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ON A FUNCTIONAL RELATION DEFINED BY THE EQUALITY OF THE CLOSURES OF GRAPHS

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ABSTRACT. Some common properties (continuity, quasicontinuity, symmetrical quasicontinuity, . . .) of functions whose graphs have the same closures are investigated.

If (X, T_X) and (Y, T_Y) are topological spaces and $f, g : X \to Y$ are functions, then we will say that

$$f \rho g \qquad \text{if and only if} \quad cl\big(Gr(f)\big) = cl\big(Gr(g)\big),$$

where cl denotes the closure operation and $Gr(f)$ denotes the graph of the function f. Evidently, ρ is an equivalence in the class of all functions from X to Y. In the case where $X = Y$ is a metric compact space, the relation ρ was investigated in [3].

Remark 1. Assume that (Y, T_Y) is a Hausdorff space and that $f, g : X \to Y$ are functions such that $f \rho g$. If f is continuous at a point x, then $f(x) = g(x)$ and

$$\Big\{y \in Y; (x, y) \in cl\big(Gr(g)\big)\Big\} = \big\{g(x)\big\}. \tag{a}$$

Moreover, if (Y, T_Y) is a regular space, then g is continuous at x.

P r o o f. Assume by contradiction that there is a point $y \in Y$ with $y \neq f(x)$ and $(x, y) \in cl\big(Gr(g)\big) = cl\big(Gr(f)\big)$. Since (Y, T_Y) is a Hausdorff space, there are sets $V_1, V_2 \in T_Y$ such that $y \in V_1$, $f(x) \in V_2$ and $V_1 \cap V_2 = \emptyset$. But f is continuous at x, so there is a set $U \in T_X$ containing x with $f(U) \subset V_2$. Consequently,

$$(U \times V_1) \cap cl\big(Gr(g)\big) = (U \times V_1) \cap cl\big(Gr(f)\big) = \emptyset,$$

and

$$(x, y) \notin cl\big(Gr(g)\big).$$

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This contradiction implies (a) and the equality $g(x) = f(x)$. For the proof of the second part, assume by contradiction that the space (Y, T_Y) is regular and g is not continuous at x. Then there is a set $V \in T_Y$ containing $g(x)$ such that for each set $U \in T_X$ containing X there exists a point $u \in U$ with $g(u) \notin V$. From the regularity of the space (Y, T_Y) it follows that there are disjoint sets $V_3, V_4 \in T_Y$ such that $g(x) \in V_3$ and $Y \setminus V \subset V_4$. Since f is continuous at x and $V_3 \ni g(x) = f(x)$, there is a set $U_1 \in T_X$ containing x with $U_1 \subset U$ and $f(U_1) \subset V_3$. There is a point $u_1 \in U_1$ with $g(u) \in Y \setminus V \subset V_4$. Since $cl\big(Gr(g)\big) = cl\big(Gr(f)\big)$, there is a point $u_2 \in U_1$ with $f(u_2) \in V_4$, a contradiction with $V_3 \cap V_4 = \emptyset$ and $f(U_1) \subset V_3$. This finishes the proof. $\square$

Observe that the hypothesis in Remark 1 that (Y, T_Y) is a Hausdorff space is important, as the example below shows.

Example 1. Let $X = Y = N$ be the set of all positive integers and let

$$T_X = T_Y = \{\emptyset\} \cup \{N \setminus A; A \quad \text{is finite}\}.$$

Then the space $(X, T_X) = (Y, T_Y)$ is not Hausdorff, but it satisfies (T_1)-axiom and the functions $f, g : X \to Y$ defined by

$$f(x) = x \quad \text{and} \quad g(x) = x + 1 \qquad \text{for} \quad x \in X,$$

are continuous, and $cl\big(Gr(f)\big) = cl\big(Gr(g)\big) = X \times Y$.

Remark 2. Let $f, g : X \to Y$ be functions such that $f \rho g$. If $x \in X$ is a point such that the point $\big(x, f(x)\big)$ is isolated in $Gr(f)$, then $f(x) = g(x)$.

P r o o f. There are sets $U \in T_X$ and $V \in T_Y$ such that $x \in U$, $f(x) \in V$ and $(U \times V) \cap Gr(f) = \big\{(x, f(x))\big\}$. So,

$$cl\Bigg(Gr(f) \setminus \Big\{\big(x, f(x)\big)\Big\}\Bigg) \subset (X \times Y) \setminus (U \times V),$$

and consequently

$$cl\Bigg(Gr(g) \setminus \Big\{\big(x, f(x)\big)\Big\}\Bigg) = cl\Bigg(Gr(f) \setminus \Big\{\big(x, f(x)\big)\Big\}\Bigg) \subset (X \times Y) \setminus (U \times V).$$

Since $\big(x, f(x)\big) \in cl\big(Gr(g)\big)$, we have $f(x) = g(x)$. $\square$

In [3] it is proved that if (X, d) is a metric compact space and $f, g : X \to X$ are such that $f \rho g$, then the quasicontinuity of f implies the quasicontinuity of g. Recall that a function $f : X \to Y$ is quasicontinuous at a point $x \in X$ if for all the sets $U \in T_X$ containing x and $V \in T_Y$ containing $f(x)$ there is a nonempty set $W \in T_X$ contained in U such that $f(W) \subset V$ ([6],[8]).

Theorem 1. *Assume that (Y, T_Y) is a regular space. Let $f, g : X \to Y$ be functions such that $f \rho g$. If f is quasicontinuous at a point x, then also g is quasicontinuous at x.*

Proof. Let $U \in T_X$ and $V \in T_Y$ be sets such that $x \in U$ and $f(x) \in V$. Since (Y, T_Y) is a regular space, there is a set $V_1 \in T_Y$ such that $x \in V_1 \subset cl(V_1) \subset V$. From the quasicontinuity of f at x it follows that there is a nonempty set $U_1 \subset U$ such that $U_1 \in T_X$ and $f(U_1) \subset V_1$. Since

$$cl\big(Gr(g)\big) \cap \big(U_1 \times cl(V_1)\big) = cl\big(Gr(f)\big) \cap \big(U_1 \times cl(V_1)\big) \subset cl\big(Gr(f)\big) \cap (U_1 \times V),$$

we obtain that $g(U_1) \subset V$. This finishes the proof. □

Now, consider the functions of two variables. Let (Z, T_Z) be a topological space and let $f : X \times Y \to Z$ be a function. Then the functions $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$, where $x \in X$ and $y \in Y$, are said to be the sections of f. A function f is said to be separately continuous (resp. separately quasicontinuous) if the sections f_x and f^y, $x \in X$ and $y \in Y$, are continuous (resp. quasicontinuous).

Example 2. Let $X = Y = Z = \mathbb{R}$ and $T_X = T_Y = T_Z = T_e$, where T_e is the natural topology in $\mathbb{R}$. Let

$$A = \left\{ (x, y);\ x > 0 \quad \text{and} \quad \frac{x}{2} \le y \le 2x \right\}$$

and let $f : \mathbb{R}^2 \to [0, 1]$ be a function such that

$$f(x, x) = 1 \qquad \text{for} \quad x > 0$$

and

$$f(x, y) = 0 \qquad \text{for} \quad (x, y) \in \mathbb{R}^2 \setminus A,$$

and f is continuous at each point $(x, y) \neq (0, 0)$. Then f is separately continuous. The function

$$g(x, y) = f(x, y) \qquad \text{for} \quad (x, y) \neq 0 \quad \text{and} \quad g(0, 0) = 1$$

is not separately continuous (it is not even separately quasicontinuous), but $f \rho g$. Note that both f and g are quasicontinuous.

However, the following is true.

Remark 3. Assume that (X, d_X) is a complete metric space, (Y, d_Y) is a compact metric space and (Z, d_Z) is a metric space. If functions $f, g : X \times Y \to Z$ are separately continuous and $f \rho g$, then $f = g$.

P r o o f. There is a residual set $A \subset X$ such that the function f is continuous at each point $(x,y) \in A \times Y$ ([10], p. 172, Exercise 6). By Remark 1 the equality $g(x,y) = f(x,y)$ is true at each point $(x,y) \in A \times Y$. Assume by contradiction that there is a point $(x_1,y_1) \in X \times Y$ at which $f(x_1,y_1) \neq g(x_1,y_1)$. Let

$$r = \frac{|f(x_1,y_1) - g(x_1,y_1)|}{2}.$$

Then $r > 0$ and from the continuity of the sections g^{y_1} and f^{y_1} at x_1 it follows that there exists an open neighbourhood $U \subset X$ of x_1 such that

$$\max |g(u,y_1) - g(x_1,y_1)|,\ |f(u,y_1) - f(x_1,y_1)| < r \qquad \text{for} \quad u \in U.$$

Since (X, d_X) is a complete metric space and A is a residual subset of X, the intersection $A \cap U \neq \emptyset$. Let w be an element of $U \cap A$. Then

$$\begin{aligned} g(w,y_1) &= f(w,y_1),\\ |g(w,y_1) - g(x_1,y_1)| &< r,\\ |f(w,y_1) - f(x_1,y_1)| &< r. \end{aligned}$$

So,

$$\begin{aligned} 2r &= |g(x_1,y_1) - f(x_1,y_1)|\\ &\le |g(x_1,y_1) - g(w,y_1)| + |g(w,y_1) - f(x_1,y_1)|\\ &< r + |f(w,y_1) - f(x_1,y_1)| < r + r = 2r, \end{aligned}$$

and this contradiction finishes the proof. □

Since there are different quasicontinuous functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f \rho g$ (for example $f(x) = g(x) = \sin\frac{1}{x}$ for $x \neq 0$, and $f(0) = 0$ and $g(0) = 1$), a quasicontinuous analogy of Remark 3 is not true.

In [8] (see also [6], [7]) Z. Piotrowski and R. Vallin investigate some very special notions of the quasicontinuity of functions of two variables. Let (Z, T_Z) be a topological space.

A function $f : X \times Y \to Z$ is said to be:

(1) quasicontinuous at a point $(x_1,y_1) \in X \times Y$ with respect to x (alternatively y) if for every set $U \times V \in T_X \times T_Y$ containing (x_1,y_1) and for each set $W \subset T_Z$ containing $f(x_1,y_1)$ there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x_1 \in U'$ (alternatively $y_1 \in V'$) and $f(U' \times V') \subset W$ ([8]);

(2) symmetrically quasicontinuous at (x_1,y_1) if it is quasicontinuous at (x_1,y_1) with respect to x and with respect to y ([8]).

Analogously as in the case of Theorem 1 we can prove the following.

Theorem 2. *Assume that (Z, T_Z) is a regular space. Let $f, g : X \times Y \to Z$ be functions such that $f \rho g$. If f is quasicontinuous at a point $(x_1, y_1) \in X \times Y$ with respect to x (alternatively y) [alternatively symmetrically quasicontinuous], then g has the same kind of quasicontinuity as f at this point.*

Some similar results are true for cliquishness. Let (M, d) be a metric space.

Recall that a function $f : X \to M$ is cliquish at a point $x \in X$ if for every real $\eta > 0$ and for every set $U \in T_X$ containing x there is a nonempty set $U_1 \in T_X$ contained in U such that the diameter $\operatorname{diam}\big(f(U_1)\big) = \sup\big\{d(f(x_1), f(x_2));\ x_1, x_2 \in U_1\big\} < \eta$ ([7]).

Theorem 3. *Let $f, g : X \to Y$ be functions such that $f \rho g$. If f is cliquish at a point x, then also g is cliquish at x.*

P r o o f. Fix a real $\eta > 0$ and a set $U \in T_X$ such that $x \in U$. Because of the cliquishness of f at x it follows that there is a nonempty set $U_1 \subset U$ belonging to T_X such that $\operatorname{diam}\big(f(U_1)\big) < \frac{\eta}{3}$. Fix a point $u \in U_1$ and let $V_1 = K\big(f(u), \frac{\eta}{3}\big)$ be the ball with the center $f(u)$ and the radius $\frac{\eta}{3}$. Since

$$cl\big(Gr(g)\big) \cap \big(U_1 \times cl(V_1)\big) = cl\big(Gr(f)\big) \cap \big(U_1 \times cl(V_1)\big)$$

and

$$\operatorname{diam}(V_1) = \operatorname{diam}\big(cl(V_1)\big) \le \frac{2\eta}{3} < \eta,$$

we have $\operatorname{diam}\big(g(U_1)\big) < \eta$. This completes the proof. $\square$

Remark 4. Let $S \subset \mathbb{R}^2$ be a Sierpiński nonmeasurable set of full outer Lebesgue measure such that for each straight line p the cardinality $card(p \cap S) \le 2$ ([9]), and let f be the characteristic function of S. Then f is separately cliquish and $cl\big(Gr(f)\big) = \mathbb{R}^2 \times \{0, 1\}$. Let Q denote the set of all rationals and let g be the characteristic functions of the set $Q \times Q$. Then $f \rho g$ and g is not separately cliquish.

A function $f : X \times Y \to M$ is said to be:

(1) cliquish at a point $(x_1, y_1) \in X \times Y$ with respect to x (alternatively y) if for every set $U \times V \in T_X \times T_Y$ containing (x_1, y_1) and for each real $\eta > 0$ there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x_1 \in U'$ (alternatively $y_1 \in V'$) and $\operatorname{diam}\big(f(U' \times V')\big) < \eta$ ([4]);

(2) symmetrically cliquish at (x_1, y_1) if it is cliquish at (x_1, y_1) with respect to x and with respect to y ([4]).

Analogously as in the case of Theorem 3 we can prove the following.

Theorem 4. *Let $f, g : X \times Y \to M$ be functions such that $f \rho g$. If f is cliquish at a point $(x_1, y_1) \in X \times Y$ with respect to x (alternatively y) [alternatively symmetrically cliquish], then g is the same at this point.*

Remark 5. Let (X, d_X) and (Y, d_Y) be metric spaces and let $\mathcal{I}$ be a proper ideal of subsets of X. Let $f : X \to Y$ be a function such that the set $D(f)$ of all its discontinuity points belongs to $\mathcal{I}$. If the relation $f \rho g$ works for a function $g : X \to Y$, then $D(g) \in \mathcal{I}$.

Proof. If f is continuous at a point $x \in X$, then by Remark 1 the function g is continuous at x and $g(x) = f(x)$. So, $D(g) \subset D(f) \in \mathcal{I}$. □

Corollary 1. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function such that the set $D(f)$ is of the first category. If the relation $f \rho g$ works for a function $g : X \to Y$, then $D(g)$ is also of the first category.*

Example 3. If $C \subset [0, 1]$ is a ternary Cantor set and $A \subset C$ is a nonborelien set, then the characteristic function $f = \kappa_C$ is in Baire 1 class (it is even in the discrete Baire 1 class), the characteristic function $g = \kappa_A$ is nonborelien and $f \rho g$.

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is in the discrete Baire 1 class if there is a sequence of continuous functions $f_n : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in \mathbb{R}^n$ there is a positive integer $n(x)$ with $f_n(x) = f(x)$ for $n > n(x)$ ([2]). f belongs to the discrete Baire 1 class if and only if for each nonempty closed set $H \subset \mathbb{R}^n$ there is an open set G such that $G \cap H \neq \emptyset$ and the restricted function $f/(G \cap H)$ is continuous.

From Remark 1 we also obtain.

Corollary 2. *If a function $f : \mathbb{R}^n \to \mathbb{R}$ is almost everywhere continuous (with respect to Lebesgue measure), then each function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f \rho g$ is also almost everywhere continuous.*

Example 4. Observe that the functions

$$f(x) = 0 \quad \text{for } x \le 0 \qquad \text{and} \qquad f(x) = 1 \quad \text{for} \quad x > 0,$$

and

$$g(x) = 0 \quad \text{for } x < 0 \qquad \text{and} \qquad g(x) = 1 \quad \text{for} \quad x \ge 0$$

are different, $D(f)$ and $D(g)$ belong to $\mathcal{I}$ according to Remark 5, and $f \rho g$.

Still consider Riemann's integral quasicontinuities. For this let T_e denote the Euclidean topology in $\mathbb{R}^n$ and T_{od} (T_{sd}) the ordinary (strong) density topology in $\mathbb{R}^n$ ([7]). Moreover, let μ denote the Lebesgue measure in $\mathbb{R}^n$.

We will say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is R-integrally quasicontinuous at a point $x \in \mathbb{R}^n$ (belongs to $Q_{r,s}(x)$) [belongs to $Q_{r,o}(x)$] if for each real $\eta > 0$

and for each set $U \in T_e$ ($U \in T_{sd}$) [$U \in T_{od}$] containing x there is a bounded Jordan measurable set $I \subset U$ such that $\mathrm{int}(I) \neq \emptyset$, f/I is integrable in the sense of Riemann, $I \subset U$ ([$\mathrm{int}(I) \cap U \neq \emptyset$] and

$$\left| \frac{\int_I f(t)\,\mathrm{d}t}{\mu(I)} - f(x) \right| < r,$$

$$\left| \frac{\int_{I \cap U} f(t)\,\mathrm{d}t}{\mu(I \cap U)} - f(x) \right| < r.$$

Remark 6. If a function $f : \mathbb{R}^n \to \mathbb{R}$ is R-integrally quasicontinuous at a point $x \in \mathbb{R}^n$, then each function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f \rho g$ and $f(x) = g(x)$ is also R-integrally quasicontinuous at x.

Proof. Fix a real $r > 0$ and a set $U \in T_e$ containing x. Since f is R-integrally quasicontinuous at x, there is a bounded Jordan measurable set $I \subset U$ such that $int(I) \neq \emptyset$, f/I is integrable in the sense of Riemann and

$$\left| \frac{\int_I f(t)\,\mathrm{d}t}{\mu(I)} - f(x) \right| < r.$$

However, the restricted function f/I is integrable in the sense of Riemann, so f is continuous almost everywhere on I and consequently, by Remark 1, we have $g = f$ almost everywhere on I and g is continuous almost everywhere on I. From the equality $f(x) = g(x)$ it follows that

$$\left| \frac{\int_I g(t)\,\mathrm{d}t}{\mu(I)} - g(x) \right| = \left| \frac{\int_I f(t)\,\mathrm{d}t}{\mu(I)} - f(x) \right| < r.$$

Analogously we can prove the following. $\square$

Remark 7. If a function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_{r.o}(x)$ ($Q_{r,s}(x)$), then each function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f \rho g$ and $f(x) = g(x)$ belongs also to $Q_{r,o}(x)$ ($Q_{r,s}(x)$).

EXAMPLE 5. The function

$$f\left(\frac{1}{n}\right) = 1 \qquad \text{for} \quad n = 1, 2, \ldots$$

and

$$f(t) = 0 \qquad \text{otherwise on} \quad \mathbb{R},$$

belongs to $Q_{s,o}(0)$ and

$$g(0) = g\left(\frac{1}{n}\right) = 1 \qquad \text{for} \quad n = 1, 2, \ldots$$

and

$$g(t) = 0 \qquad \text{otherwise on} \quad \mathbb{R},$$

is not R-integrally quasicontinuous at 0, but $f \rho g$.

Now, we will consider the Darboux property of functions from $\mathbb{R}$ to $\mathbb{R}$. Let $X = Y = \mathbb{R}$ and $T_X = T_Y = T_e$, where T_e denotes the natural topology in $\mathbb{R}$. We start with the following example.

EXAMPLE 6. Let $A \subset \mathbb{R}$ be a nowhere dense (in T_e) nonempty F_σ-set belonging to the density topology ([1]). By Zahorski's lemma ([1]) there is an approximately continuous and upper semi-continuous function $f : \mathbb{R} \to [0, 1]$ such that $f(A) = (0, 1]$, $f^{-1}(0) = \mathbb{R} \setminus A$ and f is continuous at points $x \in \mathbb{R} \setminus A$. Let $B \subset A$ be a countable set such that the set $E = \{(x, f(x)); x \in B\}$ is dense in the graph $Gr(f/A)$ of the restricted function f/A. Let $y \in (0, 1)$ be a real such that $f^{-1}(y) \cap B = \emptyset$. Put

$$g(x) = 0 \qquad \text{for} \quad x \in f^{-1}(y)$$

and

$$g(x) = f(x) \qquad \text{otherwise on} \quad \mathbb{R}.$$

Then $g \rho f$, f has Darboux property, and g does not have Darboux property.

Moreover, let

$$h(x) = \frac{f(x)}{2} \qquad \text{for} \quad x \in \mathbb{R}.$$

Then the function h is approximately continuous and $f \neq h$.

Observe that by modification of functions f and h on the components of the set $\mathbb{R} \setminus cl(A)$ we can define approximately continuous and simultaneously quasicontinuous functions ϕ and ψ such that $\phi \rho \psi$ and $\phi \neq \psi$.

THEOREM 5. *Let $\mathcal{H}$ be a class of almost everywhere (with respect to Lebesgue measure) continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the following condition*

(i) *if x is a discontinuity point of $f \in \mathcal{H}$, then either the point $(x, f(x))$ is isolated in $Gr(f)$ or for each function $g \in \mathcal{H}$, if $g/C(f) = f/C(f)$, then $g(x) = f(x)$ ($C(f)$ denotes the set of all continuity points of f).*

Then two arbitrary functions $f, g \in \mathcal{H}$ such that $f \rho g$ are equal.

Proof. Let $f, g \in \mathcal{H}$ be such that $f \rho g$. By Remark 1 we have $f(x) = g(x)$ for $x \in C(f)$. If x is a discontinuity point of f and the point $(x, f(x))$ is isolated in

$Gr(f)$, then $f(x) = g(x)$ by Remark 2. For other discontinuity points of f the equality $f(x) = g(x)$ follows from condition (i). □

Observe that the family C_{ae} of all almost everywhere continuous and everywhere approximately continuous functions or the family Δ_{ae} of all almost everywhere continuous locally integrable derivatives may be used as some examples of families $\mathcal{H}$ satisfying condition (i) of last theorem.

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