (lower) quasi-continuous if and only if its restriction $F|_B$ is upper (lower) somewhat continuous for any set $B$ belonging to a base $\beta$ of open sets in $X$.

So, it seems to be a good idea to consider the other way of generalizing the quasi-continuity.
II. A multifunction \( F : X \to Y \) is said to be \( \tau_1 \) upper inclusion quasi-continuous with respect to \( \tau_2 \) (\( \subset uQC_{\tau_1}^{\tau_2} \) for short) at \( x_0 \in X \) if for any open set \( V \subset Y \), \( F(x_0) \subset V \) and for any set \( U \in \tau_2 \), \( x_0 \in U \) there exists a non-empty set \( W \in \tau_1 \), \( W \subset U \) such that \( F(x) \subset V \) for any \( x \in W \). A multifunction \( F : X \to Y \) is said to be \( \tau_1 \) lower inclusion quasi-continuous with respect to \( \tau_2 \) (\( \subset lQC_{\tau_1}^{\tau_2} \) for short) at \( x_0 \in X \) if for any open set \( V \subset Y \), \( F(x_0) \cap V \neq \emptyset \) and for any set \( U \in \tau_2 \), \( x_0 \in U \) there exists a non-empty set \( W \in \tau_1 \), \( W \subset U \) such that \( F(x) \cap V \neq \emptyset \) for any \( x \in W \).

A multifunction \( F \) is \( \tau_1 \) upper inclusion quasi-continuous (lower inclusion quasi-continuous, inclusion quasi-continuous) with respect to \( \tau_2 \) if it is \( \tau_1 \) upper inclusion quasi-continuous (lower inclusion quasi-continuous, inclusion quasi-continuous) multifunction with respect to \( \tau_2 \) at any point of its domain.

Note that the inclusion versions of quasi continuity have been introduced by M. Matejdes in more general setting in such a way that the condition “there exists a non-empty open set \( W \in \tau_1 \), \( W \subset U \)” is replaced by the condition “there exists a non-empty set \( W \in \mathcal{E}, W \subset U \) where \( \mathcal{E} \) is not topology but a non-empty family of non-empty subset of \( X \)” ([MA1], [MA2]).

Let us mention that not any \( \tau_1 \) intersection quasi-continuous with respect to \( \tau_2 \) is \( \tau_1 \) inclusion quasi-continuous with respect to \( \tau_2 \) multifunction. Let us consider the following example.

**Example 1.** Let \( F : (X, \tau_1, \tau_2) \to \mathbb{R} \), where \( \mathbb{R} \) denotes a space of reals with natural topology, be a multifunction defined by the following formula

\[
F(x) = \begin{cases} 
[0, x] & \text{if } x > 0, \\
\{0\} & \text{if } x = 0, \\
[x, 0] & \text{if } x < 0,
\end{cases}
\]

and let \( \tau_1 \) denotes a natural topology of the real line \( X = \mathbb{R} \), \( \tau_2 \) denotes a discrete topology. It is easy to verify that \( F \) is \( \tau_1 \) intersection upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction with respect to \( \tau_2 \), however it is not \( \tau_1 \) inclusion upper quasi-continuous (lower quasi-continuous, quasi-continuous) multifunction with respect to \( \tau_2 \).

We can observe that if \( X \) is an ordinary topological space then the above definitions and definitions of upper quasi-continuity, lower quasi-continuity and quasi-continuity are the same.
If $F : X \to Y$, then for $A \subset Y$ we denote
$$F^+1(A) = \{ x : F(x) \subset A \} \quad \text{and} \quad F^{-1}(A) = \{ x : F(x) \cap A \neq \emptyset \}.$$ 

Let $(X, \tau_0)$ be a topological space. A multifunction $F : X \to Y$ is said to be $\tau_0$ upper (lower) somewhat continuous if for any open subset $V$ of topological space $Y$ such that $F^+1(V) \neq \emptyset$ $(F^{-1}(V) \neq \emptyset)$ we have $\text{int}_{\tau_0}(F^+1(V)) \neq \emptyset$ $(\text{int}_{\tau_0}(F^{-1}(V)) \neq \emptyset)$ (see [NE1]). A multifunction $F : X \to Y$ is said to be $\tau_0$ somewhat continuous if for any open subsets $V_1, V_2$ of topological space $Y$ such that $F^+1(V_1) \cap F^{-1}(V_2) \neq \emptyset$ we have $\text{int}_{\tau_0}(F^+1(V_1) \cap F^{-1}(V_2)) \neq \emptyset$ (see [EN]).

Let $A$ be a non-empty topological subspace of $(X, \tau_0)$. It is clear that a restricted multifunction $F|_A : A \to Y$ is $\tau_0^A$ upper (lower) somewhat continuous (where $\tau_0^A$ denotes the induced topology) if for any open set $V \subset Y$ such that $(F|_A)^+1(V) \neq \emptyset$ $((F|_A)^{-1}(V) \neq \emptyset)$ we have $\text{int}_{\tau_0^A}((F|_A)^+1(V)) \neq \emptyset$ $((\text{int}_{\tau_0^A}((F|_A)^{-1}(V)) \neq \emptyset)$. In the case of a restricted multifunction $F|_A$ we can also consider upper (lower) somewhat continuity with respect to topology $\tau_0$. A restricted multifunction $F|_A$ is said to be $\tau_0$ upper (lower) somewhat continuous if $\text{int}_{\tau_0}(F^+1(V) \cap A) \neq \emptyset$ $(\text{int}_{\tau_0}(F^{-1}(V) \cap A) \neq \emptyset)$ for any open set $V \subset Y$ such that $(F|_A)^+1(V) \neq \emptyset$ $((F|_A)^{-1}(V) \neq \emptyset)$. Of course, if $F|_A$ is $\tau_0$ upper (lower) somewhat continuous multifunction then $F|_A$ is $\tau_0^A$ upper (lower) somewhat continuous multifunction. If $A \in \tau_0$ then the converse theorem is also true. Analogous considerations can be provided for the case of somewhat continuity (not upper or lower). A restricted multifunction $F|_A$ is said to be $\tau_0$ somewhat continuous if $\text{int}_{\tau_0}(F^+1(V_1) \cap F^{-1}(V_2) \cap A) \neq \emptyset$ for any open sets $V_1, V_2 \subset Y$ such that $F^+1(V_1) \cap F^{-1}(V_2) \neq \emptyset$. Note that in case of single valued map the upper somewhat continuity, lower somewhat continuity and somewhat continuity become the somewhat continuity for a single valued map introduced in [GH]. It is easy to show the following two theorems which are generalizations of Neubrunn’s results.

### 3. Main results

**Theorem 1.** Let $(X, \tau_1, \tau_2)$ be a bitopological space, and $Y$—a topological space. If $F : X \to Y$ is a multifunction then the following statements are equivalent:

(a) Multifunction $F$ is $\subset uQC_{\tau_2}^+ \subset lQC_{\tau_2}^+$ multifunction.

(b) Multifunction $F|_U$ is $\tau_1$ upper (lower) somewhat continuous multifunction for any non-empty $U \in \tau_2$. 

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We will prove only the "upper case" of this theorem.

\textbf{Proof.}

\((a) \Rightarrow (b).\) Let \(F\) be \(\subset uQC_{\tau_2}^\pi\) multifunction and let \(U \in \tau_2\). Let \(V \subset Y\) be an open set such that \((F|_U)^{+1}(V) \neq \emptyset\). Therefore, there exists \(x_0 \in U\) such that \((F|_U)(x_0) \subset V\). It follows that there exists a non-empty set \(W \in \tau_1\), \(W \subset U\) such that \(F(x) \subset V\) for any \(x \in W\). It means that \(W \subset F^{+1}(V)\). Therefore, \(W \subset U \cap F^{+1}(V)\). It implies that \(\text{int}_{\tau_1}(F^{+1}(V) \cap U) \neq \emptyset\). Since \(V\) was arbitrary chosen, the multifunction \(F|_U\) is \(\tau_1\) upper somewhat continuous multifunction.

\((b) \Rightarrow (c).\) Obvious.

\((c) \Rightarrow (a).\) Now, let \(x_0 \in X\) and let \(G\) be an open subset of \(Y\) such that \(F(x_0) \subset G\). Let \(V \in \tau_2\), \(x_0 \in V\). There exists \(U \in \beta_2\) such that \(x_0 \in U \subset V\). Because of \(F(x_0) \subset G\) and \(x_0 \in U\) then \(F|_U(x_0) \subset G\) and, consequently, \((F|_U)^{+1}(G) \neq \emptyset\).

By the assumptions \(\text{int}_{\tau_1}(F^{+1}(G) \cap U) \neq \emptyset\) and therefore there exists a non-empty set \(W \in \tau_1\) such that \(W \subset F^{+1}(G) \cap U\). So, there exists a non-empty set \(W \in \tau_1\), \(W \subset V\) such that \(F^{+1}(x) \subset G\) for any \(x \in W\). We have shown that \(F\) is \(\subset uQC_{\tau_2}^\pi\) multifunction at \(x_0\). Since \(x_0\) was arbitrary chosen, the multifunction \(F\) is \(\subset uQC_{\tau_2}^\pi\) multifunction.

\(\Box\)

\textbf{Theorem 2.} Let \((X, \tau_1, \tau_2)\) be a bitopological space, and \(Y\)—a topological space, \(F : X \to Y\)—a multifunction. The following statements are equivalent:

(a) Multifunction \(F\) is \(\subset QC_{\tau_2}^\pi\) multifunction.

(b) Multifunction \(F|_U\) is \(\tau_1\) somewhat continuous multifunction for any non-empty \(U \in \tau_2\).

(c) There exists a basis \(\beta_2\) of topology \(\tau_2\) such that \(F|_U\) is \(\tau_1\) somewhat continuous multifunction for any non-empty \(U \in \beta_2\).

\textbf{Proof.}

\((a) \Rightarrow (b).\) Let \(F\) be \(\subset QC_{\tau_2}^\pi\) multifunction and let \(U \in \tau_2\). Let \(V_1, V_2\) be open subsets of the space \(Y\) such that \((F|_U)^{+1}(V_1) \cap (F|_U)^{-1}(V_2) \neq \emptyset\). Therefore there exists \(x_0 \in U\) such that \(F(x_0) \subset V_1\) and \(F(x_0) \cap V_2 \neq \emptyset\). Since \(F\) is \(\subset QC_{\tau_2}^\pi\) multifunction, it follows that there exists a non-empty set \(W \in \tau_1\), \(W \subset U\) such that \(F(x) \subset V_1\) and \(F(x) \cap V_2 \neq \emptyset\) for any \(x \in W\). It means that \(W \subset F^{+1}(V_1) \cap F^{-1}(V_2)\). As a result \(\text{int}_{\tau_1}(F^{+1}(V_1) \cap F^{-1}(V_2)) \neq \emptyset\). Since \(V_1\) and \(V_2\) were arbitrary chosen, the multifunction \(F|_U\) is \(\tau_1\) somewhat continuous multifunction.

\((b) \Rightarrow (c).\) Obvious.

\((c) \Rightarrow (a).\) Let \(\beta_2\) be a basis of topology \(\tau_2\) such that \(F|_U\) is \(\tau_1\) somewhat continuous multifunction for any \(U \in \beta_2\). Let \(x_0 \in X\) and let \(G_1, G_2\) be
open subsets of $Y$ such that $F(x_0) \subset G_1$ and $F(x_0) \cap G_2 \neq \emptyset$. Let $V \in \tau_2$, $x_0 \in V$. There exists $U \in \beta_2$ such that $x_0 \in U \subset V$. Therefore $(F|_U)^+1(G_1) \cap (F|_U)^-(G_2) \neq \emptyset$. The assumptions imply $\int_{\tau_1} ((F|_U)^+1(G_1) \cap (F|_U)^-(G_2) \cap U) \neq \emptyset$ and that there exists a non-empty set $W \in \tau_1$ such that $W \subset (F|_U)^+1(G_1) \cap (F|_U)^-(G_2) \cap U$. So, there exists a non-empty set $W \in \tau_1 W \subset V$ such that $F^+1(x) \subset G_1$ and $F^-(x) \cap G_2 \neq \emptyset$ for any $x \in W$. We have shown that $F$ is $\subset QC_{\tau_2}^{\tau_1}$ multifunction at $x_0$. Since $x_0$ was arbitrary chosen, $F$ is $\subset QC_{\tau_2}^{\tau_1}$ multifunction.

**Remark 1.** If we replace the symbol $\tau_1$ with the symbol $\tau_1^U$ in Theorem 1 (b), (c) (2 (b), (c)), then implication (c) $\Rightarrow$ (a) does not hold.

**Example 2.** Let $F : X \to \mathbf{R}$, where $X = \{1, 2, 3\}$ and $\mathbf{R}$ denotes the space of reals with natural topology, and a multifunction $F$ be defined by the following formula

$$F(x) = \begin{cases} [-1, 0] \cup \{1\} & \text{if } x \in \{1, 2\}, \\ [-2, 0] & \text{if } x = 3. \end{cases}$$

Let $\tau_1$, $\tau_2$ be topologies in $X$ defined as follows.

$$\tau_1 = \{\emptyset, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\},$$

$$\tau_2 = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}.$$

We can show that $F|_U$ is $\tau_1^U$ somewhat continuous (and lower and upper somewhat continuous at the same time) for any non-empty $U \in \tau_2$. This multifunction is not $\tau_1$ somewhat continuous and neither $\tau_1$ lower nor $\tau_1$ upper somewhat continuous multifunction. Indeed, let $U = \{2, 3\}$ and $V_1 = (\frac{-3}{2}, \frac{3}{2})$. Then the interior of the set $(F|_U)^+1(V_1) = \{2\}$ is not empty in induced topology $\tau_1^U$ but empty in topology $\tau_1$. Let now $V_2 = (\frac{1}{2}, \frac{3}{2})$. Then $(F|_U)^-(V_2) = \{2\}$ is again not empty in induced topology $\tau_1^U$ but empty in topology $\tau_1$. Let us observe that $(F|_U)^+1(V_1) \cap (F|_U)^-(V_2) = \{2\}$. Now, let $x_0 = 2$, then $F(x_0) = [-1, 0] \cup \{1\}$. It implies that $x_0 \in (F|_U)^+1(V_1)$, but there does not exist a non-empty set $W \in \tau_1$, $W \subset U$ such that $W \subset (F|_U)^+1(V_1)$. A multifunction $F$ is not $\subset lQC_{\tau_2}^{\tau_1}$ at the point $x_0$. In a similar way we can show that $F$ is not $\subset lQC_{\tau_2}^{\tau_1}$ and is not $\subset QC_{\tau_2}^{\tau_1}$ at the point $x_0$.

In the case of “intersection” we can prove the two following theorems.

**Theorem 3.** Let $(X, \tau_1, \tau_2)$ be a bitopological space, and $Y$—a topological space. If a multifunction $F : X \to Y$ is $\cap uQC_{\tau_2}^{\tau_1}$ multifunction then $F|_U$ is $\tau_1^U$ upper (lower) somewhat continuous multifunction for any non-empty $U \in \tau_2$.

We will only prove the “lower case” of this theorem.
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Proof. Let $F$ be $\cap lQC_{\tau_2}^*$ multifunction and let $U \in \tau_2$. Let $V \subset Y$ be an open set such that $(F|_U)^{-1}(V) \neq \emptyset$. Therefore there exists $x_0 \in U$ such that $F(x_0) \cap V \neq \emptyset$. It follows that there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \cap V \neq \emptyset$ for any $x \in W$. It means that $W \subset F^{-1}(V)$. Therefore, $W \cap U \subset (F|_U)^{-1}(V)$. It implies that $\text{int}_U^\tau ((F|_U)^{-1}(V)) \neq \emptyset$. Since $V$ was arbitrary chosen, the multifunction $F|_U$ is $\tau_1^U$ upper somewhat continuous multifunction. □

Theorem 4. Let $(X, \tau_1, \tau_2)$ be a bitopological space, and $Y$—a topological space. If a multifunction $F : X \rightarrow Y$ is $\cap QC_{\tau_2}^*$ multifunction then $F|_U$ is $\tau_1^U$ somewhat continuous multifunction for any non-empty $U \in \tau_2$.

Proof. Let $F$ be $\cap QC_{\tau_2}^*$ multifunction and $U \in \tau_2$. Let $V_1, V_2 \subset Y$ be open sets such that $(F|_U)^{+1} \cap (F|_U)^{-1}(V_2) \neq \emptyset$. Therefore there exists $x_0 \in U$ such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$. It follows that there exists a set $W \in \tau_1$, $W \cap U \neq \emptyset$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in W$. It means that $W \subset F^{+1}(V_1) \cap F^{-1}(V_2)$. Therefore $W \cap U \subset F^{+1}(V_1) \cap F^{-1}(V_2) \cap U$. It implies that $\text{int}_U^\tau ((F|_U)^{+1}(V_1)) \cap ((F|_U)^{-1}(V_2)) \neq \emptyset$. Since $V_1, V_2$ were arbitrary chosen, the multifunction $F|_U$ is $\tau_1^U$ somewhat continuous multifunction. □

We cannot replace the symbol $\tau_1^U$ with the symbol $\tau_1$ in Theorems 3 and 4. Indeed, a multifunction $F$ from Example 2 is $\cap uQC_{\tau_2}^*$, $\cap lQC_{\tau_2}^*$ and $\cap QC_{\tau_2}^*$ but neither $\tau_1$ upper somewhat continuous nor lower somewhat continuous, and is not somewhat continuous multifunction.

If $F : X \rightarrow Y$ is a multifunction such that $F|_U$ is $\tau_1$ somewhat continuous (upper somewhat continuous, lower somewhat continuous) multifunction for any non-empty $U \in \tau_2$, then by Theorem 1 (Theorem 2) a multifunction $F$ is $\subset QC_{\tau_2}^*$, $\subset lQC_{\tau_2}^*$, $\subset QC_{\tau_2}^*$ multifunction. It implies that $F$ is $\cap QC_{\tau_2}^*$, $\cap lQC_{\tau_2}^*$, $\cap QC_{\tau_2}^*$ multifunction. It is wondering whether $\tau_1^U$ somewhat continuity (upper somewhat continuity, lower somewhat continuity) of $F|_U$ for any non-empty $U \in \tau_2$ implies $\cap QC_{\tau_2}^*$ continuity ($\cap uQC_{\tau_2}^*$ continuity, $\cap lQC_{\tau_2}^*$ continuity) of $F$. An example below gives an negative answer. Consequently, the converse theorems of Theorems 3 and 4 are not valid.

Example 3. Let $F : X \rightarrow \mathbb{R}$, where $X = \{1, 2\}$ and $\mathbb{R}$ denotes the space of reals with natural topology and a multifunction $F$ be defined by the following formula

$$F(x) = \begin{cases} [-2, 0] & \text{if } x = 1, \\ [-1, 1] & \text{if } x = 2. \end{cases}$$

Let $\tau_1, \tau_2$ be topologies in $X$ defined as follows.

$$\tau_1 = \{\emptyset, \{1, 2\}\} \quad \text{and} \quad \tau_2 = \{\emptyset, \{2\}, \{1, 2\}\}.$$
We can show that $F|_{U}$ is $\tau_{1}^{U}$ somewhat continuous (and lower and upper somewhat continuous at the same time) for any non-empty $U \in \tau_{2}$ but neither $\cap_{u} QC_{\tau_{2}}^{1}$ nor $\cap_{l} QC_{\tau_{2}}^{1}$ (and, of course, not $\cap_{C} QC_{\tau_{2}}^{1}$). Indeed, let $x_{0} = 2$ and $U = \{2\}$. Then $F(x_{0}) = [-1, 1]$. Let $V_{1} = (-\frac{3}{2}, \frac{3}{2})$. We can see that there does not exist a set $W \in \tau_{1}$ such that $F(x) \subset V_{1}$ for any $x \in W$. Now, let $V_{2} = (\frac{1}{2}, \frac{3}{2})$. Then there does not exist a set $W \in \tau_{1}$ such that $F(x) \cap V_{2} \neq \emptyset$ for any $x \in W$.

Let us recall the notions of u-density, l-density and density of a collection of subsets.

A collection $\alpha$ of non-empty subsets of $Y$ is said to be u-dense (l-dense) in collection $\beta$ of non-empty subsets of $Y$ if for any $U \in \beta$ and any open set $V \subset Y$ such that $U \subset V$ ($U \cap V \neq \emptyset$) there exists a set $W \in \alpha$ such that $W \subset V$ ($W \cap V \neq \emptyset$) (see [NE2]).

A collection $\alpha$ of non-empty subsets of $Y$ is said to be dense in collection $\beta$ of non-empty subsets of $Y$ if for any $U \in \beta$ and any open sets $V_{1}, V_{2} \subset Y$ such that $U \subset V_{1}$ and $U \cap V_{2} \neq \emptyset$ there exists a set $W \in \alpha$ such that $W \subset V_{1}$ and $W \cap V_{2} \neq \emptyset$ (see [EN]).

We know that a multivalued map $F : X \to Y$ is upper (lower) somewhat continuous if and only if for any dense set $S \subset X$ the collection $\{F(s) : s \in S\}$ is u-dense (l-dense) in the collection $\{F(x) : x \in X\}$ (see [NE3]). We also know that a multivalued map $F : X \to Y$ is somewhat continuous if and only if for any dense set $S \subset X$ the collection $\{F(s) : s \in S\}$ is dense in the collection $\{F(x) : x \in X\}$ (see [EN]). We can generalize these results.

**Theorem 5.** Let $(X, \tau_{1}, \tau_{2})$ be a bitopological space, and $Y$—a topological space. A multifunction $F : X \to Y$ is $\subset u QC_{\tau_{2}}^{1} (\subset l QC_{\tau_{2}}^{1})$ multifunction if and only if for any dense set $S \subset X$ in topology $\tau_{1}$ and for any non-empty set $U \in \tau_{2}$ the collection $\{F(s) : s \in U \cap S\}$ is u-dense (l-dense) in the collection $\{F(x) : x \in U\}$.

We will only prove the “lower case” of this theorem.

**Proof.**

$\Rightarrow$: Let $F$ be $\subset l QC_{\tau_{2}}^{1}$ multifunction and let $S$ be a dense subset of $X$, $V$ an open subset of $Y$, and let $\emptyset \neq U \in \tau_{2}$. Let $F(x_{0})$ be a set such that $F(x_{0}) \cap V \neq \emptyset$, where $x_{0} \in U$. By the assumptions that there exists a non-empty set $W \in \tau_{1}$, $W \subset U$ such that $W \subset F^{-1}(V)$. Because of $\tau_{1}$ density of the set $S$ and relation $W \in \tau_{1}$, we have $S \cap W = S \cap W \cap U \neq \emptyset$. Let $x_{1} \in S \cap W \cap U$. It implies that $F(x_{1}) \cap V \neq \emptyset$. Moreover, $x_{1} \in S \cap U$, then $F(x_{1}) \in \{F(x) : x \in S \cap U\}$. The l-density of the family $\{F(x) : x \in S \cap U\}$ in the family $\{F(x) : x \in U\}$ was shown.
\(
\Leftrightarrow: \text{Let us assume that for any dense set } S \text{ in topology } \tau_1 \text{ and for any non-empty set } U \in \tau_2 \text{ the collection } \{F(s) : s \in U \cap S\} \text{ is l-dense in the collection } \{F(x) : x \in U\}. \text{ Suppose that } F \text{ is not } \subset QC^{\tau_1}_{\tau_2} \text{ at some point } x_0. \text{ Then there exists an open set } V \subset Y \text{ such that } F(x_0) \cap V \neq \emptyset, \text{ and there exists a non-empty set } U \in \tau_2 \text{ such that for any non-empty subset } W \in \tau_1 \text{ of } U \text{ we have } F(x_W) \cap V = \emptyset \text{ for some element } x_W \in W. \text{ Let } A = (X \setminus U) \cup \{x_W \in W : \emptyset \neq W \in \tau_1 \land W \subset U\}. \text{ We will show } \tau_1\text{-density of the set } A. \text{ Let } G \in \tau_1. \text{ If } G \cap (X \setminus U) \neq \emptyset \text{ then, of course, } A \cap G \neq \emptyset. \text{ Now, let } G \subset U. \text{ Then there exists } x_G \in G \text{ such that } F(x_G) \cap V = \emptyset. \text{ It follows that } x_G \in A \text{ and } x_G \in G, \text{ and again, } A \cap G \neq \emptyset. \text{ Let us observe that } F(x_0) \in \{F(x) : x \in U\} \text{ and } F(x_0) \cap V \neq \emptyset \text{ but } F(x) \cap V = \emptyset \text{ for any } x \in A \cap U. \qed
\)

**Theorem 6.** Let \((X, \tau_1, \tau_2)\) be a bitopological space, and \(Y\) — a topological space. Then \(F : X \to Y\) is \(\subset QC^{\tau_1}_{\tau_2}\) multifunction if and only if for any dense set \(S\) in topology \(\tau_1\) and for any non-empty set \(U \in \tau_2\) the collection \(\{F(s) : s \in U \cap S\}\) is dense in the collection \(\{F(x) : x \in U\}\).

**Proof.**

\(\Rightarrow: \text{Let } F \subset QC^{\tau_1}_{\tau_2} \text{ multifunction and let } S \text{ be a dense subset of } X, \text{ let } V_1, V_2 \text{ be an open subsets of } Y \text{ and let } \emptyset \neq U \in \tau_2. \text{ Let } F(x_0) \text{ be such that } F(x_0) \subset V_1 \text{ and } F(x_0) \cap V_2 \neq \emptyset, \text{ where } x_0 \in U. \text{ By the assumptions that there exists a non-empty set } W \in \tau_2, W \subset U \text{ such that } F(x) \subset V_1 \text{ and } F(x) \cap V_2 \neq \emptyset \text{ for any } x \in W. \text{ Because of } \tau_1 \text{ density of the set } S \text{ and relation } W \in \tau_1, \text{ we have } S \cap W = S \cap W \cap U \neq \emptyset. \text{ Let } x_1 \in S \cap W \cap U. \text{ It implies that } F(x_1) \subset V_1 \text{ and } F(x_1) \cap V_2 \neq \emptyset. \text{ Moreover, } x_1 \in S \cap U \text{ then } F(x_1) \in \{F(x) : x \in S \cap U\}. \text{ The density of the family } \{F(x) : x \in S \cap U\} \text{ in the family } \{F(x) : x \in U\} \text{ was shown.}\)

\(\Leftarrow: \text{Let us assume that for any dense set } S \text{ in topology } \tau_1 \text{ and for any non-empty set } U \in \tau_2 \text{ the collection } \{F(s) : s \in U \cap S\} \text{ is dense in the collection } \{F(x) : x \in U\}. \text{ Suppose that } F \neq \subset QC^{\tau_1}_{\tau_2} \text{ at some point } x_0. \text{ Then there exist open subsets } V_1, V_2 \text{ of the space } Y \text{ such that } F(x_0) \subset V_1 \text{ and } F(x_0) \cap V_2 \neq \emptyset, \text{ and there exists a non-empty set } U \in \tau_2 \text{ such that for any non-empty subset } W \in \tau_1 \text{ of } U \text{ we have } \neg F(x_W) \subset V_1 \text{ or } F(x_W) \cap V = \emptyset \text{ for some element } x_W \in W. \text{ Let } A = (X \setminus U) \cup \{x_W \in W : \emptyset \neq W \in \tau_1 \land W \subset U\}. \text{ We will show } \tau_1\text{-density of the set } A. \text{ Let } G \in \tau_1. \text{ If } G \cap (X \setminus U) \neq \emptyset \text{ then, of course, } A \cap G \neq \emptyset. \text{ Now, let } G \subset U. \text{ Then there exists } x_G \in G \text{ such that } \neg F(x_G) \subset V_1 \text{ or } F(x_G) \cap V_2 = \emptyset. \text{ It follows that } x_G \in A \text{ and } x_G \in G \text{ and again } A \cap G \neq \emptyset. \text{ Let us observe that } F(x_0) \in \{F(x) : x \in U\} \text{ and } F(x_0) \subset V_1 \text{ and } F(x_0) \cap V_2 \neq \emptyset \text{ but } \neg F(x) \subset V_1 \text{ or } F(x) \cap V_2 = \emptyset \text{ for any } x \in A \cap U. \qed\)
Let us consider “intersection” case.

Let $F : X \rightarrow Y$ be a multifunction. By Theorem 6 (Theorem 5) the condition “the collection $\{F(s) : s \in U \cap S\}$ is dense (u-dense, l-dense) in the collection $\{F(x) : x \in U\}$” implies that “$F$ is $\subset QC_{\tau_1} \subset uQC_{\tau_1} \subset lQC_{\tau_1}$ multifunction” and, consequently, it implies that $F$ is $\cap QC_{\tau_1} \subset \cap uQC_{\tau_1} \subset \cap lQC_{\tau_1}$ multifunction. But the last condition does not imply that the collection $\{F(s) : s \in U \cap S\}$ is dense (u-dense, l-dense) in the collection $\{F(x) : x \in U\}$. Let us consider the multifunction $F$ from Example 2. We can see that $2^X \setminus \tau_1 = \{\{1, 2, 3\}, \{2, 3\}, \{1, 3\}, \{2\}, \emptyset\}$. Let $S = \{1, 2\}$. It is obvious that $cl_{\tau_1}(\{1, 2\}) = \{1, 2, 3\} = X$. Therefore $S$ is a $\tau_1$-dense subset of $X$. Let $U = \{2, 3\}$. Then $\{F(x) : x \in U\} = \{F(2), F(3)\} = \{[-1, 0] \cup \{1\}, [-2, 0]\}$ and $\{F(s) : s \in U \cap S\} = \{F(2)\} = \{[-1, 0] \cup \{1\}\}$. Let $V_1 = \left(-\frac{3}{2}, \frac{1}{2}\right)$. Then $F(3) \subset V_1$ but $-F(2) \subset V_1$. It was shown that the collection $\{F(s) : s \in U \cap S\}$ is not u-dense in the collection $\{F(x) : x \in U\}$. Let $V_2 = \left(-\frac{5}{2}, -\frac{3}{2}\right)$. Then $F(3) \cap V_2 \neq \emptyset$ but $F(2) \cap V_2 = \emptyset$. It was shown that the collection $\{F(s) : s \in U \cap S\}$ is not l-dense in the collection $\{F(x) : x \in U\}$. Of course, the collection $\{F(s) : s \in U \cap S\}$ is not dense in the collection $\{F(x) : x \in U\}$.

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