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A HERGLOTZ THEOREM IN ORDERED VECTOR SPACES WHICH ARE NOT A LATTICE

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ABSTRACT. A generalization of the Herglotz theorem is established for a monotone σ-complete partially ordered vector spaces. Some version of this theorem may be proved without the assumption of the monotone σ-completeness.

1. Introduction

The classical Herglotz theorem states that a sequence $(a_k)_{k=-\infty}^{\infty}$ of complex numbers is the sequence of the Fourier-Stieltjes coefficients of a non-decreasing function g defined on the interval $\langle 0, 2\pi\rangle$ if and only if the sequence $(a_k)_{k=-\infty}^{\infty}$ is positive definite. Recall that $a_k = \int_0^{2\pi} e^{-\mathrm{i}kt}\, \mathrm{d}g(t)$ is said to be the kth Fourier-Stieltjes coefficient of a function g (with bounded variation defined on the interval $\langle 0, 2\pi\rangle$). A sequence $(a_k)_{k=-\infty}^{\infty}$ of complex numbers is said to be positive definite if and only if

$$0 \leq \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \bar{c}_k a_{k-j}$$

for any finite sequence of complex numbers $(c_j)_{j=-n}^{n}$. A generalization of Herglotz theorem for vector lattices was given in paper [1]. The present paper generalizes this result for a sequence $(z_k)_{k=-\infty}^{\infty}$, elements of which belong to the complexification of a monotone σ-complete partially ordered vector space, namely Theorems 5.6, 5.7 and 5.8. We also recall in Theorem 5.9 that some kind of the Herglotz theorem may be proved without the assumption of the monotone σ-completeness.

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2. Monotone σ-complete partially ordered vector spaces

In the whole paper the symbol Y denotes a real monotone σ-complete partially ordered vector space and Z the complexification of Y. A real partially ordered vector space is said to be monotone σ-complete if any increasing sequence $(a_n)_{n=1}^{\infty}$ of elements of Y, which is bounded above, has a least upper bound $\bigvee_{n=1}^{\infty} a_n$, see [5].

Proposition 2.1. *Let Y be a real monotone σ-complete partially ordered vector space. Then*

(i) *Any decreasing sequence $(a_n)_{n=1}^{\infty}$, which is bounded below, has a greatest lower bound $\bigwedge\limits_{n=1}^{\infty} a_n$ and $\bigvee\limits_{n=1}^{\infty} (-a_n) = -\bigwedge\limits_{n=1}^{\infty} a_n$.*

(ii) *Y is Archimedean, i.e., for any positive $y \in Y$ the sequence $(ny)_{n=1}^{\infty}$ is unbounded above and $\bigwedge\limits_{n=1}^{\infty} \frac{1}{n} y = 0$.*

(iii) *For any positive $y \in Y$ and any decreasing sequence $(\alpha_n)_{n=1}^{\infty}$ of reals with $\lim\limits_{n\to\infty} \alpha_n = 0$ we have $\bigwedge\limits_{n=1}^{\infty} \alpha_n y = 0$.*

(iv) *For any increasing (decreasing) sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, which are bounded above (below)*

$$\bigvee_{n=1}^{\infty} (a_n + b_n) = \bigvee_{n=1}^{\infty} a_n + \bigvee_{n=1}^{\infty} b_n,$$

$$(\textit{resp.}\quad \bigwedge_{n=1}^{\infty} (a_n + b_n) = \bigwedge_{n=1}^{\infty} a_n + \bigwedge_{n=1}^{\infty} b_n).$$

Definition 2.1. Let Y be a monotone σ-complete partially ordered vector space. We say that a sequence $(y_n)_{n=1}^{\infty}$ of elements of Y converges to $y \in Y$ if there is an non-increasing sequence $(a_n)_{n=1}^{\infty}$ such that

$$\bigwedge_{n=1}^{\infty} a_n = 0 \quad \text{and} \quad -a_n \le y_n - y \le a_n \quad \text{for all } n\,.$$

We write $\lim\limits_{n\to\infty} y_n = y$ in this case.

Proposition 2.2. *Any sequence of a monotone σ-complete partially ordered vector space can have only one limit.*

P r o o f. Assume that y and y' are limits of a sequence $(y_n)_{n=1}^{\infty}$ of elements of a monotone σ-complete partially ordered vector space Y. then there are non-increasing seqeunces $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that

$$\bigwedge_{n=1}^{\infty} a_n = 0, \ \bigwedge_{n=1}^{\infty} b_n = 0, \ -a_n \le y_n - y \le a_n$$

and

$$-b_n \le y_n - y' \le b_n \quad \text{for all} \quad n \ .$$

It means

$$-a_n - b_n \le y - y' \le a_n + b_n \ \text{ for all } \ n \ .$$

Since

$$\bigwedge_{n=1}^{\infty} (a_n + b_n) = 0 \ ,$$

we have $y - y' \le 0$ and $y' - y \le 0$, i.e., $y = y'$. $\square$

The following example shows that monotone σ-complete partially ordered vector space need not be a lattice.

EXAMPLE 1. Let H be a Hilbert space (real or complex, finite or infinite dimensional) and Y be the set of all bounded self-adjoint linear operators. Then Y is a monotone σ-complete partially ordered vector space under the natural operations and the ordering $A \ge B$, if and only if $\langle Ax, x\rangle \ge \langle Bx, x\rangle$ for all $x \in H$, see [4, pp. 261-263]. We shall show that Y is not a lattice whenever $\dim H \ge 2$. If $A \ge 0$, i.e., $\langle Ax, x\rangle \ge 0$ for all $x \in H$, then we have $|\langle Ax, y\rangle|^2 \le \langle Ax, x\rangle \langle Ay, y\rangle$ for all $x, y \in H$, see [4, p. 262]. Particularly, $\langle Ax, Ax\rangle^2 \le \langle Ax, x\rangle \langle A^2x, Ax\rangle$ for all $x \in H$. It means that for $A \ge 0$ we have $\operatorname{Ker} A = \{x \in H : \langle Ax, x\rangle \ge 0\}$. Equality $\operatorname{Ker} A = \operatorname{Im} A^{\perp}$ is valid for any self-adjoint A and $\operatorname{Im} A = \operatorname{Ker} A^{\perp}$ is true whenever $\operatorname{Im} A$ is closed, particularly for $\dim(\operatorname{Im} A) < \infty$. Let $A \ge 0$ and $\dim(\operatorname{Im} A) = 1$. Since $\operatorname{Im} A = \operatorname{Ker} A^{\perp}$, it is easy to see that $Ax = \langle x, a\rangle\, a$ for some nonzero $a \in H$. Now, let $A \ge B \ge 0$ and $\dim(\operatorname{Im} A) = 1$. Then $\operatorname{Ker} A \subset \operatorname{Ker} B$ and $\operatorname{Im} B \subset \operatorname{Im} A$, which is possible only if $B = \alpha A$. Since $A \ge B \ge 0$, we have $0 \le \alpha \le 1$. Let $e_1, e_2 \in H$ be such that $||e_1|| = ||e_2|| = 1$ and $\langle e_1, e_2\rangle = 0$. Put $A_i x = \langle x, e_i\rangle\, e_i$ for $i = 1, 2$, $B_1 = A_1 + A_2$ and $B_2 x = 2\,(\langle x, e_1\rangle\, e_1 + \langle x, e_2\rangle\, e_2) + \sqrt{2}\,(\langle x, e_1\rangle\, e_2 + \langle x, e_2\rangle\, e_1)$. We shall show that B_1 and B_2 are minimal upper bounds of the set $\{A_1, A_2\}$. Since $B_1 - A_1 = A_2$ and $B_1 - A_2 = A_1$, B_1 is an upper bound of the set $\{A_1, A_2\}$. Let C be such that $B \ge C \ge A_{1,2}$. Then $C - A_1 = \alpha\,(B_1 - A_1) = \alpha A_2$ with $0 \le \alpha \le 1$, because $\dim(\operatorname{Im}(B_1 - A_1)) = \dim(\operatorname{Im} A_2) = 1$. We have $C = A_1 + \alpha A_2$. Inequality $C \ge A_2$ is possible only if $\alpha = 1$, i.e., if $C = B_1$. Since $B_2 x - A_1 x = 2\,\langle x, e_1\rangle\, e_1 + \langle x, e_2\rangle\, e_2 + \sqrt{2}\,(\langle x, e_1\rangle\, e_2 + \langle x, e_2\rangle\, e_1) = \langle x, \sqrt{2}\, e_1 + e_2\rangle \left(\sqrt{2}\, e_1 + e_2\right)$ and $B_2 x - A_2 x = \langle x, \sqrt{2}\, e_2 + e_1\rangle \left(\sqrt{2}\, e_2 + e_1\right)$, we have $B_2 \geq A_{1,2}$ and $\dim(\operatorname{Im}(B_2 - A_{1,2})) = 1$. If $B_2 \geq C \geq A_{1,2}$ for some C, then $C - A_1 = \alpha(B_2 - A_1)$ with $0 \leq \alpha \leq 1$, i.e., $C = A_1 + \alpha(B_2 - A_1) = (1-\alpha)A_1 + \alpha B_2$. Similarly, $C = (1-\beta)A_2 + \beta B_2$ for $0 \leq \beta \leq 1$. Since A_1, A_2 and B_2 are linearly independent, we have $\beta = \alpha = 1$ and $C = B_2$. So $\sup\{A_1, A_2\}$ does not exist. It follows that Y is not a vector lattice.

3. Riemann-Stieltjes integral

Take a continuous real function f, non-decreasing Y-valued function g defined on the interval $\langle a, b\rangle$ and a partition

$$\Pi = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

of the interval $\langle a, b\rangle$. Put

$$s(f,g,\Pi) = \sum_{j=1}^{n} m_j(g(x_j) - g(x_{j-1})) \text{ and } S(f,g,\Pi) = \sum_{j=1}^{n} M_j(g(x_j) - g(x_{j-1})),$$

where

$$m_j = \min_{x \in \langle x_{j-1}, x_j\rangle} f(x) \quad \text{and} \quad M_j = \max_{x \in \langle x_{j-1}, x_j\rangle} f(x).$$

Proposition 3.1. *There is a unique element $I \in Y$ such that*

$$s(f,g,\Pi') \leq I \leq S(f,g,\Pi)$$

for any partitions Π' and Π of the interval $\langle a, b\rangle$.

Proof. The inequality $s(f,g,\Pi') \leq S(f,g,\Pi)$ is obvious. For any natural n and $0 \leq i \leq 2^n$ put $x_{n,i} = a + i(b-a)2^{-n}$ and $\Pi_n = \{x_{n,i}\}_{i=0}^{2^n}$. Obviously,

$$\bigvee_{n=1}^{\infty} s(f,g,\Pi_n) = \bigwedge_{n=1}^{\infty} S(f,g,\Pi_n).$$

Now, the element

$$I = \bigvee_{n=1}^{\infty} s(f,g,\Pi_n) = \bigwedge_{n=1}^{\infty} S(f,g,\Pi_n)$$

has the required properties.

The element I will be denoted by $\int_a^b f(x)\,\mathrm{d}g(x)$ and is said to be the Riemann-Stieltjes integral of f with respect to g on the interval $\langle a, b\rangle$. $\square$

4. Positive linear maps

We denote by symbol $\mathbf{T}$ the quotient space $\mathbf{R}/2\pi\mathbf{Z}$ which may be identified with the unit circle $\{\zeta : |\zeta| = 1\}$ of the complex plane. The set of all real continuous functions on the space $\mathbf{T}$ will be denoted by $C(\mathbf{T})$ and the set of all real (complex) continuous 2π-periodic functions on the real line by $C_{2\pi}(\mathbf{R})$ (by $C_{2\pi}(\mathbf{R},\mathbf{C})$). Clearly, the formula $\psi(e^{\mathrm{i}t}) = f(t)$ defines one-to-one correspondence between the sets $C(\mathbf{T})$ and $C_{2\pi}(\mathbf{R})$. A linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ is said to be positive if $\Phi(f) \geq 0$ for $f \geq 0$. Obviously, any non-decreasing function $g : \langle 0, 2\pi\rangle \to Y$ defines a positive linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ by the formula $\Phi(f) = \int_0^{2\pi} f(x)\ \mathrm{d}g(x)$. We shall show that it is sufficient to consider only functions g which are left o-continuous at any $x \in (0, 2\pi\rangle$, i.e., $\bigvee_{n=1}^{\infty} g(x_n) = g(x)$ for any increasing sequence $(x_n)_{n=1}^{\infty}$ converging to $x \in (0, 2\pi\rangle$.

Recall that Y-valued Baire measure m on a compact $\mathbf{X}$ is a Y-valued function defined on Baire subsets of $\mathbf{X}$ such that

$$m(\emptyset) = 0,\ m(A) \geq 0 \quad \text{and} \quad m\left(\bigcup\text{-}j = 1^{\infty} A_j\right) = \bigvee_{n=1}^{\infty} \sum_{j=1}^{n} m(A_j)$$

for any sequence $(A_j)_{j=1}^{\infty}$ of pairwise disjoint Baire subsets of $\mathbf{X}$. For more details see [5]. We will consider a Baire measure m on the interval $\langle 0, 2\pi)$, which is noncompact. Clearly, it is a Baire measure on $\langle 0, 2\pi\rangle$ such that $m(\{2\pi\}) = 0$.

Theorem 4.1. *The formulas*

$$\Phi(f) = \int_0^{2\pi} f(x)\ \mathrm{d}g(x)\ ,$$

$$\Phi(f) = \int_{\langle 0,2\pi)} f(t)\ \mathrm{d}m(t)$$

and

$$g(x) = m(\langle 0, x))\ \text{ for } x \in \langle 0, 2\pi\rangle$$

define one-to-one correspondence between the sets of all positive linear maps $\Phi : C_{2\pi}(\mathbf{R}) \to Y$, Y-valued Baire measures m on the interval $\langle 0, 2\pi)$ and non-decreasing functions $g : \langle 0, 2\pi\rangle \to Y$ which are left o-continuous at any $x \in (0, 2\pi\rangle$ and $g(0) = 0$.

Proof. Let $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ be a positive linear map. For $f \in C_{2\pi}(\mathbf{R})$ put

$$\psi(e^{\mathrm{i}t}) = f(t) \quad \text{and} \quad \Psi(\psi) = \Phi(f).$$

Obviously, we have a positive linear map $\Psi : C(\mathbf{T}) \to Y$. By [5, p. 687], for such a map there is a unique Y-valued Baire measure μ on $\mathbf{T}$ such that

$$\Psi(\psi) = \int_{\mathbf{T}} \psi(\zeta)\ d\mu(\zeta).$$

For a Baire subset $A \subset \langle 0, 2\pi)$ put $m(A) = \mu(\{e^{\mathrm{i}t} : t \in A\})$. Then

$$\Phi(f) = \int_{\langle 0,2\pi)} f(t)\ \mathrm{d}m(t) \quad \text{for any} \quad f \in C_{2\pi}(\mathbf{R}).$$

Now, put $g(t) = m(\langle 0, t))$, for $t \in \langle 0, 2\pi\rangle$. Then $g(0) = 0$ and the function g is left o-continuous at any $t \in (0, 2\pi\rangle$. Obviously,

$$s(f, g, \Pi') \le \int_{\langle 0,2\pi)} f(t)\ \mathrm{d}m(t) \le S(f, g, \Pi)$$

for any partitions Π' and Π of the interval $\langle 0, 2\pi\rangle$. It means

$$\int_{\langle 0,2\pi)} f(t)\ \mathrm{d}m(t) = \int_0^{2\pi} f(t)\ \mathrm{d}g(t) = \Phi(f)$$

and the proof is complete. □

Take a function $h : \langle a, b\rangle \to Y$ of the form $h(t) = \sum_{j=1}^{n} \varphi_j(t) y_j$, where $\varphi_j : \langle a, b\rangle \to \mathbf{R}$ are continuous and $y_j \in Y$. Then the integral

$$\int_a^b h(t)\ \mathrm{d}t = \sum_{j=1}^{n} \left(\int_a^b \varphi_j(t)\ \mathrm{d}t \right) y_j$$

is correctly defined and $\int_a^b h(t)\ \mathrm{d}t \ge 0$ whenever $h(t) \ge 0$ for all $t \in \langle a, b\rangle$.

Clearly, any linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ may be extended onto a linear map $\Phi : C_{2\pi}(\mathbf{R}, \mathbf{C}) \to Z$ by the formula $\Phi(f + i\ g) = \Phi(f) + i\Phi(g)$. For any integer j put $\chi_j(t) = e^{-\mathrm{i}jt}$. For a linear map $\Phi : C_{2\pi}(\mathbf{R}, \mathbf{C}) \to Z$ the element $z_j = \Phi(\chi_j)$ is said to be the jth Fourier coefficient of Φ. For a function $f \in C_{2\pi}(\mathbf{R}, \mathbf{C})$ put $f_s(t) = f(s - t)$ and define a function $\varphi(s) = \Phi(f_s)$. The function φ is a 2π-periodic Y-valued function. It will be denoted by $\Phi * f$ and is said to be a convolution of Φ and f. Obviously, $\Phi * f \ge 0$ whenever $f \ge 0$ and Φ is a positive map.

5. Fourier coefficients of a positive linear map

Now, we give a characterization of a positive linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ in terms of its Fourier coefficients. The main result, Theorem 5.7, uses only the following easy lemmas and is independent of Theorem 4.1.

Let $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ be a sequence of elements of Z. The sequence $\mathbf{z}$ is said to be positive definite if

$$0 \leq \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \bar{c}_k z_{k-j}$$

for any finite sequence $(c_j)_{j=-n}^{n}$ of complex numbers. The sum

$$\sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) z_j e^{\mathrm{i}jt}$$

is denoted by $\sigma_N(\mathbf{z}, t)$ and is called the Cesaro sum of the sequence $\mathbf{z}$. The sum

$$\sum_{j=-N}^{N} \left(1 - \frac{|j|}{N+1}\right) e^{\mathrm{i}jt} = \frac{\sin^2(\frac{N+1}{2}t)}{(N+1)\sin^2(\frac{1}{2}t)}.$$

is denoted by $F_N(t)$ and is called the Fejer kernel.

Lemma 5.1. *Let $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ be a positive linear map. Then the sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of the Fourier coefficients of Φ is positive definite and $\sigma_N(\mathbf{z}, t) \geq 0$ for all real t and natural N.*

Proof. Let $(c_j)_{j=-n}^{n}$ be a sequence of complex numbers.
Put

$$f(t) = \sum_{j=-n}^{n} c_j e^{\mathrm{i}jt} \text{ and } g(t) = \left|f(t)\right|^2 = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \bar{c}_k e^{\mathrm{i}(j-k)t}.$$

Then

$$0 \leq \Phi(g) = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \bar{c}_k z_{k-j}.$$

So, the sequence $\mathbf{z}$ is positive definite. Clearly, the Cesaro sum $\sigma_N(\mathbf{z}, t)$ is the convolution of Φ and the Fejer kernel F_N. Therefore $\sigma_N(\mathbf{z}, t) \geq 0$. $\square$

Lemma 5.2. *Let a sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z be positive definite. Then*

(i) $\sigma_N(\mathbf{z}, t) \geq 0$ *for all real t and natural N.*

(ii) $0 \leq z_0 \in Y$, $z_{-n} = \bar{z}_n, \pm \mathrm{Re}(z_n) \leq z_0$ *and* $\pm \mathrm{Im}(z_n) \leq z_0$ *for any integer n.*

P r o o f.

(i). Put $c_j = e^{-\mathrm{i}jt}$ for $0 \le j \le N$. Then

$$0 \le \sum_{j=0}^{N} \sum_{k=0}^{N} c_j \bar{c}_k z_{k-j} = \sum_{j=-N}^{N} (N+1-|j|) z_j e^{\mathrm{i}jt} = (N+1)\sigma_N(\mathbf{z}, t).$$

(ii). Put $c_0 = 1$ Then

$$0 \le \sum_{j=0}^{0} \sum_{k=0}^{0} c_j \bar{c}_k z_{k-j} = c_0 z_0 = z_0, \quad \text{i.e., } 0 \le z_0 \in Y.$$

Put $c_0 = c_n = 1$ and $c_j = 0$ for $0 < j < n$. Then

$$0 \le \sum_{j=0}^{n} \sum_{k=0}^{n} c_j \bar{c}_k z_{k-j} = 2z_0 + z_n + z_{-n}.$$

It means that $\mathrm{Im}(z_n + z_{-n}) = 0$ and $-\mathrm{Re}(z_n + z_{-n}) \le 2z_0$. Replacing c_n by -1, we obtain $\mathrm{Re}(z_n + z_{-n}) \le 2z_0$. Put $c_n = \pm\mathrm{i}$, then we have $\pm\mathrm{Im}(z_n) \le z_0$ and $\mathrm{Re}(z_n - z_{-n}) = 0$. Therefore $z_{-n} = \bar{z}_n, \pm\mathrm{Re}(z_n) \le z_0$ and $\pm\mathrm{Im}(z_n) \le z_0$. □

Lemma 5.3. *Let $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ be a sequence of elements of Z such that $z_j = 0$ for $|j| > m$. If $\sum\limits_{j=-m}^{m} z_j e^{\mathrm{i}jt} \ge 0$ for all t, then the sequence $\mathbf{z}$ is positive definite.*

P r o o f. Put

$$f(t) = \sum_{j=-n}^{n} c_j e^{\mathrm{i}jt}, g(t) = \left|f(t)\right|^2 = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \bar{c}_k e^{\mathrm{i}(j-k)t}$$

and

$$h(t) = \sum_{j=-n}^{n} z_j e^{\mathrm{i}jt}.$$

Then

$$0 \le \frac{1}{2\pi} \int_0^{2\pi} g(t)h(t)\,\mathrm{d}t = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_j \bar{c}_j z_{k-j}.$$

□

Lemma 5.4. *Let a sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z be such that $\sigma_N(\mathbf{z}, t) \ge 0$ for all real t and natural N. Then $\pm\mathrm{Re}(z_j) \le 2z_0$ and $\pm\mathrm{Im}(z_j) \le 2z_0$ for any integer j.*

P r o o f. Fix an integer $N \geq 0$ and put

$$\zeta_j = \begin{cases} \left(1 - \frac{|j|}{N+1}\right) z_j & \text{for } |j| < N+1 \\ \zeta_j = 0 & \text{otherwise.} \end{cases}$$

Since $\sigma_N(\mathbf{z}, t) \geq 0$ for all t, the sequence $(\zeta_j)_{j=-\infty}^{\infty}$ is positive definite by Lemma 5.3. Now, Lemma 5.2 implies $\pm \operatorname{Re}(\zeta_j) \leq \zeta_0$ and $\pm \operatorname{Im}(\zeta_j) \leq \zeta_0$ for $-N-1 < j < N+1$. It means

$$\pm \left(1 - \frac{|j|}{N+1}\right) \operatorname{Re}(z_j) \leq z_0 \text{ and } \pm \left(1 - \frac{|j|}{N+1}\right) \operatorname{Im}(z_j) \leq z_0$$

for $-N-1 < j < N+1$. Now, for any integer j take $N = 2|j| - 1$. Then

$$\pm \operatorname{Re}(z_j) \leq 2z_0 \text{ and } \pm \operatorname{Im}(z_j) \leq 2z_0$$

for any integer j. □

Lemma 5.5. *For any function $f \in C_{2\pi}(\mathbf{R})$ there are decreasing and increasing sequences $(\psi_n)_{n=1}^{\infty}$ and $(\varphi_n)_{n=1}^{\infty}$ of trigonometric polynomials converging uniformly to the function f.*

P r o o f. It is well known, see [3, Theorem 2.12] , that the convolutions $\frac{1}{2\pi} f * F_N$ are trigonometric polynomials which converge uniformly to the function f. So, for any natural n there is a trigonometric polynomial f_n such that

$$f_n - 2^{-n} < f < f_n + 2^{-n}.$$

Put

$$\psi_n = f_n + 3 \cdot 2^{-n} \quad \text{and} \quad \varphi_n = f_n - 3 \cdot 2^{-n}.$$

Then

$$\psi_{n+1} = f_{n+1} + 3 \cdot 2^{-n-1} < f + 4 \cdot 2^{-n-1} < f_n + 2^{-n} + 4 \cdot 2^{-n-1} = f_n + 3 \cdot 2^{-n} = \psi_n$$

and

$$\varphi_{n+1} = f_{n+1} - 3 \cdot 2^{-n-1} > f - 4 \cdot 2^{-n-1} > f_n - 2^{-n} - 4 \cdot 2^{-n-1} = f_n - 3 \cdot 2^{-n} = \varphi_n.$$

So, we obtain the required sequences of trigonometric polynomials. □

Theorem 5.6. *Let a sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z be such that $\sigma_N(\mathbf{z}, t) \geq 0$ for all real t and natural N. Then there is a positive linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ for which the sequence of the Fourier coefficients is the sequence $\mathbf{z}$.*

P r o o f. Take a trigonometric polynomial

$$f(t) = \sum_{j=-n}^{n} c_j e^{\mathrm{i}jt} \quad \text{and put} \quad \Phi(f) = \sum_{j=-n}^{n} c_j z_{-j}.$$

Linearity of Φ is obvious, we want to prove positivity of Φ. So, assume $f(t) \geq 0$. We have to prove that

$$\sum_{j=-n}^{n} c_j z_{-j} \geq 0.$$

Since $f(t) \geq 0$ and $\sigma_N(\mathbf{z}, t) \geq 0$ for all t, we have

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} f(t)\sigma_N(\mathbf{z}, t)\, \mathrm{d}t = \sum_{j=-n}^{n} c_j \left(1 - \frac{|j|}{N+1}\right) z_{-j},$$

whenever $n \leq 2N+1$. Denote

$$S = \sum_{j=-n}^{n} c_j z_{-j} \text{ and } S_N = \sum_{j=-n}^{n} c_j \left(1 - \frac{|j|}{N+1}\right) z_{-j}\ .$$

Then

$$S \geq S - S_N = \sum_{j=-n}^{n} c_j \frac{|j|}{N+1} z_{-j} = \sum_{j=1}^{n} 2\operatorname{Re}\left(c_j \frac{|j|}{N+1} z_{-j}\right),$$

because the elements c_j, c_{-j} and z_j, z_{-j} are conjugated. Let $c_j = a_j + ib_j$, where a_j and b_j are real, and $z_j = u_j + iv_j$, where $u_j, v_j \in Y$. Then

$$z_{-j} = u_j - iv_j \text{ and } S \geq 2\sum_{j=1}^{n} \frac{j}{N+1}(a_j\ u_j + b_j\ v_j).$$

Lemma 5.4 implies $\pm u_j \leq 2z_0$ and $\pm v_j \leq 2z_0$. The inequalities $\pm a_j \leq c_0$ and $\pm b_j \leq c_0$ follow from Lemma 5.2. Therefore

$$S \geq -8\sum_{j=1}^{n} \frac{j}{N+1} c_0 z_0 = -\frac{4n(n+1)}{N+1} c_0 z_0.$$

It means

$$S \geq \bigvee_{N=1}^{\infty} -\frac{4n(n+1)}{N+1} c_0 z_0 = 0.$$

So, Φ is positive. Now, take an arbitrary $f \in C_{2\pi}(\mathbf{R})$. By Lemma 5.5, there exist non-increasing and non-decreasing sequences $(\psi_n)_{n=1}^{\infty}$ and $(\varphi_n)_{n=1}^{\infty}$ of trigonometric polynomials converging uniformly to the function f. We have

$$\varphi_n \leq \varphi_{n+1} \leq f \leq \psi_{m+1} \leq \psi_m$$

for any integers m and n. Therefore

$$\Phi(\varphi_n) \leq \Phi(\varphi_{n+1}) \leq \Phi(\psi_{m+1}) \leq \Phi(\psi_m)$$

which implies $\Phi(\varphi_n) \le \Phi(\psi_m)$ for any integers m and n. It means

$$\bigvee_{n=1}^{\infty} \Phi(\varphi_n) \le \Phi(\psi_m) \text{ and } \bigvee_{n=1}^{\infty} \Phi(\varphi_n) \le \bigwedge_{m=1}^{\infty} \Phi(\psi_m).$$

Now, consider the sequence $(\psi_n - \varphi_n)_{n=1}^{\infty}$ which is non-increasing and converges uniformly to zero. Denote

$$\alpha_n = \sup_{t \in \langle 0, 2\pi \rangle} (\psi_n(t) - \varphi_n(t)).$$

Then

$$0 = \bigwedge_{n=1}^{\infty} \alpha_n \Phi(1) = \bigwedge_{n=1}^{\infty} \Phi(\alpha_n) \ge \bigwedge_{n=1}^{\infty} \Phi(\psi_n - \varphi_n) = \bigwedge_{n=1}^{\infty} \Phi(\psi_n) - \bigvee_{n=1}^{\infty} \Phi(\varphi_n)$$

which means

$$\bigvee_{n=1}^{\infty} \Phi(\varphi_n) \ge \bigwedge_{n=1}^{\infty} \Phi(\psi_n).$$

Therefore

$$\bigvee_{n=1}^{\infty} \Phi(\varphi_n) = \bigwedge_{n=1}^{\infty} \Phi(\psi_n).$$

So, neither $\bigvee\limits_{n=1}^{\infty} \Phi(\varphi_n)$ nor $\bigwedge\limits_{n=1}^{\infty} \Phi(\psi_n)$ does not depend on the choice of non-increasing and non-decreasing sequences $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ of trigonometric polynomials converging uniformly to the function f. Put

$$\Phi(f) = \bigvee_{n=1}^{\infty} \Phi(\varphi_n) = \bigwedge_{n=1}^{\infty} \Phi(\psi_n).$$

Then we have the required positive linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$. $\square$

From Lemmas 5.1, 5.2 and Theorem 5.7 we get

Theorem 5.7. *For any sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z the following properties are equivalent.*

(i) *There is a positive linear map $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ for which $\mathbf{z}$ is the sequence of the Fourier coefficients.*

(ii) *The sequence $\mathbf{z}$ is positive definite.*

(iii) *The Cesaro sums $\sigma_N(\mathbf{z}, t)$ are nonnegative for all natural N.*

The following theorem describes directly a positive linear map trough its Fourier coefficients.

Theorem 5.8. *Let $\Phi : C_{2\pi}(\mathbf{R}) \to Y$ be a nonnegative linear map and*

$$\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$$

be a sequence of Fourier coefficients of Φ. Then for any $\varphi \in C_{2\pi}(\mathbf{R})$

$$\Phi(\varphi) = \lim_{n\to\infty} \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) c_j z_{-j} \ ,$$

where

$$c_j = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-\mathrm{i}jt} \,\mathrm{d}t \ .$$

P r o o f. Put

$$\varphi_n = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) c_j e^{\mathrm{i}jt} \ .$$

The sequence $(\varphi_n)_{n=0}^{\infty}$ converges uniformly to φ.
Denote

$$u_n = \sup_{m\ge n} \max_{t\in\langle 0,2\pi\rangle} |\varphi_m(t) - \varphi(t)| \ .$$

Obviously,

$$\Phi(\varphi_n) = \sum_{j=-n}^{j=n} \left(1 - \frac{|j|}{n+1}\right) c_j z_{-j}$$

and

$$-u_n \Phi(1) \le \Phi(\varphi_n) - \Phi(\varphi) \le u_n \Phi(1) \ ,$$

which means

$$\Phi(\varphi) = \lim_{n\to\infty} \Phi(\varphi_n) = \lim_{n\to\infty} \sum_{j=-n}^{j=n} \left(1 - \frac{|j|}{n+1}\right) c_j z_{-j} \ .$$

□

Now, let Y be *arbitrary* partially ordered vector space, (possibly with a trivial ordering, however in this case the results are also trivial). We denote again by Z the complexification of Y and by $T_{2\pi}(\mathbf{R})$ the set of all trigonometrical polynomials. In paper [2] we have proved the following result.

Theorem 5.9. *For any sequence $\mathbf{z} = (z_j)_{j=-\infty}^{\infty}$ of elements of Z with*

$$0 \le z_0 \in Y \quad \text{and} \quad \bigwedge_{N=1}^{\infty} \frac{z_0}{N} = 0$$

the following properties are equivalent.

(i) *There is a unique positive linear map* $\Phi : T_{2\pi}(\mathbf{R}) \to Y$ *for which* $\mathbf{z}$ *is the sequence of the Fourier coefficients.*

(ii) *The sequence* $\mathbf{z}$ *is positive definite.*

(iii) *The Cesaro sums* $\sigma_N(\mathbf{z}, t)$ *are nonnegative for all natural* N.

REFERENCES

[1] M. DUCHOŇ: *On Herglotz theorem in vector lattices*, Tatra Mt. Math. Publ. **3** (1993), 231–236.

[2] D. DUCHOŇ—P.MALIČKÝ: *Positive linear maps on trigonometric polynomials*, Acta Univ. Mathaei. Belii Nat. Sci. Ser. Ser. Math. **5** (1999), 85–90.

[3] Y. KATZNELSON: *An Introduction to Harmonic Analysis*, 2nd ed., Dower Publications, New York, 1976.

[4] F. RIESZ—B. SZ.-NAGY: *Functional Analysis*, Dower Publications, New York, 1990.

[5] M. WRIGHT: *Measures with values in a partially ordered vector space*, Proc. London Math. Soc. **25** (1972), 675–688.

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