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THREE-POLAR SPACE OVER THE SEMI-FIELD OF DOUBLE NUMBERS

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ABSTRACT. Multi-polar space is a generalization of the notion of vector space. In this paper, we deal with a three-polar vector space over a semi-field of double (hyperbolic complex) numbers. We introduce and study operations of addition and multiplication such that they form a commutative ring with unit on the three-polar space.

1. Introduction

As far as we know, the idea of multi-polarity is relatively new and comes from physics, cf. [6]. There are, e.g., K-polar oscillators, magnets, $K \in \mathbb{N}$. The principle of multi-polarity can be easily observed in the case of K-phase electric current, $K \in \mathbb{N}$. Particularly, if $K = 3$, we say about three-phase current. The sum of all three phases of equal amplitude in three-phase current is zero at each moment. We call this fact to be *the cancellation law* in this paper.

Another source of motivation for multi-polarity study is a color vision. If we deal with a classical RGB model of colors, then polarities are represented by red, green and blue. The mixture of these basic colors with the same intensities makes a gray color (from white to black, depending on intensity). So, the cancellation law can be understood as "gray color" in this model.

Multi-polarity can be imagined as a "colouring" of the black-and-white mathematics. A mathematical description of multi-polarity in the plane is introduced in [4]. There is presented the complex plane as a three-polar space.

Every Euclidean space (with two standard polarities "+" and "−") is a multipolar space, where the real field is replaced by a semi-field $[0, \infty)$. Remind the semi-field definition.

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Definition 1.1. Let X be a set with two binary operations $+$ and $\cdot$. The system $(X,+,\cdot)$ is said to be a *semi-field* if $(X,+)$ is a commutative semi-group, $(X,\cdot)$ is a commutative multiplicative group, and multiplication distributes over addition.

Elements of a semi-field X are called *non-polar* or *neutral* elements.

Let $\mathbb{A} = \{A,B,C\}$ be a set of three mappings from the given semi-field X to a set Y. The mappings are called *polar operators*, or, *poles*. According to the number of polar operators we say about three-polar space.

In this paper we consider a binary operation $*$ on $\mathbb{A}$ given by the Latin square

$$\begin{array}{c|ccc} * & A & B & C \\ \hline A & A & B & C \\ B & B & C & A \\ C & C & A & B \end{array}. \tag{1}$$

There is a neutral element A with respect to the operation $*$. Next, we assume that $A(X), B(X), C(X)$ are isomorphic copies of X and

$$A(0) = B(0) = C(0), \tag{2}$$

where 0 is a neutral element in the semi-group $(X,+)$. The images $A(X), B(X)$ and $C(X)$ are called *polar axes*. The condition (2) shows that all polar axes intersect in a single point which can be viewed as an "origin" of the multi-polar space.

In the paper [4], the semi-field of all non-negative real numbers with usual addition and multiplication was used to construct the multi-polar spaces. The resulting multi-polar spaces with arithmetic operations of addition, subtraction, multiplication and division are all isomorphic to the system of complex numbers. In this paper we bring a three-polar space over a more general semi-field, a subset of the so-called double numbers.

We construct new quaternion-like systems. An interesting question is about a generalization to octonions. Octonions are used in physics, cf. [2], [3], [7].

2. A semi-field of double numbers

In this paper, we deal with the system of double numbers.[1] We will denote the system of double numbers as $\mathbb{D}$.

[1]There is no standard terminology for "double" numbers; we can find the following terms in literature: hyperbolic complex numbers, hyperbolic numbers, semi-complex numbers, split-complex numbers, perplex numbers.

Double numbers are similar to classical complex numbers. Arithmetic operations of double numbers $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$ are as follows

$$\begin{aligned}(a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2)\,,\\ (a_1, a_2) - (b_1, b_2) &= (a_1 - b_1, a_2 - b_2)\,,\\ (a_1, a_2) \cdot (b_1, b_2) &= (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1)\,,\end{aligned}$$

and if $b_1^2 \neq b_2^2$, then

$$\frac{(a_1, a_2)}{(b_1, b_2)} = \left(\frac{a_1 b_1 - a_2 b_2}{b_1^2 - b_2^2}, \frac{a_2 b_1 - a_1 b_2}{b_1^2 - b_2^2}\right).$$

All operations on the right sides are usual operations of real numbers. We will denote operations on $\mathbb{R}$ and $\mathbb{R}^2$ alike.

It is easy to check that double numbers form a commutative ring with unit but not a semi-field. If we take the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 = 0\}$, then double numbers are reduced on the semi-field of non-negative real numbers. There exists also a non-trivial subset of double numbers which is closed under addition, multiplication and division. It is a set $\mathbb{D}_+^0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > |x_2| \text{ or } x_1 = x_2 = 0\}$. Obviously, the system $(\mathbb{D}_+^0, +, \cdot)$ is a semi-field. For more information about double numbers, cf., e.g., [1], [5].

We introduce the following definition for the lemma of "existence of the square of the difference".

Definition 2.1. Let $a = (a_1, a_2) \in \mathbb{D}_+^0$ and $b = (b_1, b_2) \in \mathbb{D}_+^0$. We will say that *$a, b$ are in the relation* $\overset{+}{\sim}$ ($\overset{-}{\sim}$), $a \overset{+}{\sim} b$ ($a \overset{-}{\sim} b$), if $a_1 + a_2 = b_1 + b_2$ ($a_1 - a_2 = b_1 - b_2$).

Remark 2.2. Both relations $\overset{+}{\sim}$ and $\overset{-}{\sim}$ are equivalence relations.

Lemma 2.3. *Let $a = (a_1, a_2) \in \mathbb{D}_+^0$ and $b = (b_1, b_2) \in \mathbb{D}_+^0$. Then*

1. $(a - b)^2 \in \mathbb{D}_+^* = \mathbb{D}_+^0 \cup \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = |x_2|\}$,
2. $(a - b)^2 \in \mathbb{D}_+ = \mathbb{D}_+^0 \setminus \{(0, 0)\}$ *if and only if* $a \overset{+}{\not\sim} b$ *and* $a \overset{-}{\not\sim} b$.

Proof. There holds for $a, b \in \mathbb{D}$

$$(a - b)^2 = \left((a_1 - b_1)^2 + (a_2 - b_2)^2, 2(a_1 - b_1)(a_2 - b_2)\right).$$

1. Obviously,

$$|a_1 - b_1|^2 + |a_2 - b_2|^2 \geq 2|a_1 - b_1||a_2 - b_2| \quad \text{for all } a_1, a_2, b_1, b_2 \in \mathbb{R}$$

which implies that $(a - b)^2 \in \mathbb{D}_+^*$.

2.
$$\begin{aligned}(a - b)^2 \in \mathbb{D}_+ &\iff (a_1 - b_1)^2 + (a_2 - b_2)^2 > 2|(a_1 - b_1)(a_2 - b_2)|\\ &\iff (|a_1 - b_1| - |a_2 - b_2|)^2 > 0\\ &\iff |a_1 - b_1| \neq |a_2 - b_2|\end{aligned}$$

$$\iff a_1 + a_2 \neq b_1 + b_2 \quad \text{and} \quad a_1 - a_2 \neq b_1 - b_2 \iff a \overset{+}{\not\sim} b \quad \text{and} \quad a \overset{-}{\not\sim} b. \quad \square$$

3. Three-polar space over double numbers

The next step in construction of a three-polar space is to find a suitable system of three polar operators. Let us consider three poles A, B, C from the semi-field $\mathbb{D}^0_+$ to a set Y and an operation $*$ given by the Latin square (1). For convenience, denote

$$X := \mathbb{D}^0_+, \quad X_+ := X \setminus \{(0,0)\} = \mathbb{D}_+.$$

Now let us consider the element $(a, b, c) \in X^3$. It can be very formally written as a sum

$$(a, b, c) = Aa + Bb + Cc.$$

3.1. Cancellation law, "null of addition"

Definition 3.1. Let $(u, v, w), (x, y, z) \in X^3$. We will say that the triple (u, v, w) *is equal to the triple* (x, y, z) *in the sense of cancellation law* ("null of addition"), we write $(u, v, w) \cong (x, y, z)$, if there exists $d \in X$, such that $(u+d, v+d, w+d) = (x, y, z)$ or $(u, v, w) = (x + d, y + d, z + d)$.

Remark 3.2.

1. Relation $\cong$ is an equivalence relation.
2. If $(a, b, c) \in X^3$, then $(a, b, c) \cong (0, 0, 0) \iff a = b = c$. Formally, for all $a \in X$
$$Aa + Ba + Ca \cong (0, 0, 0).$$
3. If $a, b, c \in \mathbb{R}^2$, there exists $d \in X$ such that $(a + d, b + d, c + d) \in X^3$, for example, $d = (d_1, 0)$, where $d_1 = 1 + \max\{|x_1| + |x_2|, |y_1| + |y_2|, |z_1| + |z_2|\}$.

3.2. Operations in the three-polar space

We introduce operations analogical to the classical operations of addition, subtraction, conjugation, multiplication and division. All these operations are defined on X^3.

Addition. For (u, v, w), $(x, y, z) \in X^3$,

$$(u, v, w) \oplus (x, y, z) := (u + x, v + y, w + z). \tag{3}$$

Clearly, the operation $\oplus$ is associative and commutative.

Subtraction. For all $x, y, z \in X$ and due to the cancellation law, formally

$$\begin{aligned} -Ax &= Bx + Cx, \\ -By &= Ay + Cy, \\ -Cz &= Az + Bz. \end{aligned}$$

We define the operation of subtraction as follows

$$\begin{aligned} (u, v, w) \ominus (x, y, z) &= (u, v, w) \oplus (y + z, x + z, x + y) \\ &= (u + y + z, v + x + z, w + x + y). \end{aligned} \tag{4}$$

It is obvious that $\ominus(x,y,z)$ is just an inverse element to (x,y,z) with respect to $\oplus$ because

$$(x,y,z)\ominus(x,y,z)=(x+y+z,x+y+z,x+y+z)\cong(0,0,0).$$

Multiplication. We define the operation of multiplication polynomial-like ("each one with each one") using the table for operation $*$, i.e.,

$$\begin{aligned}&(u,v,w)\odot(x,y,z)\\&=(Au+Bv+Cw)\odot(Ax+By+Cz)\\&=(A*A)(ux)+(A*B)(uy)+(A*C)(uz)+(B*A)(vx)+(B*B)(vy)\\&\quad+(B*C)(vz)+(C*A)(wx)+(C*B)(wy)+(C*C)(wz)\\&=A(ux+vz+wy)+B(uy+vx+wz)+C(uz+vy+wx)\\&=(ux+vz+wy,uy+vx+wz,uz+vy+wx).\end{aligned}\tag{5}$$

This operation is associative and commutative and its neutral element is $\big((1,0),(0,0),(0,0)\big)$. It is also easy to see that the operation of multiplication distributes over addition.

The special case of this operation is *multiplication by scalar*,

$$k(u,v,w)=(k,0,0)\odot(u,v,w)=(ku,kv,kw),\quad\text{where}\quad k\in X.$$

Conjugation. This operation is expressed in terms of three-polar space in the following way

$$(u,v,w)^*=(u,w,v).\tag{6}$$

It can be easily to check that operation of conjugation has the following properties.

Lemma 3.3. *Let $(u,v,w)\in X^3,(x,y,z)\in X^3$. Then there holds:*

1. $\big((u,v,w)^*\big)^*=(u,v,w)$,
2. $\big((u,v,w)\oplus(x,y,z)\big)^*=(u,v,w)^*\oplus(x,y,z)^*$,
3. $\big((u,v,w)\odot(x,y,z)\big)^*=(u,v,w)^*\odot(x,y,z)^*$,
4. $(u,v,w)\odot(u,v,w)^*\cong(d,0,0)$ *for some* $d\in X_+$ *if and only if it is not true that* $u\overset{+}{\sim}v\overset{+}{\sim}w$ *or* $u\overset{-}{\sim}v\overset{-}{\sim}w$.

Proof.
1.–3. are trivial.
4. Let $(u,v,w)\in X^3$. We have

$$(u,v,w)\odot(u,v,w)^*=(uu+vv+ww,uw+vu+wv,uv+vw+wu),$$

thus it is sufficient to prove that

$$d=\big(u^2+v^2+w^2\big)-(uw+vu+wv)\in X_+$$

if and only if it is false that $u\overset{+}{\sim}v\overset{+}{\sim}w$ or $u\overset{-}{\sim}v\overset{-}{\sim}w$.

There holds that
$$d = \frac{1}{2}\left[(u-v)^2 + (v-w)^2 + (w-u)^2\right].$$
If $d \in X_+$, then at least one of elements $(u-v)^2, (v-w)^2, (w-u)^2$ belongs to X_+, for example, let there be $(u-v)^2$, and thus $u \overset{+}{\not\sim} v$ and $u \overset{-}{\not\sim} v$, by Lemma 2.3.

Conversely, let the statement $u \overset{+}{\sim} v \overset{+}{\sim} w$ or $u \overset{-}{\sim} v \overset{-}{\sim} w$ not be true. Then there exists a pair
$$a, b \in \{u, v, w\} \quad \text{such that} \quad a \overset{+}{\not\sim} b \quad \text{and} \quad a \overset{-}{\not\sim} b.$$
Let $u \overset{+}{\not\sim} w$, without loss of generality. If $u \overset{-}{\not\sim} w$, then $a = u, b = w$; if $v \overset{+}{\not\sim} w$ and $v \overset{-}{\not\sim} w$, then $a = v, b = w$. Otherwise $u \overset{+}{\not\sim} v$ and $u \overset{-}{\not\sim} v$, thus $a = u, b = v$. By Lemma 2.3, $(a-b)^2$ belongs to X_+ and also $d \in X_+$. □

Division. Due to the previous lemma, and the fact $(u, v, w) \odot (u, v, w)^* \in X_+$ (if it holds), there exists an inverse element to (u, v, w) with respect to the operation $\odot$.

Definition 3.4. The triple $(u, v, w) \in X^3$ is called to be *invertible* if $(u, v, w) \odot (u, v, w)^*$ belongs to X_+. We denote the set of all invertible elements by X_i^3.

There is given an inverse element with respect to multiplication $\odot$ for an invertible element $(u, v, w) \in X_i^3$ as follows
$$\begin{aligned}(u, v, w)^{-1} &= \big((u, v, w) \odot (u, v, w)^*\big)^{-1}(u, v, w)^* \\ &\cong \frac{2}{(u-v)^2 + (v-w)^2 + (w-u)^2}(u, w, v),\end{aligned} \tag{7}$$
where the fraction is in the sense of double numbers multiplication.

Now we define the operation of division in the following way:
$$(u, v, w) \oslash (x, y, z) \;=\; (u, v, w) \odot (x, y, z)^{-1} \tag{8}$$
for
$$(u, v, w) \in X^3 \quad \text{and} \quad (x, y, z) \in X_i^3.$$

3.3. Equivalence classes with respect to the relation $\cong$

Definition 3.5. The system
$$\mathcal{M} := \left(X^3 \backslash_{\cong}, \oplus, \odot\right)$$
is called to be *the three-polar space* over the semi-field X.

In this part it will be proved that all operations in the introduced system $\mathcal{M}$ are well defined.

Theorem 3.6. *Let* (u, v, w), (x, y, z), $(a, b, c) \in X^3$ *and* $(u, v, w) \cong (a, b, c)$. *Then the following:*

(i) $(u, v, w) \cong (0, 0, 0)$ *if and only if* $(u, v, w) = (d, d, d)$ *for some* $d \in X$,

(ii) $(x, y, z) \oplus (u, v, w) \cong (x, y, z) \oplus (a, b, c)$,

(iii) $(x, y, z) \ominus (u, v, w) \cong (x, y, z) \ominus (a, b, c)$,

(iv) $(x, y, z) \odot (u, v, w) \cong (x, y, z) \odot (a, b, c)$,

(v) $(u, v, w)^* \cong (a, b, c)^*$,

(vi) (u, v, w) *is invertible if and only if* (a, b, c) *is so,*

(vii) *if* (u, v, w) *is invertible, then* $(x, y, z) \oslash (u, v, w) \cong (x, y, z) \oslash (a, b, c)$.

Proof.

(i) Directly from the definition of the relation $\cong$.

(ii), (iii), (v) are obvious.

(iv) Without loss of generality, let $(a, b, c) = (u + d, v + d, w + d)$ for some $d \in X$ in the rest of the proof. Then from distributivity of multiplication over addition

$$\begin{aligned}(x, y, z) \odot (a, b, c) &= (x, y, z) \odot (u + d, v + d, w + d) \\ &= \big[(x, y, z) \odot (u, v, w)\big] \oplus [(x, y, z) \odot (d, d, d)] \\ &\cong (x, y, z) \odot (u, v, w),\end{aligned}$$

since, by (i),

$$\begin{aligned}(x, y, z) \odot (d, d, d) &= (xd + yd + zd, xd + yd + zd, xd + yd + zd) \\ &\cong (0, 0, 0).\end{aligned}$$

(vi) Let (u, v, w) not be invertible. Then

$$u_1 + u_2 = v_1 + v_2 = w_1 + w_2 \quad \text{or} \quad u_1 - u_2 = v_1 - v_2 = w_1 - w_2\,.$$

The proof follows by

$$\begin{aligned}a_1 &= u_1 + d_1, \quad b_1 = v_1 + d_1, \quad c_1 = w_1 + d_1, \\ a_2 &= u_2 + d_2, \quad b_2 = v_2 + d_2, \quad c_2 = w_2 + d_2.\end{aligned}$$

(vii) We have

$$\begin{aligned}&(a, b, c)^{-1} \\ &= \frac{2(u + d, w + d, v + d)}{[(u + d) - (v + d)]^2 + [(v + d) - (w + d)]^2 + [(w + d) - (u + d)]^2} \\ &= \frac{2(u, v, w)}{(u - v)^2 + (v - w)^2 + (w - u)^2} \oplus \frac{2(d, d, d)}{(u - v)^2 + (v - w)^2 + (w - u)^2} \\ &\cong \frac{2}{(u - v)^2 + (v - w)^2 + (w - u)^2}(u, v, w) \\ &= (u, v, w)^{-1} \qquad \text{for} \quad (a, b, c) = (u + d, v + d, w + d), \quad d \in X.\end{aligned}$$

□

By the previous lemma, binary operations of addition, subtraction and multiplication are all independent of the choice of representatives, and thus they are all well defined. The properties of the system $\mathcal{M}$ are collected in the following statement.

Lemma 3.7. *The system $\mathcal{M}$ is a commutative ring with unit.*

4. Cancellation law in split-quaternions

In this section, we suppose the operators A, B, C in the form

$$B_i : \mathbb{R}^4 \to \mathbb{R}^4, \qquad i = 1, 2, 3,$$

where $B_{1,2,3} = A, B, C$ are square matrices 4×4 with the real coefficients.

According to (1), we obtain the following equations:

$$\begin{array}{ccccccc} B_1B_1 & = & B_2B_3 & = & B_3B_2 & = & B_1, \\ B_1B_2 & = & B_2B_1 & = & B_3B_3 & = & B_2, \\ B_1B_3 & = & B_2B_2 & = & B_3B_1 & = & B_3, \end{array} \tag{9}$$

where multiplication of matrices is understood in the usual sense. We assume that $A = B_1$ is an identity matrix. This way, the system (9) is reduced to the following four equations:

$$B_2B_3 = B_1, \quad B_3B_2 = B_1, \quad B_3B_3 = B_2, \quad B_2B_2 = B_3. \tag{10}$$

These equations lead to a system of 48 non-linear equations with 32 unknown. We do not have a complete solution of this equation system. We considered only a special cases of the so-called split-quaternions.

The operation of multiplication of two split-quaternions $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ can be expressed in the "matrix-vector" notation as

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = \begin{pmatrix} x_1 & -x_2 & x_3 & x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & -x_4 & x_1 & x_2 \\ x_4 & x_3 & -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

Then the operators A_2, A_3 can be represented by quadruples of real numbers (b_1, b_2, b_3, b_4) and (c_1, c_2, c_3, c_4), respectively. The operator A_1 is represented by $(1, 0, 0, 0)$.

In this case, the system (10) has only 8 unknown and is easy solvable. It was solved by *Mathematica*. All solutions are of two kinds. The first one is not interesting trivial solution

$$A_2 = A_3 = (1, 0, 0, 0).$$

The second type of solution is parametrizable

$$\begin{aligned} A_2(r,s,t) &= \left(-\frac{1}{2}, \frac{r}{2}\sqrt{3+4s^2+4t^2}, s, t\right), \\ A_3(r,s,t) &= \left(-\frac{1}{2}, -\frac{r}{2}\sqrt{3+4s^2+4t^2}, -s, -t\right), \end{aligned} \tag{11}$$

where $r \in \{1, -1\}$, $s \in \mathbb{R}$, $t \in \mathbb{R}$.

If $s = t = 0$, then operators A_2, A_3 correspond to rotation in the complex plane by angle $2\pi/3$ and $4\pi/3$, respectively.

We observe that for all $x \in \mathbb{R}^4$,

$$A_1 \cdot x + A_2(r,s,t) \cdot x + A_3(r,s,t) \cdot x = (0,0,0,0) \tag{12}$$

for all admissible r, s, t. Thus the equation (12) has a form of the cancellation law.

5. Commutative rings on the set $\mathbb{R}^4$

In the Section 3, we have constructed a class of commutative rings of three--polar spaces over the semi-field of double numbers. In the previous section, we have found the examples of polar operators that can be used to construct a three-polar space. In this section, we show how the operations from the three-polar space can be transfered to operations on the set $\mathbb{R}^4$.

Let $m\colon \mathcal{M} = X^3 \to \mathbb{R}^4$,

$$(a,b,c) \mapsto A_1 \cdot (a_1, a_2, 0, 0) + A_2(r,s,t) \cdot (b_1, b_2, 0, 0) + A_3(r,s,t) \cdot (c_1, c_2, 0, 0),$$

where $a = (a_1, a_2) \in X$, $b = (b_1, b_2) \in X$, $c = (c_1, c_2) \in X$. The mapping m is independent of the choice of representatives of equivalence classes $\mathbb{M} = \mathcal{M} \backslash_{\cong}$. Thus it can be considered as a mapping $\mathbb{M} \to \mathbb{R}^4$ that is injective. If $s^2 + t^2 > 0$, then the mapping m is onto. Moreover, m is linear, i.e.,

$$\begin{aligned} m\big((u,v,w) \oplus (x,y,z)\big) &= m(u,v,w) + m(x,y,z), \\ m\big(k \odot (u,v,w)\big) &= k_1 m(u,v,w), \end{aligned}$$

for all $(u,v,w), (x,y,z) \in X^3, \quad k = (k_1, 0) \in X, \quad k_1 \in \mathbb{R}$.

If $s^2 + t^2 > 0$, then m is a bijection, thus there is an inverse mapping, denoted by $n : \mathbb{R}^4 \to \mathbb{M}$. Also this mapping is linear. Through these two mappings we can introduce new operations, say addition and multiplication, on $\mathbb{R}^4$ as follows:

$$\begin{aligned} &p \boxplus q = m\big(n(p) \oplus n(q)\big), \\ &p \boxdot q = m\big(n(p) \odot n(q)\big), \quad p, q \in \mathbb{R}^4. \end{aligned} \tag{13}$$

Immediately, by linearity of both mappings,

$$p \boxplus q = p + q, \tag{14}$$

where $+$ is usual coordinate-wise addition in $\mathbb{R}^4$.

More interesting situation arises from the operation of multiplication. Let us denote four base elements in $\mathbb{R}^4$ as e_1, e_2, e_3, e_4 and let

$$p = \sum_{i=1}^{4} p_i e_i, \qquad q = \sum_{j=1}^{4} q_j e_j.$$

By properties of operations $\oplus, \odot$ and mappings m, n,

$$\begin{aligned} m\big(n(p) \odot n(q)\big) &= m\left(n\left(\sum_{i=1}^{4} p_i e_i \right) \odot n\left(\sum_{j=1}^{4} q_j e_j \right)\right) \\ &= m\left(\bigoplus_{i,j=1}^{4} p_i q_j \big[n(e_i) \odot n(e_j)\big] \right) \\ &= \sum_{i,j=1}^{4} p_i q_j m\big(n(e_i) \odot n(e_j)\big), \end{aligned}$$

where $\bigoplus$ means the sum with respect to $\oplus$. Therefore

$$p \boxdot q = \sum_{i,j=1}^{4} p_i q_j (e_i \boxdot e_j)\,,$$

and thus it suffices to find the elements $e_i \boxdot e_j = m\big(n(e_i) \odot n(e_j)\big)$.

Remark 5.1. It follows from the properties of the three-polar operations $\oplus$ and $\odot$, and the mappings m and n that the system $\big(\mathbb{R}^4, \boxplus, \boxdot\big)$ is a commutative ring with unit $(1,0,0,0)$ regardless of the parameters r, s, t, except for the condition $s^2 + t^2 > 0$. If $s = t = 0$, then the mapping $m\colon X^3 \to \mathbb{R}^4$ is not surjective and thus the mapping n is not well defined.

Example 5.2. For $r = 1, s = 1, t = 0$, c.f. (11), we have

$$A = (1,0,0,0), \quad B = \left(-\frac{1}{2}, \frac{\sqrt{7}}{2}, 1, 0 \right), \quad C = \left(-\frac{1}{2}, -\frac{\sqrt{7}}{2}, -1, 0 \right)$$

and the operation of multiplication in $\mathbb{R}^4$ possesses the following form

$$\begin{aligned} p \boxdot q \quad = \quad \Big(& p_1q_1 + p_2q_2 + \frac{9}{2}p_3q_3 - \frac{5}{2}p_4q_4 - \sqrt{7}(p_3q_2 + p_2q_3), \\ & p_1q_2 + p_2q_1 - \sqrt{7}(p_2q_4 + p_4q_2) + \frac{5}{2}(p_3q_4 + p_4q_3), \\ & p_1q_3 + p_3q_1 - p_2q_4 - p_4q_2, \\ & p_1q_4 + p_4q_1 - p_2q_3 - p_3q_2 + \sqrt{7}(p_3q_3 - p_4q_4)\Big). \end{aligned}$$

6. Conclusions

Multi-polarity has applications in physics. There are known multi-phase electric current, multi-polar magnets, oscillators, etc. Colours are another possible field of application of multi-polarity. The mathematical structure of multi-polar spaces is described in the paper [4]. Here is presented a multi-polar space over the semi-field of all non-negative real numbers. In this paper, we investigated a multi-polar space over a more general semi-field—the semi-field of double numbers. We studied a special case when polar operators are presented as split-quaternion multiplication. In this case we proved that the multi-polar space possesses structure of commutative ring with unit. Moreover, this system can be isomorphically transferred to $\mathbb{R}^4$. The operation of addition is usual coordinate-wise addition. Converse problem, to obtain an arbitrary commutative and associative operation on $\mathbb{R}^4$ distributing over coordinate-wise addition from a multi-polar space, remains open.

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