Tatra Mt. Math. Publ. **40** (2008), 161–169

Tatra Mountains
Mathematical Publications

THE ONE-TO-ONE RESTRICTIONS OF FUNCTIONS

Aleksandra Karasińska

ABSTRACT. A continuous and nowhere monotone function for which every set having the Baire property, on which it is one-to-one, is of first category, is constructed here. Further, it is shown that every continuous and nowhere monotone function has the above property. We also show that an analogous result fails to hold if one does not assume that a set on which the function is one-to-one is a Baire set.

In the second part of the paper we investigate the Lebesgue measure of a set on which a continuous nowhere monotone function is one-to-one. Here the situation turns out to be more varied. For each $\eta \in [0,1)$ we construct a continuous function non-monotone on any interval with the following property: there exists a set of measure η on which this function is one-to-one, and every set on which the function is one-to-one has measure smaller or equal to η.

We construct in the proof of the following theorem a continuous and nowhere monotone function f for which each set having the Baire property, on which f is one-to-one, is of first category. Further, we will show that each continuous and nowhere monotone function has this property; the construction below has, however, some independent interest, and will be used in the next part.

Let $f : [a,b] \to \mathbb{R}$. We will say that a function f is nowhere monotone if it is not monotone on any interval $(c,d) \subset [a,b]$.

Theorem 1. *There exists a continuous and nowhere monotone function $f : [0,1] \to \mathbb{R}$ for which each set $E \subset [0,1]$ having the Baire property and such that $f_{|E}$ is one-to-one, is of first category.*

Proof. Let us construct by induction a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ as follows. Let θ denote the Cantor function on interval $[0,1]$. It is defined in a following way: θ is constant on open component intervals of the complement of Cantor set C to interval $[0,1]$ and is equal to $\frac{1}{2}$ on interval $\left(\frac{1}{3},\frac{2}{3}\right)$, is equal to $\frac{1}{4}$ and $\frac{3}{4}$ on intervals $\left(\frac{1}{9},\frac{2}{9}\right)$ and $\left(\frac{7}{9},\frac{8}{9}\right)$, respectively. Proceed inductively. On the 2^{n-1} open

2000 Mathematics Subject Classification: 26A15, 26A48.
Keywords: nowhere monotone function, the set of first category, restriction of function.

intervals appearing at the n-th stage θ, takes the values

$$\frac{1}{2^n}, \frac{3}{2^n}, \dots, \frac{2^n-1}{2^n}$$

on these intervals and if x and y are members of different nth stage intervals with $x<y$, then $f(x)<f(y)$. This description defines θ on the complement of Cantor set C. Extend θ to all points of $[0,1]$ by defining $\theta(0)=0$ and for $x\neq 0$, $\theta(x)=\sup\big\{\theta(t): t<x,\ t\in[0,1]\setminus C\big\}$.
Put

$$f_1(x)=\begin{cases}\frac{1}{2}\theta(2x) & \text{if} \quad x\in\left[0,\frac{1}{2}\right],\\ \frac{1}{2}\theta(2-2x) & \text{if} \quad x\in\left(\frac{1}{2},1\right].\end{cases}$$

For every n, let $\mathcal{I}_n$ be a collection of all intervals on which the function f_n is constant. Put

$$f_{n+1}(x)=\begin{cases}(b-a)f_1\left(\frac{x-a}{b-a}\right)+f_n(a) & \text{if} \quad x\in(a,b) \quad \text{for some} \quad [a,b]\in\mathcal{I}_n,\\ f_n(x) & \text{otherwise.}\end{cases}$$

Let us note first, that if $[a,b]$ is an interval, where the function f_n is constant, then for each $x\in[a,b]$ we have

$$|f_{n+1}(x)-f_n(x)|\leq\left|f_{n+1}\left(\frac{a+b}{2}\right)-f_n(a)\right|=\frac{1}{2}(b-a). \tag{1}$$

It is not difficult to prove that for each $n\in\mathbb{N}$ the function f_n is continuous and the sequence $\{f_n\}_{n\in\mathbb{N}}$ fulfils the Cauchy condition of uniform convergence on interval $[0,1]$. Let us denote by f the limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$. We show that f is a required function. Observe that f is nowhere monotone. It follows from the fact that the union of intervals on which the Cantor function θ is constant is a dense subset of interval $[0,1]$ and the lengths of intervals on which f_n is constant tend to zero when n tends to infinity. Consequently, in each interval there exists some subinterval (a,b) on which some function f_n is constant. Thus

$$f(a)=f_n(a)<f_{n+1}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right)$$

and

$$f(b)=f_n(b)<f_{n+1}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right).$$

We will show that each set $E\subset[0,1]$ having the Baire property such that $f_{|E}$ is one-to-one is of first category. Let us take any set E with the required property. Then there exist an open set G and a set P of first category such that $E=G\bigtriangleup P$. We will prove that $G=\emptyset$. Suppose, on the contrary, that $G\neq\emptyset$.

As it was observed earlier there exists an nonempty interval $(a,b) \subset G$ on which some function f_n is constant. Then

$$f_m\left(\frac{a+b}{2}+t\right) = f_m\left(\frac{a+b}{2}-t\right)$$

for all $m > n$ and $t \in \left(0, \frac{b-a}{2}\right)$. Consequently,

$$f\left(\frac{a+b}{2}+t\right) = f\left(\frac{a+b}{2}-t\right)$$

for $t \in \left(0, \frac{b-a}{2}\right)$. Let P_1 be a set symmetrical to P with respect to the point $\frac{a+b}{2}$, i.e., $P_1 = \left\{(a+b-x) : x \in P\right\}$. Then $(a,b)\setminus(P \cup P_1)$ is nonempty and symmetrical with respect to the point $\frac{a+b}{2}$. Moreover, $(a,b) \setminus (P \cup P_1) \subset G \setminus P \subset E$. Let $x \in (a,b) \setminus (P \cup P_1)$ and $x \neq \frac{a+b}{2}$. Then $a+b-x \in (a,b) \setminus (P \cup P_1)$ and $f(x) = f(a+b-x)$ which is not possible, because $f_{|E}$ is one-to-one. $\square$

Obviously, if $f : [a,b] \to \mathbb{R}$ is continuous and strictly increasing function on some interval $[c,d]$, then there exists a set E having the Baire property and of the second category such that $f_{|E}$ is one-to-one. Moreover, if $f : [a,b] \to \mathbb{R}$ is continuous and the union of intervals on which f is constant is a residual set, then every set E such that $f_{|E}$ is one-to-one is of first category and even nowhere dense.

Lemma 2 ([BDL], Lemma 1)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function, not constant on any interval. If $A \subset \mathbb{R}$ is a nowhere dense set then $f^{-1}(A)$ is nowhere dense in the interval $[a,b]$.*

Corollary 3 ([BDL], Lemma 1)**.** *If a function $f : [a,b] \to [c,d]$ is continuous and not constant on any interval, then $f^{-1}(E)$ is residual in $[a,b]$ if E is residual in $[c,d]$.*

Lemma 4 ([BDL], Lemma 2)**.** *Let $f : [a,b] \to [c,d]$ be a continuous function, not constant on any interval. If $E \subset [a,b]$ is a G_δ set dense in $[a,b]$, then $f(E)$ is residual in $f([a,b])$.*

Corollary 5. *If a function $f : [a,b] \to \mathbb{R}$ is continuous, not constant on any interval and $D \subset [a,b]$ is residual in $[a,b]$, then $f(D)$ is residual in $f\big([a,b]\big)$.*

P r o o f. The set D is residual in $[a,b]$, so it contains a G_δ set E which is dense in $[a,b]$. By Lemma 4 we have that $f(E)$ is residual in $f\big([a,b]\big)$. Moreover, $f(E) \subset f(D)$. Thus $f(D)$ is residual in $f\big([a,b]\big)$. $\square$

Now, we will present the result which will reduce the essence of Theorem 1 by showing an example of a continuous and nowhere monotone function with the desired property.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be continuous and nowhere monotone. Then each set E having the Baire property such that $f_{|E}$ is one-to-one, is of first category.*

P r o o f. Let E be a set having the Baire property such that $f_{|E}$ is one-to-one and suppose that E is not of first category. Of course, $E = G \triangle P$, where G is a nonempty open set and P is of first category. Let $P = \bigcup_{n\in\mathbb{N}} P_n$, where P_n is nowhere dense for every $n \in \mathbb{N}$. There exists a nonempty interval $(c,d) \subset G$. The function f is not monotone on (c,d), so there exist three points $x_1, x_2, x_3 \in (c,d)$ such that $x_1 < x_2 < x_3$, and

a) $f(x_1) < f(x_2), f(x_3) < f(x_2)$ or

b) $f(x_1) > f(x_2), f(x_3) > f(x_2)$.

Let us consider case a) and assume additionally that $f(x_1) < f(x_3)$. Let $z_2 \in (x_1,x_3)$ be a point at which f attains its maximum on interval (x_1,x_3). Let

$$z_1 = \sup\{x \in [x_1,z_2] : f(x) = f(x_3)\},$$
$$z_3 = \inf\{x \in [z_2,x_3] : f(x) = f(x_3)\},$$
$$z_4 = \inf\{x \in [z_1,z_2] : f(x) = f(z_2)\},$$
$$z_5 = \sup\{x \in [z_2,z_3] : f(x) = f(z_2)\}.$$

Such points exist by the continuity and Darboux property of the function f. Then

$$f\big([z_1,z_4]\big) = f\big([z_5,z_3]\big) = \big[f(z_1), f(z_2)\big] = \big[f(x_3), f(z_2)\big].$$

Let us put

$$E_1 = (c,d) \setminus \bigcup_{n\in\mathbb{N}} \overline{\big(P_n \cap (c,d)\big)}.$$

We have

$$E_1 \subset (c,d) \setminus \bigcup_{n\in\mathbb{N}} \big(P_n \cap (c,d)\big) = (c,d) \setminus P \subset G \setminus P \subset E.$$

Obviously, E_1 is a G_δ set dense in interval (c,d). Clearly, $E_1 \cap [x_1,z_4]$ and $E_1 \cap [z_5,z_3]$ are G_δ sets which are dense, the first one in $[x_1,z_4]$, the second one in $[z_5,z_3]$. Thus by Lemma 4 we obtain that $f\big(E_1 \cap [x_1,z_4]\big)$ is residual in $\big[f(x_3), f(z_2)\big]$ and $f\big(E_1 \cap [z_5,z_3]\big)$ is residual in $\big[f(x_3), f(z_2)\big]$. The intersection of these sets is a residual set, so it has at least two points, which is contrary to the fact that f is one-to-one on E_1. □

Let $f : [a,b] \to \mathbb{R}$ be a continuous and non-constant function. Let us put

$$B_f = \Big\{y \in f\big([a,b]\big) : f^{-1}\big(\{y\}\big) \quad \text{is of the first category}\Big\}$$

and

$$C_f = f^{-1}(B_f).$$

Obviously, since the set $f^{-1}(\{y\})$ is closed, it is of first category if and only if it does not contain an interval. If θ is Cantor function, then C_θ is nowhere dense because it is contained in Cantor set C. If f is a function from Theorem 1 then $C_f = [a, b]$.
If $f : [a, b] \to \mathbb{R}$ is constant, then each set E such that $f_{|E}$ is one-to-one, is of first category, because it is a singleton.

Let us put

$$D_f = \Big\{ y \in f\big([a, b]\big) : f^{-1}\big(\{y\}\big) \quad \text{is not of the first category} \Big\}.$$

If $y \in D_f$, then $f^{-1}\big(\{y\}\big)$ contains a non-degenerate interval. For points $y_1, y_2 \in D_f$ such that $y_1 \neq y_2$ we have $f^{-1}\big(\{y_1\}\big) \cap f^{-1}\big(\{y_2\}\big) = \emptyset$, so D_f is countable.

Lemma 7. *The set C_f is either nowhere dense or residual in some interval.*

Proof. If C_f is not residual in I then the set $I \setminus C_f = \bigcup_{y \in D_f} I \cap f^{-1}\big(\{y\}\big)$ is not of first category. Since D_f is countable, one of the sets $I \cap f^{-1}\big(\{y\}\big)$, $y \in D_f$ must contain an interval J, and C_f is disjoint with J. Therefore, if there is no interval I such that C_f is residual in I then C_f is nowhere dense. □

Theorem 8. *If a function $f : [a, b] \to \mathbb{R}$ is continuous and the set C_f is of first category, then each set E such that $f_{|E}$ is one-to-one, is of first category.*

Proof. Obviously, $f^{-1}(D_f) = [a, b] \setminus C_f$. Thus $f^{-1}(D_f)$ is residual in the interval $[a, b]$. Let us take a set E such that $f_{|E}$ is one-to-one. Thus $E = E_1 \cup E_2$, where $E_1 = E \cap C_f$ and $E_2 = E \cap (C_f)'$. Obviously, E_1 is of first category and the set E_2 has at most one common point with each set $f^{-1}\big(\{y\}\big)$ for $y \in D_f$, so E_2 is countable, because $card(D_f) \leq \chi_0$. Consequently, E is of first category. □

The following theorem shows that the assumpion in Theorem 6 that E has the Baire property cannot be removed. If we consider the function f from Theorem 1, we can obtain the set $E \subset [0, 1]$ such that E is not of first category and $f_{|E}$ is one-to-one. Obviously, the set E does not have the Baire property.

Theorem 9 (CH). *If the set C_f is not of first category, then there exists a set $E \subset [a, b]$ such that E is not of first category and $f_{|E}$ is one-to-one.*

Proof. Let $\{F_\alpha\}_{\alpha < \omega_1}$ be a well-ordering of all F_σ sets contained in interval $[a, b]$ which are of first category (ω_1 is the first uncountable ordinal number). As a result every set F_α has only countably many predecessors. Let $x_0 \in C_f \setminus F_0$ and let $\alpha < \omega_1$. Assume we have already chosen x_γ for $\gamma < \alpha$. Choose a point $x_\alpha \in C_f$ such that

$$x_\alpha \notin \bigcup_{\gamma < \alpha} F_\gamma \cup \bigcup_{\gamma < \alpha} f^{-1}\Big(\big\{f(x_\gamma)\big\}\Big).$$

It is possible because the set C_f is not of first category and $x_\gamma \in C_f$ for $\gamma < \alpha$, thus $f(x_\gamma) \in B_f$ and $f^{-1}\big(\{f(x_\gamma)\}\big)$ is of first category for $\gamma < \alpha$. The set $\bigcup_{\gamma<\alpha} F_\gamma \cup \bigcup_{\gamma<\alpha} f^{-1}\big(\{f(x_\gamma)\}\big)$ is of first category since it is a countable union of sets of first category.
Let us put $E = \{x_\alpha : \alpha < \omega_1\}$. Clearly, E is not of first category because $E \not\subset F_\alpha$ for any $\alpha < \omega_1$ and $f_{|E}$ is one-to-one. □

The previous theorem is also true if we replace (CH) by Martin's Axiom (MA) or by the assumption that the additivity of an ideal of the sets of first category equals to the cardinality of continuum ($\mathrm{add}(M) = \mathfrak{c}$). That is in the proof of Theorem 9, where we only need the assumption that the union of less than continuum many sets of first category is also a set of first category.

Let $f : [a,b] \to \mathbb{R}$. Put

$$A_1 = \Big\{x \in [a,b] : x = \sup\big\{t \in [a,b] : f(x) = f(t)\big\}\Big\}.$$

Lemma 10. *If f is a continuous and nowhere monotone function, then the set A_1 is nowhere dense.*

P r o o f. Consider an interval I contained in $[a,b]$ and assume that $\mathrm{card}(A_1 \cap I) \geq 2$. Let $x_1, x_2 \in A_1 \cap I$ and $x_1 < x_2$. Without loss of generality we can assume that $f(x_1) < f(x_2)$. Let us put

$$x_3 = \inf\big\{x \in [x_1, x_2] : f(x) = f(x_2)\big\}.$$

By the definition of the points x_1, x_3 and Darboux property we have $f(x_1) < f(x) < f(x_3)$ for $x \in (x_1, x_3)$. Moreover, the function f is nowhere monotone, so there exist points $x_4, x_5 \in (x_1, x_3)$ such that $x_4 < x_5$ and $f(x_4) > f(x_5)$. Let us put

$$x_6 = \sup\big\{x \in [x_4, x_5] : f(x) = f(x_4)\big\}$$

and

$$x_7 = \inf\big\{x \in [x_6, x_5] : f(x) = f(x_5)\big\}.$$

Obviously, $x_4 \leq x_6 < x_7 \leq x_5$. Moreover, for $x \in (x_6, x_7)$ we have $f(x_7) < f(x) < f(x_6)$. Of course, $(x_6, x_7) \subset I$ and $(x_6, x_7) \neq \emptyset$. We will show that $(x_6, x_7) \cap A_1 = \emptyset$. Suppose that there exists a point x_0 such that $x_0 \in A_1 \cap (x_6, x_7)$. Clearly,

$$f(x_5) = f(x_7) < f(x_0) < f(x_6) = f(x_4) < f(x_3).$$

Since $f(x_0) \in \big(f(x_5), f(x_3)\big)$ and the function f is continuous, there exists a point $x'' \in (x_5, x_3)$ such that $f(x'') = f(x_0)$ and $x_0 < x''$. This is impossible because $x_0 \in A_1$. Consequently, A_1 is nowhere dense. □

We will say that a function $f : \mathbb{R} \to \mathbb{R}$ fulfils (N)-Lusin condition for category if $f(E)$ is a set of first category for each set $E \subset \mathbb{R}$ of first category.

Theorem 11. *Let $f : [a,b] \to \mathbb{R}$ be a continuous and nowhere monotone function. Then f does not fulfil (N)-Lusin condition for category.*

Proof. By Lemma 10 we obtain that A_1 is the set of first category. By [HS], Ex. 6.101 we have $f(A_1) = f\big([a,b]\big)$. The set $f\big([a,b]\big)$ is not of first category, so the function f does not fulfil (N)-Lusin condition for category. $\square$

It is natural to ask what one can say about the Lebesgue measure of a set on which a continuous nowhere monotone function is one-to-one. It turns out that for the case of measure the situation is more complicated than for category.

Theorem 12. *There exists a continuous and nowhere monotone function $f : [0,1] \to \mathbb{R}$ for which each measurable set E such that $f_{|E}$ is one-to-one has measure zero.*

Proof. The function f constructed in the proof of Theorem 1 posesses this property. Indeed, let $E \subset [0,1]$ be a measurable set such that $f_{|E}$ is one-to-one. Let us denote by D the union of all Cantor-like sets used in the construction of f. Clearly $m(D) = 0$. We have $E = (E \cap D) \cup (E \setminus D)$. It suffices to examine the measure of $E \setminus D$. Let $x_0 \in E \setminus D$. Let us note that $\{x_0\} = \bigcap_{n=1}^{\infty} I_n$, where I_n is an interval with the function f_n constant. Let us take any positive integer n. Let $I_n = (a,b)$. Obviously, $x_0 \neq \frac{a+b}{2}$.
Assume that $x_0 \in \left(a, \frac{a+b}{2}\right)$ and let (c,d) denote the longest interval contained in $\left(\frac{a+b}{2}, b\right)$ on which f_{n+1} is constant. The set $E \cap (c,d)$ is measurable and has the following property: if $z \in (c,d) \cap E$ then the point symmetrical to z with respect to the point $\frac{c+d}{2}$ does not belong to $(c,d) \cap E$. Let us put

$$A_1 = E \cap \left(c, \frac{c+d}{2}\right), \quad A_2 = E \cap \left(\frac{c+d}{2}, d\right).$$

The sets A_1, A_2 are measurable and disjoint. Let us remark that

$$m(A_1 \cup A_2) = m(A_1) + m(A_2) \leq \frac{1}{2}\,(d-c).$$

Indeed, suppose that $m(A_1 \cup A_2) > \frac{1}{2}(d-c)$ and let A_2^* be symmetrical reflection of the set A_2 with respect to the point $\frac{c+d}{2}$. We have $m(A_2^*) = m(A_2)$. Obviously, $A_1 \cap A_2^* = \emptyset$ and $A_1 \cup A_2^* \subset \left(c, \frac{c+d}{2}\right)$. Then

$$m(A_1 \cup A_2^*) = m(A_1) + m(A_2^*) = m(A_1) + m(A_2) = m(A_1 \cup A_2) > \frac{1}{2}(d-c),$$

which is impossible. Thus we have shown that $m(A_1 \cup A_2) \leq \frac{1}{2}(d-c)$. Then

$$m\big(E' \cap (c,d)\big) \geq \frac{1}{2}(d-c) = \frac{1}{12}(b-a).$$

Consequently,

$$\frac{m\big(E' \cap (x_0, d)\big)}{d - x_0} > \frac{m\big(E' \cap (c, d)\big)}{d - a} \geq \frac{\frac{1}{12}(b-a)}{\frac{5}{6}(b-a)} = \frac{1}{10}.$$

Hence, we obtain

$$\limsup_{h \to 0^+} \frac{m\big(E' \cap [x_0, x_0 + h]\big)}{h} \geq \frac{1}{10}.$$

Thus, x_0 is not a density point of the set E, and so the set $E \setminus D$ has no density point. The same we can obtain if $x_0 \in \big(\frac{a+b}{2}, b\big)$. Finally, the set $E \setminus D$ does not have density points, so by the Lebesgue density theorem, $E \setminus D$ is a set of measure zero. □

Theorem 13. *For each number $\eta \in (0,1)$ there exist a continuous and nowhere monotone function $g : [0,1] \to \mathbb{R}$ and a set $F \subset [0,1]$ such that $m(F) = \eta$, the function $g_{|F}$ is one-to-one and for each measurable set $E \subset [0,1]$, if $g_{|E}$ is one-to-one then $m(E) \leq \eta$.*

Proof. Let $\eta \in (0,1)$ and let C_η denote the Cantor-like set with measure η constructed on the interval $[0,1]$. Let θ_η denote the Cantor-type function on interval $[0,1]$ connected with the set C_η. Put

$$g(x) = \begin{cases} \theta_\eta(x) & \text{for} \quad x \in C_\eta, \\ (b-a) f\big(\frac{x-a}{b-a}\big) + \theta_\eta(a) & \text{for} \quad x \in (a,b), \end{cases}$$

where (a,b) is an arbitrary contiguous interval of the set C_η and f is the function constructed in the proof of Theorem 1. It is not difficult to prove that the function g is continuous on interval $[0,1]$. The function g is nowhere monotone because the union of intervals where θ_η is constant is dense on interval $[0,1]$, and the function f by Theorem 1 is nowhere monotone.

Let H be an arbitrary measurable set such that $g_{|H}$ is one-to-one. Obviously,

$$H = \big(H \cap C_\eta\big) \cup \big(H \cap C'_\eta\big),$$

and $m(H) = m\big(H \cap C_\eta\big) + m\big(H \cap C'_\eta\big)$. Moreover, $m\big(H \cap C_\eta\big) \leq \eta$. We will show that $m\big(H \cap C'_\eta\big) = 0$. Notice that

$$H \cap C'_\eta = \bigcup_{(c,d) \subset C'_\eta} H \cap (c,d),$$

where (c,d) is an interval on which θ_η is constant. Then

$$m\big(H \cap C'_\eta\big) = \sum_{(c,d) \subset C'_\eta} m\big(H \cap (c,d)\big).$$

Since we define the function g on interval (c,d) as the composition of the function f and a linear function, by a property of the Lebesgue measure and by the proof of Theorem 1 we have $m\big(H \cap C'_\eta\big) = 0$. Consequently, $m(H) \le \eta$.

Let A be the set of all right ends of intervals on which the function θ_η is constant. Let us put $E = C_\eta \setminus A$. Obviously, $g_{|E} = \theta_{\eta|E}$. Therefore the function $g_{|E}$ is one-to-one. Moreover, $m(E) = m(C_\eta) = \eta$. $\square$

By [BBT, Ex. 10:6.6] we have that the set of all continuous functions which are one-to-one almost everywhere on interval $[a,b]$ is residual in the space of the continuous function on interval $[a,b]$, so the typical (in the sense of Baire) continuous function defined on $[a,b]$ is one-to-one a.e. on $[a,b]$. It is well-known that the typical continuous function is nowhere differentiable, so it is nowhere monotone. Consequently, the typical continuous function is nowhere monotone and one-to-one almost everywhere on $[a,b]$.

REFERENCES

[BBT] BRUCKNER, A. M.—BRUCKNER, J. B.—THOMSON, B. S.: *Real Analysis.* Prentice-Hall International, Upper Saddle River, New Jersey, 1997.

[BDL] BROWN, J. B.—DARJI, U. B.—LARSEN, E .P.: *Nowhere monotone functions and functions of nonmonotonic type,* Proc. Amer. Math. Soc. **127** (1999), 173–182.

[HS] HEWITT, E.—STROMBERG, K.: *Real and Abstract Analysis.* Springer-Verlag, Berlin, 1965.

[K] KUNEN, K.: *Set Theory. An Introduction to Independence Proofs.* Stud. Logic Found. Math., Vol. 102, North-Holland, Amsterdam, 1980.

Received March 2, 2007

Faculty of Mathematics
and Informatics
University of Łódź
Banacha 22
PL–90-238 Łódź
POLAND

E-mail: karasia@math.uni.lodz.pl