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ON THE DENSITY TOPOLOGIES GENERATED BY FUNCTIONS

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ABSTRACT. The paper contains investigation of the density topologies generated by functions f such that $\limsup_{x\to 0^+} \frac{x}{f(x)} > 0$.

The papers [FF1] and [FF2] contain the concept of density topologies in the family of Lebesgue measurable sets with respect to functions having some desired properties. In this paper the concept of symmetrical density point and density topology, respectively, with respect to a function $f : R_+ \to R_+$, where $R_+ = (0; +\infty)$, assuming only $\limsup_{x\to 0^+} \frac{x}{f(x)} > 0$ is introduced. The definition of so-called symmetrical f-density point is less restrictive then f-density point considered in [FF1] and [FF2]. Some properties of topologies obtained via the operators of symmetrical f-density points are investigated. Throughout the paper we will use the standard notations: R is the set of real numbers, $\mathbb{N}$ the set of positive integers, ℓ the Lebesgue measure on R, $\mathcal{L}$ the family of Lebesgue measurable sets, $\mathbb{L}$ the family of Lebesgue null sets. If $\ell(A \triangle B) = 0$ for the sets $A, B \in \mathcal{L}$, then we use notation $A \sim B$. Let $\Phi_d(A)$, where $A \in \mathcal{L}$, be the set of all Lebesgue density points of the set A and by $\mathcal{T}_d$ we will denote the density topology on R. Let T_{nat} be the natural topology on R.

Let $\Phi : \mathcal{L} \to 2^R$ be an operator such that for any sets $A, B \in \mathcal{L}$, we infer that

1° $\Phi(\emptyset) = \emptyset,\ \Phi(R) = R\,,$

2° $A \sim B \implies \Phi(A) = \Phi(B)\,,$

3° $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)\,,$

4° $\ell(\Phi(A) - A) = 0\,.$

Theorem 1 (cf. [H]). *Let $\Phi : \mathcal{L} \to 2^R$ be an operator satisfying conditions $1° - 4°$ above. Then the family $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ is a topology on R.*

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Theorem 2 (cf. [H], [W]). *Let* $\mathcal{T}_\Phi = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ *be a topology obtained via operator* $\Phi : \mathcal{L} \to 2^R$ *having properties* $1^\circ - 4^\circ$. *Then for every set* $E \subset R$ *the following properties hold:*

a) *if* $E \in \mathbb{L}$, *then* E *is* $\mathcal{T}_\Phi$*-nowhere dense*,
b) $E \in \mathbb{L}$ *if and only if the set* E *is* $\mathcal{T}_\Phi$*-closed and* $\mathcal{T}_\Phi$*-discrete*,
c) *the set* E *is compact if and only if* E *is finite*,
d) *the space* $(R, \mathcal{T}_\Phi)$ *is neither Lindelöf space nor the first countable and separable*,
e) *if* $\mathcal{T}_{nat} \subset \mathcal{T}_\Phi$, *then* $\mathcal{L} = \mathcal{B}(\mathcal{T}_\Phi) = F_\sigma(\mathcal{T}_\Phi)$, *where* $\mathcal{B}(\mathcal{T}_\Phi)$ *and* $F_\sigma(\mathcal{T}_\Phi)$ *is the family of all Borel sets and* F_σ*-type sets with respect to topology* $\mathcal{T}_\Phi$, *respectively.*

Now, we give an example of operator $\Phi : \mathcal{L} \to 2^R$ having properties $1^\circ - 4^\circ$ in retard to the density points with respect to functions. Let $\mathcal{A}$ be a family of functions $f : R_+ \to R_+$ such that

a) $\lim_{x \to 0^+} f(x) = 0$,
b) f is nondecreasing,
c) f is continuous,
d) $\limsup_{x \to 0^+} \frac{x}{f(x)} > 0$.

Let $f \in \mathcal{A}$. Following the concept of a density point with respect to the function introduced in paper [FF1] we will give

Definition 3. The point $x \in R$ is an f-density point of a set $A \in \mathcal{L}$ if

$$\lim_{\substack{h \to 0, k \to 0 \\ h \geq 0, k \geq 0 \\ h+k > 0}} \frac{\ell\big(A' \cap [x-h,\, x+k]\big)}{f(h+k)} = 0 .$$

Let $\Phi_{\langle f \rangle}(A) = \{x \in R : x \text{ is an } f\text{-density point of } A\}$.

It was proved (see [FF1]) that $\Phi_{\langle f \rangle}(A) \in F_{\sigma\delta}$ and the family $\mathcal{T}_{\langle f \rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle f \rangle}(A)\}$ forms a topology on R. Some observations pointed in [FF2] prove that condition c) is not necessary to get that the family $\mathcal{T}_{\langle f \rangle}$ is the topology on R.

Now, we will extend the class $\mathcal{A}$ to the class $\mathcal{B}$ assuming that

$$\mathcal{B} = \left\{ f : R_+ \to R_+,\ \limsup_{x \to 0^+} \frac{x}{f(x)} > 0 \right\}.$$

Property 4. *Let* $f \in \mathcal{B}$. *Then the operator* $\Phi_{\langle f \rangle} : \mathcal{L} \to 2^R$ *satisfies conditions* $1^\circ - 4^\circ$.

P r o o f. It is clear that conditions $1^\circ - 3^\circ$ are satisfied. To prove condition 4°, let us notice that

$$\underset{A\in\mathcal{L}}{\forall}\ \Phi_{\langle f\rangle}(A) \subset R - \Phi_d(R-A)\,.$$

Let

$$x \notin \big(R - \Phi_d(R-A)\big).$$

Then

$$x \in \Phi_d(R-A)$$

and

$$\limsup_{\substack{h\to 0^+,\,k\to 0^+\\ h+k>0}} \frac{\ell\big(A' \cap [x-h, x+k]\big)}{g(h+k)} = \lim_{\substack{h\to 0^+,\,k\to 0^+\\ h+k>0}} \frac{\ell\big(A' \cap [x-h, x+k]\big)}{h+k} \times \limsup_{\substack{h\to 0^+,\,k\to 0^+\\ h+k>0}} \frac{h+k}{g(h+k)} > 0\,.$$

Hence $x \notin \Phi_{\langle f\rangle}(A)$. Thus

$$\Phi_{\langle f\rangle}(A) - A \subset (R-A) - \Phi_d(R-A)\,.$$

By the Lebesgue density theorem we conclude that

$$\ell\big(\Phi_{\langle f\rangle}(A) - A\big) = 0\,.$$

□

Corollary 5. *The family* $\mathcal{T}_{\langle f\rangle} = \big\{A \in \mathcal{L} : A \subset \Phi_{\langle f\rangle}(A)\big\}$ *forms a topology on* R *and* $\mathcal{T}_{nat} \subset \mathcal{T}_{\langle f\rangle}$.

Theorem 6. *Let* $f : R_+ \to R_+$. *Then the family* $\mathcal{T}_{\langle f\rangle} = \big\{A \in \mathcal{L} : A \subset \Phi_{\langle f\rangle}(A)\big\}$ *forms a topology on* R *if and only if* $f \in \mathcal{B}$.

P r o o f. Necessity. Let us suppose that $f \notin \mathcal{B}$. Then

$$\underset{A\in\mathcal{L}}{\forall}\ \Phi_{\langle f\rangle}(A) = R\,.$$

Hence

$$\underset{x\in R}{\forall}\ \{x\} \in \mathcal{T}_{\langle f\rangle}\,.$$

Let $X \notin \mathcal{L}$. Then $X = \bigcup_{x\in X}\{x\} \notin \mathcal{T}_{\langle f\rangle}$. It contradicts the fact that the family $\mathcal{T}_{\langle f\rangle}$ is a topology on R.

Sufficiency is the consequence of Property 3 and Theorem 1. □

Now, we will consider the concept of the symmetrical density with respect to a function.

Let $f : R_+ \to R_+$ be an arbitrary function.

Definition 7. We will say that a point $x \in R$ is a symmetrical f-density point of the set $A \in \mathcal{L}$ if

$$\lim_{h\to 0^+} \frac{\ell\big(A' \cap [x-h,\, x+h]\big)}{f(2h)} = 0\,.$$

This definition generalizes a classical density and ψ-density (see [W], [TW-B]).

Let $A \in \mathcal{L}$ and $\Phi^s_{\langle f\rangle}(A) = \{x \in R : x$ is a symmetrical f-density point of $A\}$.

It is clear that for the function $f(x) = x$ and $A \in \mathcal{L}$ we have

$$\Phi_{\langle f\rangle}(A) = \Phi^s_{\langle f\rangle}(A) = \Phi_d(A).$$

However, it is not true that the concepts of f-density and symmetrical f-density are identical.

Theorem 8. *There exists a set $A \in \mathcal{L}$ and a function $f \in \mathcal{B}$ such that*

$$\lim_{h\to 0} \frac{\ell\big(A' \cap [-h,h]\big)}{f(2h)} = 0$$

and

$$\limsup_{h\to 0^+} \frac{\ell\big(A' \cap [0,h]\big)}{f(h)} > 0\,.$$

P r o o f. Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $\lim_{n\to 0} a_n = 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Let

$$f(x) = \begin{cases} x & x = a_n\,, \\ 1 & x \neq a_n\,. \end{cases}$$

It is clear that $f \in \mathcal{B}$.

Putting

$$B = \bigcup_{n=1}^{\infty} \left[\frac{1}{2}\, a_n;\, a_n\right] \cup \left[-a_n;\, -\frac{1}{2}\, a_n\right] \quad \text{and} \quad A = R - B$$

we have $A \in \mathcal{L}$. Moreover, $\lim \frac{\ell(A'\cap[-h,h])}{f(2h)} = 0$. If $h \neq \frac{1}{2}\, a_n$ for $n \in \mathbb{N}$ then

$$\frac{\ell\big(A' \cap [-h,h]\big)}{f(2h)} = \frac{2\ell\big(B \cap [0;h]\big)}{f(2h)} \leq 2h\,.$$

If $h = \frac{1}{2}\, a_n$ for some $n \in \mathbb{N}$, then

$$\frac{\ell\big(A' \cap [-\frac{1}{2}\, a_n,\, \frac{1}{2}\, a_n]\big)}{f(a_n)} = \frac{2\ell\big(B \cap [0; \frac{1}{2}\, a_n]\big)}{a_n} = \frac{2\ell\big([0;\, a_{n+1}]\big)}{a_n} = \frac{2a_{n+1}}{a_n}\,.$$

The two above inequalities imply that

$$\lim_{h\to 0^+} \frac{\ell\big(A' \cap [-h,h]\big)}{f(2h)} = 0\,.$$

Simultaneously,

$$\limsup_{h\to 0^+} \frac{\ell\big(A'\cap[0,h]\big)}{f(h)} \ge \limsup_{n\to\infty} \frac{\ell\big(B\cap[0;a_n]\big)}{f(a_n)} \ge \limsup_{n\to\infty} \frac{\ell\Big(\big[\frac{1}{2}a_n, a_n\big]\Big)}{a_n} = \frac{1}{2}\,.$$

□

Similarly, as Property 4, we get the following one:

Property 9. *Let $f \in B$. Then the operator $\Phi^s_{\langle f\rangle} = \mathcal{L} \to 2^R$ satisfies conditions* $1^\circ - 4^\circ$.

Corollary 10. *The family $\mathcal{T}^s_{\langle f\rangle} = \big\{A \in \mathcal{L} : A \subset \Phi^s_{\langle f\rangle}\big\}$ is a topology on R such that $\mathcal{T}_{nat} \subset \mathcal{T}^s_{\langle f\rangle}$ and $\mathcal{T}^s_{\langle f\rangle} \subset \mathcal{T}_{\langle f\rangle}$.*

Let us observe that there exists a function $f \in \mathcal{B}$ such that

$$\mathcal{T}_{\langle f\rangle} \not\subseteq \mathcal{T}^s_{\langle f\rangle}\,.$$

Let the function f and the set A be the same as in the proof of Theorem 8. Then

$$A\cup\{0\} \notin \mathcal{T}_{\langle f\rangle} \quad \text{and} \quad A\cup\{0\} \in \mathcal{T}^s_{\langle f\rangle}\,.$$

In the same way to that of Theorem 6, we get the following one

Theorem 11. *Let $f : R_+ \to R_+$. Then the family $\mathcal{T}^s_{\langle f\rangle} = \big\{A \in \mathcal{L} : A \subset \Phi_{\langle f\rangle}(A)\big\}$ is a topology on R if and only if $f \in \mathcal{B}$.*

Corollary 12. *Let $f : R_+ \to R_+$. Then the following conditions are equivalent:*

a) $f \in \mathcal{B}$,

b) *the family $\mathcal{T}_{\langle f\rangle}$ forms topology on R,*

c) *the family $\mathcal{T}^s_{\langle f\rangle}$ forms topology on R.*

Theorem 13. *Let $f \in \mathcal{B}$. Then $\mathcal{T}_{\langle f\rangle} = \mathcal{T}^s_{\langle f\rangle}$ if and only if for every set $A \in \mathcal{L}$ we have that $\Phi_{\langle f\rangle}(A) = \Phi^s_{\langle f\rangle}(A)$.*

Proof. Sufficiency is obvious.

Let us prove necessity. Let $f \in \mathcal{B}$ and $\mathcal{T}_{\langle f\rangle} = \mathcal{T}^s_{\langle f\rangle}$. It is sufficient to show that for every set $A \in \mathcal{L}$ we have that $\Phi^s_{\langle f\rangle}(A) \subset \Phi_{\langle f\rangle}(A)$. For the simplicity, let us assume that there exists a set $A \in \mathcal{L}$ such that $0 \in \Phi^s_{\langle f\rangle}(A) - \Phi_{\langle f\rangle}(A)$. There exist $\alpha > 0$, sequences $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$ tending to 0 and such that $\alpha_n \ge 0$, $\beta_n \ge 0$, $\alpha_n + \beta_n > 0$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\ell\big(A'\cap[-\alpha_n,\beta_n]\big)}{f(\alpha_n+\beta_n)} \ge \alpha > 0\,.$$

Hence there exist decreasing sequences $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ and $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\frac{\ell\big(A'\cap[-\alpha_{n_k},0]\big)}{f(\alpha_{n_k}+\beta_{n_k})}\geq\frac{\alpha}{2}$$

or

$$\lim_{k\to\infty}\frac{\ell\big(A'\cap[0,\beta_{n_k}]\big)}{f(\alpha_{n_k}+\beta_{n_k})}\geq\frac{\alpha}{2}\,.$$

Let us suppose that

$$\lim_{n\to\infty}\frac{\ell\big(A'\cap[0,\beta_n]\big)}{g(\alpha_n+\beta_n)}=\beta>0\,.$$

Putting

$$b_n=\beta_n-\ell\big((\beta_{n+1},\beta_n)\cap A'\big)\qquad\text{for}\quad n\in\mathbb{N}\,,$$

and

$$B=(-\infty;0]\cup\bigcup_{n=1}^{\infty}(\beta_{n+1},b_n)$$

we have that

$$\ell\big((0,\beta_n)\cap B'\big)=\ell\big((0,\beta_n)\cap A'\big)\,,$$

and for $h>0$

$$\ell\big((0,h)\cap B'\big)\leq\ell\big((0,h)\cap A'\big)\,.$$

It implies that

$$\begin{aligned}\frac{\ell\big(B'\cap[-h,h]\big)}{f(2h)}&=\frac{\ell\big(B'\cap[0,h]\big)}{f(2h)}\\&\leq\frac{\ell\big(A'\cap[0,h]\big)}{f(2h)}\\&\leq\frac{\ell\big(A'\cap[-h,h]\big)}{f(2h)}\underset{h\to0^+}{\longrightarrow}0\end{aligned}$$

because $0\in\Phi^s_{\langle f\rangle}(A)$. Hence $0\in\Phi^s_{\langle f\rangle}(B)$ and $B\in\mathcal{T}^s_{\langle f\rangle}$. On the assumption that $\mathcal{T}^s_{\langle f\rangle}=\mathcal{T}_{\langle f\rangle}$, we get $B\in\mathcal{T}_{\langle f\rangle}$. Simultaneously,

$$\frac{\ell\big(B\cap[-\alpha_n,\beta_n]\big)}{f(\alpha_n+\beta_n)}\geq\frac{\ell\big(B'\cap[0,\beta_n]\big)}{f(\alpha_n+\beta_n)}=\frac{\ell\big(A'\cap[0,\beta_n]\big)}{f(\alpha_n+\beta_n)}\,,$$

and

$$\limsup_{n\to\infty}\frac{\ell\big(B'\cap[-\alpha_n,\beta_n]\big)}{f(\alpha_n+\beta_n)}\geq\lim_{n\to\infty}\frac{\ell\big(A'\cap[0,\beta_n]\big)}{f(\alpha_n+\beta_n)}=\beta>0\,.$$

It means that $B\notin\mathcal{T}_{\langle f\rangle}$. This contradiction completes the proof. □

Theorem 14. *Let $f\in\mathcal{B}$. Then the topology $\mathcal{T}^s_{\langle f\rangle}$ is invariant with respect to the translation and for every $\alpha\geq1$ the topology $\mathcal{T}_{\langle f\rangle}$ is invariant with respect to the multiplication by α.*

Proof. The translation property is a simple consequence of the definition of symmetrical f-density. Let us notice that for every $\alpha \geq 1$ and $A \in \mathcal{L}$ the inequality

$$\frac{\ell\big((\alpha A)' \cap [-h,h]\big)}{f(2h)} = \frac{\alpha\ell\big(A' \cap \big[-\frac{h}{\alpha},\frac{h}{\alpha}\big]\big)}{f(2h)} \leq \frac{\alpha\ell\big[A' \cap [-h,h]\big)}{f(2h)}$$

implies that, if $0 \in \Phi^s_{\langle f\rangle}(A)$, then $0 \in \Phi^s_{\langle f\rangle}(\alpha A)$. Let $E \in \mathcal{T}^s_{\langle f\rangle}$. We show that $\alpha E \in \mathcal{T}_f^{\langle s\rangle}$. If $y \in \alpha E$, then $\frac{y}{\alpha} \in E \subset \Phi^s_{\langle f\rangle}(E)$ and $0 \in \Phi^s_{\langle f\rangle}\big(E - \frac{y}{\alpha}\big)$. Hence $0 \in \Phi^s_{\langle f\rangle}(\alpha E - y)$ and $y \in \Phi^s_{\langle f\rangle}(\alpha E)$. Thus $\alpha E \subset \Phi^s_{\langle f\rangle}(\alpha E)$. It implies that $\alpha E \in \mathcal{T}^s_{\langle f\rangle}$. □

Theorem 15. *For every $\alpha \in (0;1)$ there exists a function $g \in \mathcal{B}$ such that the topology $\mathcal{T}^s_{\langle g\rangle}$ is not invariant with respect to the multiplication by α.*

Proof. Proposition 6 in [FF1] gives an example of the function $f \in \mathcal{B}$ and the set $E \in \mathcal{T}_{\langle f\rangle}$, such that $0 \in E$ and

$$\limsup_{h\to 0^+} \frac{\ell\big((\alpha E)' \cap [0,h]\big)}{f(h)} > 0\,.$$

Hence $0 \in \alpha E - \Phi_{\langle f\rangle}(\alpha E)$ and $\alpha E \notin \mathcal{T}_{\langle f\rangle}$. Since $\mathcal{T}_{\langle f\rangle} \subset \mathcal{T}^s_{\langle f\rangle}$ we get that $E \in \mathcal{T}^s_{\langle f\rangle}$. Putting $g(h) = f\big(\frac{h}{2}\big)$ we have that $g \in \mathcal{B}$, $E \in \mathcal{T}^s_{\langle g\rangle}$, and moreover,

$$\frac{\ell\big((\alpha E)' \cap [-h,h]\big)}{g(2h)} = \frac{\ell\big((\alpha E)' \cap [-h,h]\big)}{f(h)} \geq \frac{\ell\big((\alpha E)' \cap [0,h]\big)}{f(h)}\,.$$

We get

$$\limsup_{h\to 0^+} \frac{\ell\big((\alpha E)' \cap [-h,h]\big)}{g(2h)} > 0 \quad \text{and} \quad 0 \in \alpha E - \Phi^s_{\langle g\rangle}(\alpha E)\,.$$

Hence

$$\alpha E \notin \mathcal{T}^s_{\langle g\rangle}.$$

Let

$$\mathcal{B}_1 = \left\{ f \in \mathcal{B} : \limsup_{x\to 0^+} \frac{x}{f(x)} < \infty \right\},$$

$$\mathcal{B}_2 = \left\{ f \in \mathcal{B} : \limsup_{x\to 0^+} \frac{x}{f(x)} = \infty \right\}.$$

□

Proposition 16. For every function $f \in \mathcal{B}_1$ and every set $A \in \mathcal{L}$ we have

$$\Phi^s_{\langle f\rangle}(A) \sim A\,.$$

P r o o f. Due to Property 9, it is sufficient to prove that $\ell\big(A - \Phi^s_{\langle f\rangle}(A)\big) = 0$. Let us notice that $\Phi_d(A) \subset \Phi^s_{\langle f\rangle}(A)$. Let $x \in \Phi_d(A)$ then

$$\limsup_{x\to 0^+} \frac{\ell\big(A' \cap [x-h, x+h]\big)}{f(2h)} = \lim_{h\to 0} \frac{\ell\big(A' \cap [x-h, x+h]\big)}{2h} \cdot \limsup_{h\to 0^+} \frac{2h}{f(2h)} = 0\,.$$

Therefore, $A - \Phi^s_{\langle f\rangle}(A) \subset A - \Phi_d(A)$, and by the Lebesgue density theorem, $\ell\big(A - \Phi^s_{\langle f\rangle}(A)\big) = 0$. □

Corollary 17. *If* $f \in \mathcal{B}_1$, *then* $\underset{A\in\mathcal{L}}{\forall}\, \Phi^s_{\langle f\rangle}(A) \sim \Phi_d(A)\,.$

Proposition 18. Let $f \in \mathcal{B}_1$ and $\liminf_{x\to 0^+} \frac{x}{f(x)} > 0$. Then

$$\underset{A\in\mathcal{L}}{\forall}\, \Phi^s_{\langle f\rangle}(A) = \Phi_d(A)\,.$$

P r o o f. Let $A \in \mathcal{L}$. According to the proof of Proposition 16, $\Phi_d(A) \subset \Phi^s_{\langle f\rangle}(A)$. Let $x \in \Phi^s_{\langle f\rangle}(A)$. Then

$$\begin{aligned}\limsup_{x\to 0^+} \frac{\ell\big(A' \cap [x-h, x+h]\big)}{2h} &\le \limsup_{h\to 0^+} \frac{\ell\big(A' \cap [x-h, x+h]\big)}{f(2h)} \\ &\quad \times \limsup_{x\to 0^+} \frac{f(2h)}{2h} = 0\,.\end{aligned}$$

□

Corollary 19.

If $f : R_+ \to R_+$ *and* $0 < \liminf_{x\to 0^+} \dfrac{x}{f(x)} \le \limsup_{x\to 0^+} \dfrac{x}{f(x)} < \infty$, *then* $\mathcal{T}^s_{\langle f\rangle} = \mathcal{T}_d$.

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