Tatra Mt. Math. Publ. **40** (2008), 81–89

TATRA MOUNTAINS
Mathematical Publications

CONTINUITY PROPERTIES OF CLUSTER MULTIFUNCTIONS AND CLOSED GRAPH THEOREMS

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ABSTRACT. The notion of a cluster system and a cluster operation acting on multifunctions, as a base tool for investigation of continuity properties of multifunctions is introduced. The main results concern the lower quasi continuity and the lower semi continuity of a cluster multifunction and the properties of a multifunction having the closed graph with respect to a given cluster system.

1. Terminology and basic definitions in cluster setting

The main goal of the presented paper is to introduce a unified concept for studying some problems concerning the continuity properties of multifunctions. The crucial notion of the paper is a cluster operator acting on multifunctions $F : X \to Y$ with respect to a cluster system $\mathcal{E} \subset 2^X$. Our investigation is focused on the mutual connection between an original multifunction F and a resultant cluster multifunction $\mathcal{E}_F$. The base properties of cluster multifunctions, their lower quasi continuity, upper semi continuity and connection to closed graph are main goals of the paper.

In the sequel, X, Y are topological spaces, $\overline{A}$, $\mathbb{N}$, $\mathbb{R}$ denote the closure of A, the natural numbers $\{0, 1, 2, \dots\}$ and the reals with their usual topology, respectively. A compact/locally compact topological space is understood to be T_2 space. Hence a locally compact topological space is regular (even completely regular). A set S is quasi open, if for any open set H intersecting S there is a non-empty open set $H_0 \subset H$ such that $H_0 \subset S$, or equivalently, it is of the form $G \cup A$, where G is an open and A is a nowhere dense subset of $\overline{G}$.

A multifunction F is a non-empty subset of cartesian product $X \times Y$ with the values $\{y \in Y : [x, y] \in F\} := F(x)$ (it can be empty valued at some points). By $\mathrm{Dom}(F)$, we denote the domain of F, i.e., the set of all arguments x in which

2000 Mathematics Subject Classification: 54C60, 54C65, 26E25.
Keywords: cluster system, multifunction, quasi continuity, selection, cluster set.

$F(x)$ is non-empty. For a multifunction with domain A we will use the notation $F : A \to Y$. If $\mathrm{Dom}(F)$ is a dense set, F is said to be densely defined. By $\overline{F}$, we denote a multifunction given by the closure of F in $X \times Y$. A function $f : A \to Y$ stands for a strictly single valued multifunction with values $\{f(x)\}$ for any $x \in A \subset X$. A multifunction F is locally bounded at x, if there is an open set G containing x and a compact set C such that $F(x) \subset C$ for any $x \in G$.

A multifunction $S : A \to Y$ is a submultifunction of F, $S \subset F$, if

$$S(x) \subset F(x), \qquad \text{for all} \quad x \in A,$$

and a function $f : A \to Y$ is a selection of a multifunction F on A, if

$$f(x) \in F(x), \qquad \text{for all} \quad x \in A.$$

For any set $W \subset Y$ the upper resp. lower inverse images are defined as

$$F^+(W) = \{x \in X : F(x) \subset W\},$$

$$F^-(W) = \{x \in X : F(x) \cap W \neq \emptyset\}$$

and the equations $X \setminus F^+(W) = F^-(Y \setminus W)$, $X \setminus F^-(W) = F^+(Y \setminus W)$ hold.

The basic types of continuities are the lover and the upper semi continuity, briefly *lsc* and *usc*.

Definition 1. A multifunction F is *lsc* (*usc*) at $x \in \mathrm{Dom}(F)$ if for any open set V for which $F(x) \cap V \neq \emptyset$ $\big(F(x) \subset V\big)$ there is an open set $U \ni x$ such that $F(u) \cap V \neq \emptyset$ $\;(F(u) \subset V)$ for any $u \in U$. A multifunction F is *lsc* (*usc*) if it is such at any point $x \in \mathrm{Dom}(F)$. That means $F^-(V)\big(F^+(V)\big)$ is open for any open set $V \subset Y$. Finally, F is *usco* at x if $F(x) \neq \emptyset$ is compact and F is *usc* at x.

The next two definitions introduce the notion of an $\mathcal{E}$-cluster point and a lower and upper $\mathcal{E}$-continuity as a basic tool for investigating many properties of multifunctions. In this form it was firstly studied in [9], later in [4], [8], [11]. As it was said, a cluster system is any non-empty system $\mathcal{E}$ of non-empty subset of X. For some special cluster systems we will use a special notation. For example, $\mathcal{O} = \{O : O \text{ is open non-empty}\}$, $\mathcal{B}r = \{B : B \text{ is of second category with the Baire property}\}$, $\mathcal{B}$ denotes a system such that $\mathcal{O} \subset \mathcal{B} \subset \mathcal{O} \cup \mathcal{B}r$, $\mathcal{A} = \{A : A \text{ is not nowhere dense}\}$, $\mathcal{E}^\circ = 2^X \setminus \{\emptyset\}$, $\mathcal{D} = \{A : A \text{ is of second category}\}$, $\mathcal{B}^* = \{A : A \text{ is not nowhere dense with the Baire property}\}$ and the last but not the least $\mathcal{E} = \mathcal{F}$, where $\mathcal{F}$ is a filter in X. Let us stress, $\mathcal{E}$ is always a non-empty system, and any set $E \in \mathcal{E}$ is non-empty.

Definition 2. A point $y \in Y$ is an $\mathcal{E}$-cluster point of F at a point x, if for any open sets $V \ni y$ and $U \ni x$ there is a set $E \in \mathcal{E},\ E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. The set of all $\mathcal{E}$-cluster points of F at x is denoted by $\mathcal{E}_F(x)$.

A multifunction $\mathcal{E}_F$ with the values $\mathcal{E}_F(x)$ is called an $\mathcal{E}$-cluster multifunction of F, provided there exists at least one point x for which $\mathcal{E}_F(x)$ is non-empty.

Remark 1. Note that $\mathcal{E}_F$ is empty valued outside of the closure of $\mathrm{Dom}(F)$. At the points from $\overline{\mathrm{Dom}(F)}$, it can also be empty valued. For example, if $F \subset \mathbb{R}\times\mathbb{R}$ is defined as $F(x) = \{1\}$ for $x > 0$, F is identical with Dirichlet function for $x < -1$ and is empty valued otherwise, then $\mathcal{O}_F(x) = \{1\}$ for $x \geq 0$ and empty valued otherwise, where $\mathcal{O}$ is a cluster system of all non-empty open sets in $\mathbb{R}$.

Definition 3. (Semi-$\mathcal{E}$-continuity) A multifunction F is l-$\mathcal{E}$-continuous (u-$\mathcal{E}$-continuous) at $x \in \mathrm{Dom}(F)$ if for any open sets V, U such that $V \cap F(x) \neq \emptyset$ $\big(F(x) \subset V\big)$ and $x \in U$ there is a set $E \in \mathcal{E}$, $E \subset U \cap \mathrm{Dom}(F)$ such that $F(e) \cap V \neq \emptyset$ $\big(F(e) \subset V\big)$ for any $e \in E$. The global definitions are given by local ones at any point of $\mathrm{Dom}(F)$.

Using cluster concept, l-$\mathcal{E}$-continuity at x can be characterized by the inclusion $\mathcal{E}_F(x) \supset F(x) \neq \emptyset$ and global l-$\mathcal{E}$-continuity by the inclusion $\mathcal{E}_F \supset F$. For a system $\mathcal{B}r$, the l-$\mathcal{B}r$-continuity (u-$\mathcal{B}r$-continuity) will be called lower (upper) Baire continuity, and for $\mathcal{O}$ a lower (upper) quasi continuity. Note, the sets from $\mathcal{O}$ (from $\mathcal{B}r$) are open (of second category with the Baire property) in X, not relatively open (relatively of second category with the Baire property) in the domain of F. For example, if F is semi-$\mathcal{O}$-continuous at least one point, then the domain of F has a non-empty interior. Any multifunction F with respect to the cluster system $\mathcal{E}^\circ = 2^X \setminus \{\emptyset\}$ is l-$\mathcal{E}^\circ$-continuous, since $F \subset \mathcal{E}^\circ_F = \overline{F}$.

Formally l-$\mathcal{E}$-continuity is motivated by the notion of a lower quasi continuity (if $\mathcal{E} = \mathcal{O}$, then l-$\mathcal{O}$-continuity coincides with the lower quasi continuity). Considering $\mathcal{E}^\circ = 2^X \setminus \{\emptyset\}$, $\mathcal{E}^\circ$-cluster point is a classical cluster one. In function setting, many types of cluster points have been defined and the achieved results have many applications from topological and measure theory point of view (see comprehensive book [14]).

Definition of the semi-$\mathcal{B}r$-continuity and the basic properties of $\mathcal{B}r$-cluster multifunctions were studied in [9], where the structure of the set of all usc/lsc points of lover/upper Baire continuous multifunction and the existence of a quasi continuous selection of upper Baire continuous multifunction were investigated. Recently, the existence of quasi continuous selection has been generalized in [1]. For the systems $\mathcal{O}$ and $\mathcal{B}r$, the cluster multifunctions are often minimal, which are nice alternative of selections, see [2], [3], [5], [7]. A system B^* was investigated in [4].

In the paper, system $\mathcal{A}$ will play very important role, since its cluster multifunction is often lower quasi continuous. Lower $\mathcal{A}$-continuity is also called lower demicontinuity, see [13].

2. Basic properties of a cluster multifunction

Lemma 1. *For any net $\{x_t\}$ converging to x and $y_t \in \mathcal{E}_F(x_t)$, $\mathcal{E}_F(x)$ contains all the accumulation points of the net $\{y_t\}$.*

Proof. Let y be an accumulation point of $\{y_t\}$. Then for any open sets $V \ni y$ and $U \ni x$ there are frequently given indexes t' such that $x_{t'} \in U$ and $y_{t'} \in V \cap \mathcal{E}_F(x_{t'})$. Hence there is $E \in \mathcal{E}, E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. That means, $y \in \mathcal{E}_F(x)$. □

Remark 2. (a) From Lemma 1 it follows that the multifunction $\mathcal{E}_F$ has the closed graph, hence it has the closed values, too. That means, $\mathcal{E}_F^-(K)$ is closed for any compact set K or equivalently, $\mathcal{E}_F^+(G) = X \setminus \mathcal{E}_F^-(Y \setminus G)$ is open for any open G with compact complement, hence $\mathcal{E}_F$ is *usc* with respect to topology given on Y by open sets with compact complement (also called c-upper semicontinuity [6]).

(b) If Y is Hausdorff σ-compact ($Y = \bigcup_{n\in\mathbb{N}} K_n$, K_n compact), then for any closed set $F \subset Y$, $\mathcal{E}_F^-(F) = \bigcup_{n\in\mathbb{N}} \mathcal{E}_F^-(F \cap K_n)$ is F_σ set. Consequently, $\mathrm{Dom}(\mathcal{E}_F) = \mathcal{E}_F^-(Y)$ is a F_σ set.

(c) If $\mathrm{Dom}(\mathcal{E}_F) = \bigcup_{n\in\mathbb{N}} F_n$ is F_σ and of second category at any point, then $X \setminus \mathrm{Dom}(\mathcal{E}_F)$ is a nowhere dense set. If not, there is an open set G and a set $S \subset X \setminus \mathrm{Dom}(\mathcal{E}_F)$ dense in G and there is also a set of second category $Z \subset \mathrm{Dom}(\mathcal{E}_F) \cap G$. Hence there is n such that $F_n \cap G$ is of second category. That means $\mathrm{int}(F_n) \cap G$ is a non-empty open subset of $\mathrm{Dom}(F)$, contradiction with the fact that $S \subset X \setminus \mathrm{Dom}(\mathcal{E}_F)$ is dense in G.

(d) By (a), $\mathcal{E}_F^-(K)$ is closed for any compact set K, hence if $\mathcal{E}_F^-(K)$ is dense in an open set G, then $G \subset \mathcal{E}_F^-(K)$.

(e) If Y is compact and $\mathrm{Dom}(\mathcal{E}_F)$ is dense, then $\mathrm{Dom}(\mathcal{E}_F) = X$ (see item (d)) and $\mathcal{E}_F$ is *usc*.

(f) If $\mathcal{E}^1 \subset \mathcal{E}^2$, then $\mathcal{E}_F^1 \subset \mathcal{E}_F^2$ and if $F_1 \subset F_2$, then $\mathcal{E}_{F_1} \subset \mathcal{E}_{F_2}$.

(g) $\mathcal{E}_F \subset \mathcal{E}_F^\circ = \overline{F}$, hence any multifunction is l-$\mathcal{E}^\circ$-continuous.

Lemma 2. *If F is l-$\mathcal{E}$-continuous, then $\mathcal{E}_F$ is l-$\mathcal{E}$-continuous and $\mathcal{E}_{\mathcal{E}_F} = \mathcal{E}_F$.*

Proof. From the inclusion $F \subset \mathcal{E}_F$ it follows $\mathcal{E}_F \subset \mathcal{E}_{\mathcal{E}_F}$ (see Remark 2(f)). Hence $\mathcal{E}_F$ is l-$\mathcal{E}$-continuous. Let $y \in \mathcal{E}_{\mathcal{E}_F}(x)$, $U \ni x$, $V \ni y$. Then there is $E \in \mathcal{E}$ such that $E \subset U \cap \mathcal{E}_F^-(V)$. Let $y_0 \in \mathcal{E}_F(e) \cap V$, $e \in E$. Hence there is $E_0 \in E$ such that $E_0 \subset U \cap F^-(V)$. That means, $y \in \mathcal{E}_F(x)$ and $\mathcal{E}_F = \mathcal{E}_{\mathcal{E}_F}$. □

Since the multifunction $\mathcal{E}_F$ has the closed graph (Remark 2(a)), we get the next global characterization of l-$\mathcal{E}$-continuity.

Lemma 3. *A multifunction F is l-$\mathcal{E}$-continuous if and only if $\mathcal{E}_F = \overline{F}$.*

Corollary 1. *If F is l-$\mathcal{E}^1$-continuous and l-$\mathcal{E}^2$-continuous, then $\mathcal{E}^1_F = \mathcal{E}^2_F$.*

Lemma 4. *Let Y be a locally compact topological space and F be l-$\mathcal{E}$-continuous. Suppose that any set from $\mathcal{E}$ is not nowhere dense. Then $\mathcal{E}_F(x) \neq \emptyset$ for any $x \in \overline{\mathrm{Dom}(F)} \setminus S$, where S is nowhere dense and the domain of $\mathcal{E}_F$ is a quasi open set.*

Proof. Let $x \in \overline{\mathrm{Dom}(F)}$. Then for any open set G containing x there is $a \in G \cap \mathrm{Dom}(F)$. Since $\emptyset \neq F(a) \subset \mathcal{E}_F(a)$, we can take $y \in \mathcal{E}_F(a)$. Then for an open set V containing y and $\overline{V}$ compact, there is $E \in \mathcal{E}$, $E \subset G$, such that $F(e) \cap \overline{V} \neq \emptyset$ for any $e \in E \subset \mathrm{Dom}(F)$. Since E is not nowhere dense, there is an open set $H \subset G$ such that E is dense in H. Then for any $z \in H \cap E$ we have $\emptyset \neq F(z) \cap \overline{V} \subset \mathcal{E}_F(z) \cap \overline{V}$. Then $\mathcal{E}_F(z) \neq \emptyset$ for any $z \in H \subset G$, by Remark 2(d). It means, $\mathcal{E}_F$ is non-empty valued at any point of $\overline{\mathrm{Dom}(F)}$ except for a nowhere dense set. Simultaneously, we have proved that the domain of $\mathcal{E}_F$ is quasi open. □

Corollary 2. *Let Y be a locally compact topological space and F be l-$\mathcal{E}$-continuous, where any $E \in \mathcal{E}$ is not nowhere dense. If $\mathrm{Dom}(F)$ is dense, then the domain of $\mathcal{E}_F$ is quasi open with a nowhere dense complement.*

3. Continuity properties of cluster a multifunction

In this paragraph we present the basic properties of a cluster multifunction, namely its lower quasi continuity and lower semi continuity as well as the structure of the set of all points at which $\mathcal{E}_F$ is *usco*.

Lemma 5. *Let Y be a locally compact topological space. If $\mathcal{E}_F$ is locally bounded at $x \in \mathrm{Dom}(\mathcal{E}_F)$, then $\mathcal{E}_F$ is usco on a relatively open set in $\mathrm{Dom}(\mathcal{E}_F)$ containing x. That means, the set of all points at which $\mathcal{E}_F$ is usco is relatively open in the domain of $\mathcal{E}_F$.*

Proof. Since $\mathcal{E}_F$ is locally bounded at x, there are an open set $G \ni x$ and a compact set C such that $\mathcal{E}_F(g) \subset C$ for any $g \in G$. Let $x_0 \in G \cap \mathrm{Dom}(\mathcal{E}_F)$, $\mathcal{E}_F(x_0) \subset V$, where V is open. The set $(Y \setminus V) \cap C$ is compact and its complement $V \cup (Y \setminus C)$ is open containing $\mathcal{E}_F(x_0)$. Since $\mathcal{E}_F$ is *usc* with respect to topology given on Y by the open sets with compact complement (see Remark 2(a)), there is an open set $H \subset G$ containing x_0 such that $\mathcal{E}_F(h) \subset V \cup (Y \setminus C)$ for any $h \in H$. Since $\mathcal{E}_F(h) \subset C$, $\mathcal{E}_F(h) \subset V$ for any $h \in H$. That means, $\mathcal{E}_F$ is *usco* at x_0, hence *usco* at any point of $G \cap \mathrm{Dom}(\mathcal{E}_F)$.

If $\mathcal{E}_F$ is *usco* at x', then there is an open set $W \supset \mathcal{E}_F(x')$ with a compact closure $\overline{W} =: C$ ($\mathcal{E}_F(x')$ is compact and Y locally compact) and an open set G containing x' such that $\mathcal{E}_F(g) \subset W \subset C$ for any $g \in G$. So $\mathcal{E}_F$ is bounded by compact set C on G and due to the result above, $\mathcal{E}_F$ is *usco* on $G \cap \mathrm{Dom}(\mathcal{E}_F)$, hence the set of all points at which $\mathcal{E}_F$ is *usco* is relatively open in $\mathrm{Dom}(\mathcal{E}_F)$. □

Lemma 6. *Let Y be a locally compact topological space and let $A = \{x \in H : \mathcal{E}_F(x)$ is non-empty compact and $\mathcal{E}_F$ is upper quasi continuous at $x\}$, where H is open. If A is dense in H, then there is a set $S \subset H$ such that S is a nowhere dense set and $\mathcal{E}_F$ is usco at any point of $H \setminus S$.*

Proof. Let $G \subset H$ be a non-empty open and let $a \in A \cap G$, $V \supset \mathcal{E}_F(a)$, V open. Since Y is locally compact T_1 regular and $\mathcal{E}_F(a)$ compact, there is an open set V_0 such that $V \supset \overline{V_0} \supset V_0 \supset \mathcal{E}_F(a)$ and $\overline{V}_0$ is compact. Since $\mathcal{E}_F$ is upper quasi continuous at a, there is a non-empty open set $U \subset G \cap \mathrm{Dom}(\mathcal{E}_F)$, such that $\mathcal{E}_F(x) \subset V_0 \subset \overline{V_0}$ for any $x \in U$. Hence $\mathcal{E}_F$ is bounded on U by the compact set $\overline{V_0}$ and by Lemma 5, $\mathcal{E}_F$ is *usco* on U. That means, $\mathcal{E}_F$ is *usco* on H except for a nowhere dense set. □

Corollary 3. *Let X be a Baire space and Y be a locally compact topological one. If $\mathcal{E}_F$ is compact valued and upper quasi continuous on a dense set, then $\mathcal{E}_F$ is usco except for a nowhere dense set and $\mathcal{E}_F$ has a quasi continuous selection $f : G \to Y$ defined on an open set G for which $X \setminus G$ is nowhere dense. Moreover, if Y is metric (second countable), then $\mathcal{E}_F$ is lsc except for a set of first category.*

Proof. Using the lemma above, $\mathcal{E}_F$ is *usco* except for a nowhere dense set S. By [1], $\mathcal{E}_F$ has quasi a continuous selection on an open set $G := X \setminus \overline{S}$. If Y is metric (second countable), the assertion follows from [10, Theorem 1] ([9, Theorem 2.1]). □

Next, we will solve some properties of the cluster multifunction of an l-$\mathcal{E}$--continuous multifunction with respect to a cluster system in which any set from $\mathcal{E}$ is not nowhere dense, i.e., $\mathcal{E} \subset \mathcal{A}$.

Theorem 1. *Let Y be a locally compact topological space and F be l-$\mathcal{E}$-continuous. Suppose any set from $\mathcal{E}$ is not nowhere dense. Then $\mathcal{E}_F$ is lower quasi continuous defined on a quasi open set.*

Proof. Let U, V be open, $a \in U, V \cap \mathcal{E}_F(a) \neq \emptyset$. Then there is an open set V_0 with the compact closure, $\overline{V_0} \subset V$ and $V_0 \cap \mathcal{E}_F(a) \neq \emptyset$. That means, there is $E \in \mathcal{E}$, $E \subset U \cap F^-(\overline{V_0})$. A multifunction F is l-$\mathcal{E}$-continuous (i.e., $\emptyset \neq F(x) \subset \mathcal{E}_F(x)$ for all $x \in \mathrm{Dom}(F)$), hence $E \subset F^-(\overline{V_0}) \subset \mathcal{E}_F^-(\overline{V_0})$. Since E is not nowhere dense, there is an open set $G \subset U$ such that E is dense in G. By

Remark 2(d), $G \subset \mathcal{E}_F^-(\overline{V_0}) \subset \mathcal{E}_F^-(V)$, which means, $\mathcal{E}_F$ is lower quasi continuous at a. Domain of $\mathcal{E}_F$ is quasi open by Lemma 4. $\square$

Corollary 4. *Let Y be a locally compact topological space and let any set from $\mathcal{E}$ not be nowhere dense. If a densely defined multifunction F is l-$\mathcal{E}$-continuous, then $\mathcal{E}_F$ is defined and lower quasi continuous on a quasi open set with a nowhere dense complement.*

The next definition generalizes the notion of almost continuity (very nearly and nearly continuity) of function and it will be utilized in closed graph theorems generalizing results from [12].

Definition 4. (Semi-$\mathcal{E}$-dense continuity) A multifunction F is l-$\mathcal{E}$-dense continuous (u-$\mathcal{E}$-dense continuous) at x if $F(x) \neq \emptyset$, and for any open set V such that $V \cap F(x) \neq \emptyset$ $\big(F(x) \subset V\big)$, there is an open set G containing x such that for any non-empty open set $H \subset G$, there is a set $E \in \mathcal{E}$, $E \subset H \cap \mathrm{Dom}(F)$ and $F(e) \cap V \neq \emptyset$ $\big(F(e) \subset V)\big)$ for any $e \in E$. l-$\mathcal{E}$-continuity (u-$\mathcal{E}$-continuity) follows from l-$\mathcal{E}$-dense continuity (u-$\mathcal{E}$-dense continuity).

A multifunction F is lower (upper) nearly/almost/very nearly continuous at x, if it is l-$\mathcal{A}$-dense (u-$\mathcal{A}$-dense) / l-$\mathcal{D}$-dense (u-$\mathcal{D}$-dense) / l-$\mathcal{B}r$-dense (u-$\mathcal{B}r$-dense) continuous at x (those notions for functions were investigated in [12]).

Theorem 2. *Let Y be a locally compact topological space and F be l-$\mathcal{E}$-dense continuous. Then $\mathcal{E}_F$ is lsc.*

P r o o f. Let V be open, $a \in X, V \cap \mathcal{E}_F(a) \neq \emptyset$. Then there is an open set V_0 with compact closure, $\overline{V_0} \subset V$ and $V_0 \cap \mathcal{E}_F(a) \neq \emptyset$. A multifunction F is l-$\mathcal{E}$-dense continuous at a, hence there is an open set G containing a such that $F^-(\overline{V_0})$ is dense in G. Since $F \subset \mathcal{E}_F$ (inclusion follows from l-$\mathcal{E}$-continuity of F), $\mathcal{E}_F^-(\overline{V_0})$ is dense in G. By Remark 2(d), $G \subset \mathcal{E}_F^-(\overline{V_0}) \subset \mathcal{E}_F^-(V)$, that means, $\mathcal{E}_F$ is lsc at a. $\square$

4. Closed graph theorem in cluster setting

Definition 5. A multifunction F has $\mathcal{E}$-closed graph at x if $\mathcal{E}_F(x) \subset F(x)$, and F has $\mathcal{E}$-closed graph if it has $\mathcal{E}$-closed graph at any point from X. If F has $\mathcal{E}_F^\circ$-closed graph/$\mathcal{E}_F^\circ$-closed graph at x (i.e., $\overline{F} = \mathcal{E}_F^\circ \subset F$ / $\overline{F}(x) = \mathcal{E}_F^\circ(x) \subset F(x)$), then we use usual terminology, that F has closed graph/closed graph at x.

Remark 3. The notion of $\mathcal{E}$-closedness of graph is more general than closedness of graph, because if F has the closed graph, then $\mathcal{E}_F \subset \overline{F} = F$. On the other hand, the multifunction G from $\mathbb{R}$ to $\mathbb{R}$, defined as $G(x) = \langle 0, 1\rangle$ for x rational

and $G(x) = \{0\}$ otherwise, is u-$\mathcal{B}r$-continuous with $\mathcal{B}r$-closed graph ($\mathcal{B}r_G(x) = \{0\}$ for all x), but its graph is not closed.

The next theorem is a quasi variant of results from [12] for the multifunctions and cluster systems contained in $\mathcal{A}$ (for example $\mathcal{E} \in \{\mathcal{A}, \mathcal{D}, \mathcal{B}r, \mathcal{B}^*, \mathcal{B}\}$).

Theorem 3. *Let Y be a locally compact topological space. If F with $\mathcal{E}$-closed graph is l-$\mathcal{E}$-continuous (densely defined) and any set $E \in \mathcal{E}$ is not nowhere dense, i.e., $\mathcal{E} \subset \mathcal{A}$, then F is lower quasi continuous with quasi open domain (with quasi open domain having a nowhere dense complement).*

Proof. It follows from Theorem 1 (Corollary 4) and from equation $\mathcal{E}_F = F$ ($\mathcal{E}_F \supset F$ from l-$\mathcal{E}$-continuity and $\mathcal{E}_F \subset F$ from $\mathcal{E}$-closedness of graph). $\square$

Similarly, using Theorem 2, we get a multifunction variant of Moors results from [12].

Theorem 4. *Let Y be a locally compact topological space. If F is l-$\mathcal{E}$-dense continuous with $\mathcal{E}$-closed graph, then F is lsc.*

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Received December 8, 2006

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