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ON EXTREMAL I-LIMIT POINTS OF DOUBLE SEQUENCES

Pratulananda Das — Prasanta Malik

ABSTRACT. In this paper the concepts of I-limit points, I-cluster points and I-limit superior and limit inferior of double sequences are introduced. We prove some basic properties.

1. Introduction

After Fast [6] introduced the theory of statistical convergence of a real sequence, it has become popular among mathematicians ([2], [7]–[9], [17]). The ideas of statistical limit superior and limit inferior were first extensively studied by Fridy and Orhan [9]. After Kostyrko *et al.* [10] extended the idea of statistical convergence to I-convergence using the concept of an ideal I of the set of positive integers, much work has been done on different aspects of this convergence including I-limit points, I-cluster points, I-limit superior and limit inferior (see [2], [4], [10]–[13]).

Recently Mursaleen and Edely [14] have introduced the concept of statistical convergence of double sequences and proved several basic properties. This was followed by Das, Kostyrko, Wilczyński and Malik [3] who introduced I and I^*-convergence of double sequences. As a natural consequence, in this paper, we introduce the concepts of I-limit points, I-cluster points, I-limit superior and limit inferior (automatically including the corresponding ideas with respect to statistical convergence) for double sequences, and we prove several results.

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2. Preliminaries

Throughout the paper, $\mathbf{N}$ and $\mathbf{R}$ denote the set of all positive integers and the set of all real numbers, respectively.

The idea of convergence of a double sequence was introduced by Pringsheim in [16]. A double sequence $x = (x_{jk})$ of real numbers is said to converge to $\xi \in \mathbf{R}$ in Pringsheim's sense if for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbf{N}$ such that $|x_{jk} - \xi| < \epsilon$, whenever both $j, k \geq n_\epsilon$. It is denoted by $P - \lim\limits_{j,k} x_{jk} = \xi$.

Now, we recall the concept of double natural density. Let $K \subset \mathbf{N} \times \mathbf{N}$. Let $K(n,m)$ be the numbers of $(j,k) \in K$ such that $j \leq n, k \leq m$. If the sequence $\left(\frac{K(n,m)}{nm}\right)$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is denoted by

$$d_2(K) = P - \lim_{n,m} \frac{K(n,m)}{nm} \,.$$

Definition 1 ([14]). A double sequence $x = (x_{jk})$ of real numbers is said to be statistically convergent to $\xi \in \mathbf{R}$, if for any $\epsilon > 0$, we have $d_2\big(A(\epsilon)\big) = 0$, where

$$A(\epsilon) = \big\{(j,k) \in \mathbf{N} \times \mathbf{N};\, |x_{jk} - \xi| \geq \epsilon\big\}.$$

We now recall the following definitions, where X represents an arbitrary set.

Definition 2. Let $X \neq \phi$. A class I of subsets of X is said to be an ideal on X provided

(i) $\phi \in I$,

(ii) $A, B \in I$ implies $A \bigcup B \in I$,

(iii) $A \in I, B \subset A$ implies $B \in I$.

I is called a nontrivial ideal if $X \notin I$.

Definition 3 ([3]). A nontrivial ideal I on X is called admissible if $\{x\} \in I$ for each $x \in X$.

Throughout the paper I stands for a nontrivial ideal of $\mathbf{N} \times \mathbf{N}$.

Definition 4 ([3]). A nontrivial ideal I on $\mathbf{N} \times \mathbf{N}$ is called strongly admissible if $\{i\} \times \mathbf{N}$ and $\mathbf{N} \times \{i\}$ belong to I for each $i \in \mathbf{N}$.

It is evident that a strongly admissible ideal is also admissible.

Definition 5 ([3]). A double sequence $x = (x_{jk})$ of real numbers is said to converge to $\xi \in \mathbf{R}$ with respect to the ideal I, if for every $\epsilon > 0$ the set

$$A(\epsilon) = \big\{(j,k) \in \mathbf{N} \times \mathbf{N};\, |x_{jk} - \xi| \geq \epsilon\big\} \in I.$$

In this case we say that x is I-convergent and we write $I\text{–}\lim_{j,k} x_{jk} = \xi$.

If I is strongly admissible, then clearly P-convergence of x implies I-convergence of x. However, the converse is not true. If we take $I = I_0 = \{A \subset \mathbf{N} \times \mathbf{N};\ \exists\ m(A) \in \mathbf{N}$ such that $(i,j) \notin A$ whenever both of $i, j \geq m(A)\}$, then I-convergence coincides with P-convergence and, if we take $I = \{A \subset \mathbf{N} \times \mathbf{N};\ d_2(A) = 0\}$, then I-convergence becomes statistical convergence.

3. I-limit points and I-cluster points

In [15], the concept of an ordinary limit point for a single sequence was generalized for Pringsheim limit point of a double sequence in $\mathbf{R}$. In this paper, we extend this concept to statistical and I-limit points and cluster points for double sequences. We also consider the underlying space to be a metric space (X, d).

Definition 6. Let K be a subset of $\mathbf{N} \times \mathbf{N}$ such that for each $(i,j) \in \mathbf{N} \times \mathbf{N}$, there exists $(m,n) \in K$ such that $(m,n) > (i,j)$ with respect to the dictionary ordering. If $x = (x_{jk})$ is a double sequence in (X, d), then we define $\{x\}_K = \{x_{mn}; (m,n) \in K\}$ as a subsequence of x.

Definition 7. An element $l \in X$ is said to be Pringsheim limit point of a double sequence $x = (x_{jk})$ in a metric space (X, d) if there exists a subsequence of x which is P-convergent to l.

Definition 8. Let (X, d) be a metric space and $x = (x_{jk})$ be a double sequence in X. An element $\beta \in X$ is said to be an I-limit point of x if there exists a set $M = \{(m_j, m_k);\ j, k \in \mathbf{N}\} \subset \mathbf{N} \times \mathbf{N}$ such that $M \notin I$ and $P - \lim\limits_{m_j, m_k} x_{m_j m_k} = \beta$.

We now introduce the notations L_x^2 and $I(\wedge_x)$ to denote the set of all Pringsheim limit points and I-limit points of $x = (x_{jk})$, respectively. In general, L_x^2 and $I(\wedge_x)$ may be quite different as can be seen from the following example.

Example 1. Let $I = \{A \subset \mathbf{N} \times \mathbf{N}; d_2(A) = 0\}$. We define a double sequence $x = (x_{jk})$ in the following way

$$x_{jk} = \begin{cases} 1 & \text{if } \quad j = k, \\ k & \text{otherwise.} \end{cases}$$

Then $L_x^2 = \{1\}$. However, I-limit point does not exist, i.e., $I(\wedge_x) = \phi$.

Definition 9. An element $\alpha \in X$ is said to be an I-cluster point of a double sequence $x = (x_{jk})$ in a metric space (X, d) if and only if for each $\epsilon > 0$ the set $\{(j,k);\ d(x_{jk}, \alpha) < \epsilon\} \notin I$.

We denote the set of all I-cluster points of x by $I(\Gamma_x)$. We now study the relationship between $I(\wedge_x)$ and $I(\Gamma_x)$.

Theorem 1. *Let I be a strongly admissible ideal. Then for any double sequence $x = (x_{jk})$ in (X, d) we have $I(\wedge_x) \subset I(\Gamma_x)$.*

P r o o f. Let $\alpha \in I(\wedge_x)$. Then there exists a set

$$M = \{(m_j, m_k) \in \mathbf{N} \times \mathbf{N}; j, k \in \mathbf{N}\} \notin I$$

such that

$$P - \lim_{m_j, m_k} x_{m_j m_k} = \alpha. \tag{1}$$

Let $\epsilon > 0$. Then by (1), there exists $k_0 \in \mathbf{N}$ such that for $m_j \geq k_0$, $m_k \geq k_0$, we have $d(x_{m_j m_k}, \alpha) < \epsilon$. So, we have $\{(j,k);\, d(x_{jk}, \alpha) < \epsilon\} \supset M \setminus \{(m_j, m_k),$ either $m_j \leq (k_0 - 1)$ or $m_k \leq (k_0 - 1)\}$. Since I is strongly admissible, so

$$\{(j,k);\, d(x_{jk}, \alpha) < \epsilon\} \notin I.$$

This implies $\alpha \in I(\Gamma_x)$, which completes the proof. □

Theorem 2. *Let I be a strongly admissible ideal of $\mathbf{N} \times \mathbf{N}$. Then*

(i) *The set $I(\Gamma_x)$ is closed in X for each double sequence x in (X, d).*

(ii) *Let (X, d) be a separable metric space and let there exist a disjoint sequence of the sets (A_n) such that $A_n \subset \mathbf{N} \times \mathbf{N}$ and $A_n \notin I$; $n \in \mathbf{N}$. Then for each closed set $P \subset X$, there exists a sequence $x = (x_{jk}) \in X$ such that $P = I(\Gamma_x)$.*

P r o o f. The proof is similar to the proof of Theorem 4.1 ([10]) and so is omitted. □

4. I-limit superior and limit inferior

The concept of I-limit superior and limit inferior for single sequences of real numbers was introduced in [4]. In this paper we generalize this concept for double sequences of real numbers and call it I-limit superior and I-limit inferior.

Definition 10 ([15]). Let $x = (x_{jk})$ be a double sequence of real numbers, and let $\alpha_n = \sup\,\{x_{jk};\, j, k \geq n\}$ for each n. Then Pringsheim limit superior of x is defined as follows:

(i) if $\alpha_n = +\infty$ for each n, then $P - \limsup\, x = \infty$,

(ii) if $\alpha_n < \infty$ for some n, then $P - \limsup\, x = \inf_n \alpha_n$.

Similarly, let $\beta_n = \inf\{x_{jk};\, j, k \geq n\}$. Then Pringsheim limit inferior of x is defined as follows:

(i) if $\beta_n = -\infty$ for each n, then $P - \liminf\, x = -\infty$,

(ii) if $\beta_n > -\infty$ for some n, then $P - \liminf\ x = \sup_n \beta_n$.

We now introduce the definitions of I-limit superior and I-limit inferior.

Let I be a strongly admissible ideal of $\mathbf{N} \times \mathbf{N}$ and let $x = (x_{jk})$ be a double sequence of real numbers. Let

$$B_x = \Big\{b \in \mathbf{R};\ \big\{(j,k);\ x_{jk} > b\big\} \notin I\Big\},$$

and

$$A_x = \Big\{a \in \mathbf{R};\ \big\{(j,k);\ x_{jk} < a\big\} \notin I\Big\}.$$

Then I-limit superior and I-limit inferior of x are defined as follows:

$$I - \limsup x = \begin{cases} \sup B_x & \text{if} \quad B_x \neq \phi, \\ -\infty & \text{if} \quad B_x = \phi, \end{cases}$$

$$I - \liminf x = \begin{cases} \inf A_x & \text{if} \quad A_x \neq \phi, \\ \infty & \text{if} \quad A_x = \phi. \end{cases}$$

If $I = I_0$ then I-limit superior and I-limit inferior coincide with P-limit superior and P-limit inferior.

Throughout the section, I stands for a nontrivial strongly admissible ideal of $\mathbf{N} \times \mathbf{N}$, (x_{jk}), (y_{jk}) etc. are double sequences of real numbers and are denoted by x, y etc, for short.

Theorem 3.

(i) $I - \lim \text{supx} = \alpha\ (finite)$ *if and only if for any* $\epsilon > 0$,

$$\big\{(j,k);\ x_{jk} > \alpha - \epsilon\big\} \notin I \quad \textit{and} \quad \big\{(j,k); x_{jk} > \alpha + \epsilon\big\} \in I.$$

(ii) $I - \lim \text{infx} = \beta(finite)$ *if and only if for any* $\epsilon > 0$,

$$\big\{(j,k);\ x_{jk} < \beta + \epsilon\big\} \notin I \quad \textit{and} \quad \big\{(j,k);\ x_{jk} < \beta - \epsilon\big\} \in I.$$

P r o o f. The proof is straightforward. □

Theorem 4. *The inequality*

$$I - \liminf\ x \leq I - \limsup x$$

holds for each double sequence $x = (x_{jk})$ *of real numbers.*

P r o o f. The proof is similar to the proof of Theorem 3 ([4]) and is omitted. □

Theorem 5. *Let* $x = (x_{jk})$ *be a double sequence of real numbers. Then*

$$P - \liminf x \leq I - \liminf x \leq I - \limsup x \leq P - \limsup x.$$

P r o o f. We first prove that $P - \liminf x \leq I - \liminf x$. If $P - \liminf x = -\infty$, then it is obvious. Let $P - \liminf x = \alpha > -\infty$. Then

$$\alpha = \sup_n \alpha_n,$$

where

$$\alpha_n = \inf\{x_{jk};\ j,k \geq n\}.$$

Then

$$\big\{(j,k);\ x_{jk} < \alpha_n\big\} \subset \big\{(j,k),\ \text{either } j \leq (n-1) \text{ or } k \leq (n-1)\big\}.$$

Since I is strongly admissible, then

$$\{(j,k);\ \text{either } j \leq (n-1) \text{ or } k \leq (n-1)\} \in I,$$

so

$$\{(j,k);\ x_{jk} < \alpha_n\} \in I.$$

Now, let $\beta = I - \liminf x = \inf A_x$, where

$$A_x = \big\{a \in \mathbf{R}; \{(j,k); x_{jk} < a\} \notin I\big\}.$$

Now, if $\beta < \alpha_n$, then there exists $a' \in A_x$ such that $\beta \leq a' < \alpha_n$. However,

$$\big\{(j,k); x_{jk} < a'\big\} \subset \big\{(j,k); x_{jk} < \alpha_n\big\} \in I,$$

which yields $a' \notin A_x$, which is a contradiction. Then $\beta \geq \alpha_n$ for all n. Therefore,

$$\alpha \leq \beta, \quad \text{i.e.,} P - \liminf x \leq I - \liminf x.$$

Similarly we can show $I - \limsup x \leq P - \limsup x$.

Combining these two results with Theorem 4 we get the desired result.

Recall that the core of a single sequence $x = (x_n)$ is defined by

$$\text{core}\{x\} = [\,\liminf x,\ \limsup x].$$

In [11] this idea was generalized for I-convergence. In this paper we extend this idea for double sequences of real numbers. □

Definition 11 ([15])**.** Let $x = (x_{jk})$ be a double sequence of real numbers. Then Pringsheim core of x is defined by

$$P - \text{core}\{x\} = [P - \liminf x,\ P - \limsup x].$$

Definition 12. Let $x = (x_{jk})$ be a double sequence of real numbers. Then I-core of x is defined by

$$I - \text{core}\{x\} = [I - \liminf x,\ I - \limsup x].$$

Then, by Theorem 5, we have the following result.

Corollary 1. *For any double sequence x of real numbers we have*

$$I - \text{core}\{\text{x}\} \subset \text{P} - \text{core}\{\text{x}\}.$$

Definition 13. A double sequence $x = (x_{jk})$ is said to be I-bounded if there exists a real number $M > 0$ such that $\{(j,k); | x_{jk} |> M\} \in I$.

Theorem 6. *An I-bounded double sequence $x = (x_{jk})$ is I-convergent if and only if $I - \limsup x = I - \liminf x$.*

Proof. The proof is similar to that of Theorem 4 ([4]). □

We now introduce the following definition which will be useful to prove the next theorem.

Definition 14. A double sequence $x = (x_{jk})$ is said to be I-convergent to ∞ (or $-\infty$) if for every real number $G > 0$,

$$\{(j,k);\, x_{jk} \le G\} \in I \quad \text{or} \quad \{(j,k); x_{jk} \ge -G\} \in I.$$

Theorem 7. *If $I - \limsup x = p$, then there exists a subsequence of x that is I-convergent to p.*

Proof. Since $\phi \in I$ and I is strongly admissible ideal of $\mathbf{N} \times \mathbf{N}$, we consider the double sequence $x = (x_{jk})$ to be a non constant double sequence of which x_{jk} are distinct whenever both of j, k run over the infinite subsets of N.

Now, p has three possibilities:

(i) $p = -\infty$,

(ii) $p = \infty$,

(iii) $-\infty < p < \infty$.

Case (i). When $p = -\infty$, then $B_x = \phi$. So, for any $M > 0$, we have $\{(j,k);\, x_{jk} \ge -M\} \in I$. This implies $I - \lim x = -\infty$.

Case (ii). When $p = \infty$, then $B_x = \mathbf{R}$. Hence for any $b \in \mathbf{R}$,

$$\{(j,k); x_{jk} > b\} \notin I.$$

Let $x_{n_1 m_1}$ be an arbitrary member of x and let

$$A_{n_1 m_1} = \{(j,k); x_{jk} > x_{n_1 m_1} + 1\}.$$

Then $A_{n_1 m_1} \notin I$, so $A_{n_1 m_1} \ne \phi$. Now, there exists $(n_2, m_2) \in A_{n_1 m_1}$ such that $n_2 > n_1,\ m_2 > m_1$, otherwise,

$$A_{n_1 m_1} \subset \{(j,k);\ \text{either } j \le n_1 \ \text{ or } k \le m_1\} \in I,$$

a contradiction. Proceeding in this way, we obtain a subsequence $x' = (x_{n_k m_k})$ of x with $x_{n_k m_k} > x_{n_{k-1} m_{k-1}} + 1$ for all $k > 1$. Then for any $L > 0$,

$$\{(n_k, m_k);\, x_{n_k m_k} \le L\} \in I,$$

since I is strongly admissible. Hence $I - \lim x' = \infty$.

Case (iii). When $-\infty < p < \infty$, then by Theorem 3 $\{(j,k);\, x_{jk} > p-1\} \notin I$, so $\{(j,k);\, x_{jk} > p-1\} \neq \phi$. Now, there exists at least one element, say (n_1, m_1) in $\{(j,k);\, x_{jk} > p-1\}$ for which $x_{n_1m_1} \leq p + \frac{1}{2}$, otherwise,

$$\{(j,k);\, x_{jk} > p-1\} \subset \Big\{(j,k);\, x_{jk} > p + \tfrac{1}{2}\Big\} \in I$$

which gives a contradiction. Hence, we have

$$p - 1 < x_{n_1m_1} \leq p + \tfrac{1}{2} < p + 1.$$

Now, we proceed to choose an element $x_{n_2m_2}$ from x with $n_2 > n_1$, $m_2 > m_1$ such that $p - \frac{1}{2} < x_{n_2m_2} < p + \frac{1}{2}$. We claim that there is at least one (j,k) with $j > n_1$ and $k > m_1$ for which $x_{jk} > p - \frac{1}{2}$. For otherwise,

$$\Big\{(j,k);\, x_{jk} > p - \tfrac{1}{2}\Big\} \subset \Big\{(j,k);\quad \text{either } j \leq n_1 \text{ or } k \leq m_1\Big\} \in I,$$

which yields a contradiction to Theorem 3. So, the set

$$A'_{n_1m_1} = \{(j,k);\, j > n_1, k > m_1 \quad \text{and } x_{jk} > p - \frac{1}{2}\} \neq \phi.$$

Now, we claim that there is at least one $(j,k) \in A'_{n_1m_1}$, such that $x_{jk} < p + \frac{1}{2}$. For otherwise,

$$A'_{n_1m_1} \subset \Big\{(j,k);\, x_{jk} \geq p + \tfrac{1}{2}\Big\} \subset \Big\{(j,k);\, x_{jk} > p + \tfrac{1}{4}\Big\}.$$

Now, by Theorem 3, $\Big\{(j,k);\, x_{jk} > p + \frac{1}{4}\Big\} \in I$, so $A'_{n_1m_1} \in I$. Again, since

$$\Big\{(j,k);\, x_{jk} > p - \tfrac{1}{2}\Big\} \subset \{(j,k);\quad \text{either} \quad j \leq n_1 \quad \text{or} \quad k \leq m_1\} \bigcup A'_{n_1m_1},$$

and I is strongly admissible, then the union on the right hand side is in I giving $\Big\{(j,k);\, x_{jk} > p - \frac{1}{2}\Big\} \in I$, which is a contradiction to Theorem 3. Hence, our claim is established. We put $j = n_2$ and $k = m_2$. Thus there are $n_2 > n_1$, $m_2 > m_1$ such that

$$p - \tfrac{1}{2} < x_{n_2m_2} < p + \tfrac{1}{2}.$$

Proceeding in this way we obtain a subsequence $x' = (x_{n_km_k})$ of x with $n_k > n_{k-1}$, $m_k > m_{k-1}$ such that $p - \frac{1}{k} < x_{n_km_k} < p + \frac{1}{k}$ for each k. The subsequence x' is P-convergent to p and hence I-convergent to p. This proves the theorem. □

Theorem 8. *If $I - \liminf x = m$, then there exists a subsequence of x that is I-convergent to m.*

P r o o f. The proof is similar to the proof of Theorem 7 and is omitted. □

The following example shows that I-limit point and I-limit superior of a double sequence are quite different.

EXAMPLE 2. Let $I_P = \{A \subset N \times N;\, d_2(A) = 0\}$ then it is a nontrivial ideal on $\mathbf{N} \times \mathbf{N}$. Now, let

$$A_p = \{2^{p-1}(2k-1);\, k \in N\}, \qquad p = 1, 2, \ldots$$

Then clearly,

$$A_p \bigcap A_q = \phi \quad \text{for} \quad p \neq q.$$

Now, we define

$$D_{pq} = A_p \times A_q.$$

Then

$$D_{pq} \bigcap D_{rs} = \phi \quad \textit{for}\ (p,q) \neq (r,s) \quad \textit{and} \quad d_2(D_{pq}) = \frac{1}{2^p 2^q}\ (p,q = 1,2,\ldots).$$

Now we define a double sequence $x = (x_{mn})$ as follows

$$x_{mn} = 1 - \tfrac{1}{pq}, \quad (m,n) \in D_{pq}, \quad (p,q = 1,2,\ldots).$$

Then each number $1 - \frac{1}{pq}$ is an I_P-limit point of x. Again from the definition of I-limit superior we have $I_P - \limsup x = 1$.

Now, we show that 1 is not I_P-limit point of x. If possible, let 1 be an I_P-limit point of x. Then there is a set

$$M = \{(m_j, m_k);\, j,k \in N\} \subset \mathbf{N} \times \mathbf{N}$$

such that $M \notin I_P$ and

$$\lim_{m_j, m_k} x_{m_j m_k} = 1. \tag{A}$$

The definition of x and (A) imply that there is $r \in \mathbf{N}$ such that

$$M \bigcap D_{pq} = \{(j,k);\ \text{either } j \leq r \text{ or } k \leq r\},\ (p,q = 1,2,\ldots).$$

Since

$$\mathbf{N} \times \mathbf{N} = \bigcup_{p,q=1}^{\infty} D_{pq},$$

we have

$$M = \left[\bigcup_{p,q=1}^{k}\left(D_{pq}\bigcap M\right)\right]\bigcup\left[\bigcup_{p=1}^{k}\bigcup_{q=k+1}^{\infty}\left(D_{pq}\bigcap M\right)\right]$$
$$\bigcup\left[\bigcup_{q=1}^{k}\bigcup_{p=k+1}^{\infty}\left(D_{pq}\bigcap M\right)\right]$$
$$\bigcup\left[\bigcup_{p,q=k+1}^{\infty}\left(D_{pq}\bigcap M\right)\right].$$

This holds for each k. Now, we have

$$d_2(M) \leq \sum_{p,q=1}^{k} d_2\left(M\bigcap D_{pq}\right) + \sum_{p=1}^{k} d_2(E_p) + \sum_{q=1}^{k} d_2(E_q) + d_2(E),$$

where

$$E_p = \bigcup_{q=k+1}^{\infty}\left(D_{pq}\bigcap M\right),$$
$$E_q = \bigcup_{p=k+1}^{\infty}\left(D_{pq}\bigcap M\right)$$

and

$$E = \bigcup_{p,q=k+1}^{\infty}\left(D_{pq}\bigcap M\right).$$

Since $E_p \subset \left\{(s,q);\, q \text{ is multiple of } 2^k\right\}$, we have $d_2(E_p) \leq 2^{-k}$. Similarly, $d_2(E_q) \leq 2^{-k}$ and $d_2(E) \leq 2^{-2k}$. Since this inequality is true for each $k = 1, 2, \dots$, then $d_2(M) = 0$ which is a contradiction to $M \notin I_P$. Thus 1 is not a I_P-limit point of x.

This shows that the double sequence $x = (x_{jk})$ has no greatest I_P-limit point though it has $I_P - \limsup x = 1$.

Theorem 9. *Let $x = (x_{jk})$ be a bounded double sequence of real numbers, then*

(i) $I - \limsup x = \max I(\Gamma_x)$,

(ii) $I - \liminf x = \min I(\Gamma_x)$.

Proof. (i) Let $\alpha = I - \limsup x$. Let us take a number $\alpha^{'} > \alpha$. Now, we have

$$\alpha = \sup B_x,$$

where

$$B_x = \big\{b \in R; \big\{(j,k);\, x_{jk} > b\big\} \notin I\big\}.$$

Now, we choose $\epsilon > 0$, such that $\alpha < \alpha^{'} - \epsilon < \alpha^{'}$. Then $\alpha^{'} - \epsilon \notin B_x$ and so

$$\big\{(j,k); x_{jk} > \alpha^{'} - \epsilon\big\} \in I.$$

Then by the definition of I-cluster point we have $\alpha^{'} \notin I(\Gamma_x)$. Thus any number greater than α cannot be a I-cluster point of x.

Now, we show that $\alpha \in I(\Gamma_x)$. Let $\epsilon > 0$. Then by the definition of I-limit superior, there exists $r \in B_x$ such that $\alpha - \epsilon < r \leq \alpha$. Therefore

$$\big\{(j,k);\, x_{jk} > r\big\} \notin I. \tag{2}$$

Now, since $\alpha + \frac{\epsilon}{2} \notin B_x$, we have,

$$\left\{(j,k); x_{jk} > \alpha + \frac{\epsilon}{2}\right\} \in I. \tag{3}$$

From (2) and (3) we get

$$\{(j,k);\mid x_{jk} - \alpha \mid < \epsilon\} \notin I \quad \text{ and } \quad \alpha \in I(\Gamma_x).$$

This completes the proof.

□

(ii) The proof is similar to the proof of (i) and is omitted.

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REFERENCES

[1] ČERVEŇANSKÝ, J.—ŠALÁT, T.—TOMA, V.: *Remarks on statistical and I-convergence of series,* Math. Bohem. **130** (2005), 177–184.

[2] ČINČURA, J.—ŠALÁT, T.—SLEZIAK, M.—TOMA, V.: *Sets of statistical cluster points and I-cluster points,* Real Anal. Exchange **30** (2004/2005), 565–580.

[3] DAS, P.—KOSTYRKO, P.—WILCZYŃSKI, W.—MALIK, P.: *I and I^*-convergence of double sequences,* Math. Slovaca **58** (2008) (to appear).

[4] DEMIRCI, K.: *I-limit superior and limit inferior,* Math. Commun. **6** (2001), 165–172.

[5] DINDOŠ, M.—ŠALÁT, T.—TOMA, V.: *Statistical convergence of infinite series,* Czechoslovak Math. J. **53** (2003), 989–1000.

[6] FAST, H.: *Sur la convergence statistique,* Colloq. Math. **2** (1951), 241–244.

[7] FRIDY, J. A.: *On statistical convergence,* Analysis **5** (1985), 301–313.

[8] FRIDY, J. A.: *Statistical limit points,* Proc. Amer. Math. Soc. **118** (1993), 1187–1192.

[9] FRIDY, J. A.—ORHAN, C.: *Statistical limit superior and limit inferior,* Proc. Amer. Math. Soc. **125** (1997), 3625–3631.

[10] KOSTYRKO, P.—ŠALÁT, T.—WILCZYŃSKI, W.: *I-convergence,* Real Anal. Exchange **26** (2000/2001), 669–686.
[11] KOSTYRKO, P.—MAČAJ, M.—ŠALÁT, T.—SLEZIAK, M.: *I-convergence and extremal I-limit points,* Math. Slovaca **55** (2005), 443–464.
[12] LAHIRI, B. K.—DAS, P.: *Further remarks on I-limit superior and I-limit inferior,* Math. Commun. **8** (2003), 151–156.
[13] LAHIRI, B. K.—DAS, P.: *I and I^*-convergence in topological spaces,* Math. Bohem. **130** (2005), 153–160.
[14] MURSALEEN, M.—EDELY, O. H. O.: *Statistical convergence of double sequences,* J. Math. Anal. Appl. **288** (2003), 223–231.
[15] PATTERSON, R. F.: *Double sequence core theorems,* Internat. J. Math. Math. Sci. **22** (1999), 785–793.
[16] PRINGSHEIM, A.: *Zur Theorie der zweifach unendlichen Zahlenfolgen,* Math. Ann. **53** (1900) 289–321.
[17] ŠALÁT, T.: *On statistically convergent sequences of real numbers,* Math. Slovaca **30** (1980), 139–150.

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Department of Mathematics
Jadavpur University
Kolkata-700032
West Bengal
INDIA
E-mail: pratulananda@yahoo.co.in
pmjupm@yahoo.co.in