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SUBGROUPS OF FINITE ABELIAN GROUPS HAVING RANK TWO VIA GOURSAT'S LEMMA

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ABSTRACT. Using Goursat's lemma for groups, a simple representation and the invariant factor decompositions of the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$ are deduced, where m and n are arbitrary positive integers. As consequences, explicit formulas for the total number of subgroups, the number of subgroups with a given invariant factor decomposition, and the number of subgroups of a given order are obtained.

1. Introduction

Let $\mathbb{Z}_m$ denote the additive group of residue classes modulo m and consider the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$, where $m, n \in \mathbb{N} := \{1, 2, \ldots\}$ are arbitrary. Note that this group is isomorphic to $\mathbb{Z}_{\gcd(m,n)} \times \mathbb{Z}_{\mathrm{lcm}(m,n)}$. If $\gcd(m, n) = 1$, then it is cyclic, isomorphic to $\mathbb{Z}_{mn}$. If $\gcd(m, n) > 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n$ has rank two. We recall that a finite abelian group of order > 1 has rank r if it is isomorphic to $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$, where $n_1, \ldots, n_r \in \mathbb{N} \setminus \{1\}$ and $n_j \mid n_{j+1}$ $(1 \le j \le r-1)$, which is the invariant factor decomposition of the given group. Here the number r is uniquely determined and represents the minimal number of generators of the group. For general accounts on finite abelian groups see, e.g., [10], [14].

In this paper we apply Goursat's lemma for groups, see Section 2, to derive a simple representation and the invariant factor decompositions of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ (Theorem 3.1). These are new results, as far as we know. Then, we deduce as consequences, by purely number theoretical arguments, explicit formulas for the total number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ (Theorem 4.1), the number of its subgroups of a given order (Theorem 4.3) and the number of subgroups with a given invariant factor decomposition (Theorem 4.5, which is another new result). The number of cyclic subgroups (of a given order) is

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also treated (Theorems 4.6 and 4.7). Furthermore, in Section 5 a table for the subgroups of the group $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$ is given to illustrate the applicability of our identities.

The results of Theorems 4.1 and 4.3 generalize and put in more compact forms those of G. Călugăreanu [4], J. Petrillo [13] and M. Tărnăuceanu [15], obtained for p-groups of rank two, and included in Corollaries 4.2 and 4.4. We remark that both the papers [4] and [13] applied Goursat's lemma for groups (the first one in a slightly different form), while the paper [15] used a different approach based on properties of certain attached matrices.

Another representation of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, and the formulas of Theorems 4.1, 4.3 and 4.6, but not Theorem 4.5, were also derived in [7] using different group theoretical arguments. That representation and the formula of Theorem 4.1 was generalized to the case of the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_r$ $(m, n, r \in \mathbb{N})$ [8], using similar arguments, which are different from those of the present paper.

Note that in the case $m = n$ the subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$ play an important role in the field of applied time-frequency analysis (cf. [7]). See [11] for asymptotic results on the number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$.

Throughout the paper we use the following standard notations: $\tau(n)$ is the number of the positive divisors of n, ϕ denotes Euler's totient function, μ is the Möbius function, $*$ is the Dirichlet convolution of arithmetic functions.

2. Goursat's lemma for groups

Goursat's lemma for groups [6, p. 43–48] can be stated as follows:

Proposition 2.1. *Let G and H be arbitrary groups. Then there is a bijection between the set S of all subgroups of $G \times H$ and the set T of all 5-tuples (A, B, C, D, Ψ), where $B \trianglelefteq A \leq G$, $D \trianglelefteq C \leq H$ and $\Psi : A/B \to C/D$ is an isomorphism (here $\leq$ denotes subgroup and $\trianglelefteq$ denotes normal subgroup). More precisely, the subgroup corresponding to (A, B, C, D, Ψ) is*

$$K = \big\{(g, h) \in A \times C : \Psi(gB) = hD\big\}. \tag{1}$$

Corollary 2.2. *Assume that G and H are finite groups and that the subgroup K of $G \times H$ corresponds to the 5-tuple $(A_K, B_K, C_K, D_K, \Psi_K)$ under this bijection. Then one has $|A_K| \cdot |D_K| = |K| = |B_K| \cdot |C_K|$.*

For the history, proof, discussion, applications and a generalization of Goursat's lemma see [1], [2], [5], [9], [12], [13]. Corollary 2.2 is given in [5, Cor. 3].

3. Representation of the subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$

For every $m, n \in \mathbb{N}$ let

$$J_{m,n} := \left\{(a,b,c,d,\ell) \in \mathbb{N}^5 \colon a \mid m, b \mid a, c \mid n, d \mid c, \frac{a}{b} = \frac{c}{d}, \ell \le \frac{a}{b}, \gcd\left(\ell, \frac{a}{b}\right) = 1\right\}. \quad (2)$$

Using the condition

$$a/b = c/d$$

we deduce

$$\operatorname{lcm}(a,c) = \operatorname{lcm}(a, ad/b) = \operatorname{lcm}(ad/d, ad/b) = ad/\gcd(b,d).$$

That is, $\gcd(b,d) \cdot \operatorname{lcm}(a,c) = ad$. Also, $\gcd(b,d) \mid \operatorname{lcm}(a,c)$.

For $(a,b,c,d,\ell) \in J_{m,n}$ define

$$K_{a,b,c,d,\ell} := \left\{\left(i\frac{m}{a}, i\ell\frac{n}{c} + j\frac{n}{d}\right) : 0 \le i \le a-1, 0 \le j \le d-1\right\}. \quad (3)$$

Theorem 3.1. *Let $m, n \in \mathbb{N}$.*

i) *The map $(a,b,c,d,\ell) \mapsto K_{a,b,c,d,\ell}$ is a bijection between the set $J_{m,n}$ and the set of subgroups of $(\mathbb{Z}_m \times \mathbb{Z}_n, +)$.*

ii) *The invariant factor decomposition of the subgroup $K_{a,b,c,d,\ell}$ is*

$$K_{a,b,c,d,\ell} \simeq \mathbb{Z}_{\gcd(b,d)} \times \mathbb{Z}_{\operatorname{lcm}(a,c)}. \quad (4)$$

iii) *The order of the subgroup $K_{a,b,c,d,\ell}$ is ad and its exponent is $\operatorname{lcm}(a,c)$.*

iv) *The subgroup $K_{a,b,c,d,\ell}$ is cyclic if and only if $\gcd(b,d) = 1$.*

Figure 1 represents the subgroup $K_{6,2,18,6,1}$ of $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$. It has order 36 and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{18}$.

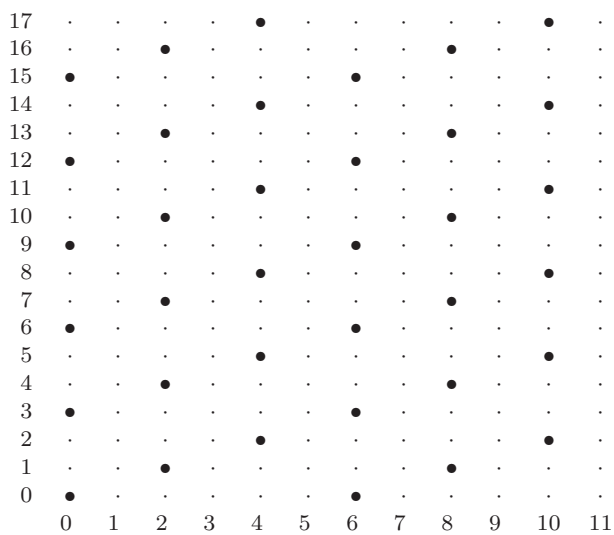

Figure 1.

P r o o f. i) Apply Goursat's lemma for the groups $G = \mathbb{Z}_m$ and $H = \mathbb{Z}_n$. We only need the following simple additional properties:

• all subgroups and all quotient groups of $\mathbb{Z}_n$ ($n \in \mathbb{N}$) are cyclic;

• for every $n \in \mathbb{N}$ and every $a \mid n$, $a \in \mathbb{N}$, there is precisely one (cyclic) subgroup of order a of $\mathbb{Z}_n$;

• the number of automorphisms of $\mathbb{Z}_n$ is $\phi(n)$ and they can be represented as $f : \mathbb{Z}_n \to \mathbb{Z}_n$, $f(x) = \ell x$, where $1 \le \ell \le n$, $\gcd(\ell, n) = 1$.

With the notations of Proposition 2.1, let $|A| = a$, $|B| = b$, $|C| = c$, $|D| = d$, where $a \mid m$, $b \mid a$, $c \mid n$, $d \mid c$. Writing explicitly the corresponding subgroups and quotient groups we deduce:

$$A = \langle m/a \rangle = \left\{0, \frac{m}{a}, 2\frac{m}{a}, \dots, (a-1)\frac{m}{a}\right\} \le \mathbb{Z}_m,$$
$$B = \langle m/b \rangle = \left\{0, \frac{m}{b}, 2\frac{m}{b}, \dots, (b-1)\frac{m}{b}\right\} \le A,$$
$$A/B = \left\langle \frac{m}{a} + B \right\rangle = \left\{B, \frac{m}{a} + B, 2\frac{m}{a} + B, \dots, \left(\frac{a}{b} - 1\right)\frac{m}{a} + B\right\},$$

and similarly

$$C = \langle n/c \rangle = \left\{0, \frac{n}{c}, 2\frac{n}{c}, \dots, (c-1)\frac{n}{c}\right\} \le \mathbb{Z}_n,$$
$$D = \langle n/d \rangle = \left\{0, \frac{n}{d}, 2\frac{n}{d}, \dots, (d-1)\frac{n}{d}\right\} \le C,$$
$$C/D = \left\langle \frac{n}{c} + D \right\rangle = \left\{D, \frac{n}{c} + D, 2\frac{n}{c} + D, \dots, \left(\frac{c}{d} - 1\right)\frac{n}{c} + D\right\}.$$

Now, in the case $a/b = c/d$ the values of the isomorphisms $\Psi : A/B \to C/D$ are

$$\Psi\left(i\frac{m}{a} + B\right) = i\ell\frac{n}{c} + D, \qquad 0 \le i \le \frac{a}{b} - 1,$$

where $1 \le \ell \le a/b$, $\gcd(\ell, a/b) = 1$. Using (1) we deduce that the corresponding subgroup is

$$K = \left\{\left(i\frac{m}{a}, k\frac{n}{c}\right) \in A \times C : \Psi\left(i\frac{m}{a} + B\right) = k\frac{n}{c} + D\right\}$$
$$= \left\{\left(i\frac{m}{a}, k\frac{n}{c}\right) : 0 \le i \le a-1, 0 \le k \le c-1, i\ell\frac{n}{c} + D = k\frac{n}{c} + D\right\},$$

where the last condition is equivalent, in turn, to $kn/c \equiv i\ell n/c \pmod{n/d}$, $k \equiv i\ell \pmod{c/d}$, and finally $k = i\ell + jc/d$, $0 \le j \le d-1$. Hence,

$$K = \left\{\left(i\frac{m}{a}, \left(i\ell + j\frac{c}{d}\right)\frac{n}{c}\right) : 0 \le i \le a-1, 0 \le j \le d-1\right\},$$

and the proof of the representation formula is complete.

ii-iii) It is clear from (3) that $|K_{a,b,c,d,\ell}| = ad = bc$ (or cf. Corollary 2.2). Next we deduce the exponent of $K_{a,b,c,d,\ell}$. According to (3) the subgroup $K_{a,b,c,d,\ell}$ is generated by the elements $(0, n/d)$ and $(m/a, \ell n/c)$. Here the order of $(0, n/d)$ is d. To obtain the order of $(m/a, \ell n/c)$ note the following properties:

(1) $m \mid r(m/a)$ if and only if $m/\gcd(m, m/a) \mid r$ if and only if $a \mid r$, and the least such $r \in \mathbb{N}$ is a,

(2) $n \mid t(\ell n/c)$ if and only if $n/\gcd(n, \ell n/c) \mid t$ if and only if $c/\gcd(\ell, c) \mid t$, and the least such $t \in \mathbb{N}$ is $c/\gcd(\ell, c)$.

Therefore the order of $(m/a, \ell n/c)$ is $\operatorname{lcm}(a, c/\gcd(\ell, c))$. We deduce that the exponent of $K_{a,b,c,d,\ell}$ is

$$\begin{aligned}
&\operatorname{lcm}\left(d, \operatorname{lcm}\left(a, \frac{c}{\gcd(\ell, c)}\right)\right) = \operatorname{lcm}\left(d, a, \frac{c}{\gcd(\ell, c)}\right) \\
&= \operatorname{lcm}\left(\frac{ac}{ac/d}, \frac{ac}{c}, \frac{ac}{a\gcd(\ell, c)}\right) = \frac{ac}{\gcd(ac/d, c, a\gcd(\ell, c))} \\
&= \frac{ac}{\gcd(c, a\gcd(c/d, \gcd(\ell, c)))} = \frac{ac}{\gcd(c, a\gcd(\ell, \gcd(c/d, c)))} \\
&= \frac{ac}{\gcd(c, a\gcd(\ell, c/d))} = \frac{ac}{\gcd(a, c)} = \operatorname{lcm}(a, c),
\end{aligned}$$

using that $\gcd(\ell, c/d) = 1$, cf. (2).

Now $K_{a,b,c,d,\ell}$ is a subgroup of the abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$ having rank ≤ 2. Therefore, $K_{a,b,c,d,\ell}$ has also rank ≤ 2. That is,

$$K_{a,b,c,d,\ell} \simeq \mathbb{Z}_u \times \mathbb{Z}_v$$

for unique u and v, where $u \mid v$ and $uv = ad$. Hence the exponent of $K_{a,b,c,d,\ell}$ is $\operatorname{lcm}(u, v) = v$. We obtain

$$v = \operatorname{lcm}(a, c) \quad \text{and} \quad u = ad/\operatorname{lcm}(a, c) = \gcd(b, d).$$

This gives (4).

iv) Clear from ii). This follows also from a general result given in [2, Th. 4.2]. □

Remark 3.2. One can also write $K_{a,b,c,d,\ell}$ in the form

$$\left\{\left(i\frac{m}{a}, i\ell\frac{n}{c} + j\frac{n}{d}\right) : 0 \leq i \leq a-1, j_i \leq j \leq j_i + d - 1\right\},$$

where $j_i = -\lfloor i\ell d/c \rfloor$. Then for the second coordinate one has

$$0 \leq i\ell\frac{n}{c} + j\frac{n}{d} \leq n - 1$$

for every given i and j.

4. Number of subgroups

According to Theorem 3.1 the total number $s(m,n)$ of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be obtained by counting the elements of the set $J_{m,n}$, which is now a purely number theoretical question.

Theorem 4.1. *For every $m,n \in \mathbb{N}$, $s(m,n)$ is given by*

$$s(m,n) = \sum_{i|m, j|n} \gcd(i,j) \tag{5}$$

$$= \sum_{t|\gcd(m,n)} \phi(t)\tau\left(\frac{m}{t}\right)\tau\left(\frac{n}{t}\right). \tag{6}$$

P r o o f. We have

$$s(m,n) = |J_{m,n}| = \sum_{\substack{a|m\\ b|a}} \sum_{\substack{c|n\\ d|c}} \sum_{a/b=c/d=e} \phi(e). \tag{7}$$

Let $m = ax$, $a = by$, $n = cz$, $c = dt$. Then, by the condition $a/b = c/d = e$ we have $y = t = e$. Rearranging the terms of (7),

$$s(m,n) = \sum_{bxe=m} \sum_{dze=n} \phi(e) = \sum_{\substack{ix=m\\ jz=n}} \sum_{\substack{be=i\\ de=j}} \phi(e)$$

$$= \sum_{\substack{i|m\\ j|n}} \sum_{e|\gcd(i,j)} \phi(e) = \sum_{\substack{i|m\\ j|n}} \gcd(i,j), \tag{8}$$

finishing the proof of (5). To obtain the formula (6) write (8) as follows:

$$s(m,n) = \sum_{\substack{ek=m\\ e\ell=n}} \phi(e) \sum_{\substack{bx=k\\ dz=\ell}} 1 = \sum_{\substack{ek=m\\ e\ell=n}} \phi(e)\tau(k)\tau(\ell)$$

$$= \sum_{e|\gcd(m,n)} \phi(e)\tau\left(\frac{m}{e}\right)\tau\left(\frac{n}{e}\right).$$

□

Note that (5) is a special case of an identity deduced in [3] by different arguments.

Corollary 4.2 ([4], [13, Prop. 2], [15, Th. 3.3]). *The total number of subgroups of the p-group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ $(1 \le a \le b)$ of rank two is given by*

$$s(p^a, p^b) = \frac{(b-a+1)p^{a+2} - (b-a-1)p^{a+1} - (a+b+3)p + (a+b+1)}{(p-1)^2}. \tag{9}$$

Now consider $s_\delta(m,n)$, denoting the number of subgroups of order δ of $\mathbb{Z}_m \times \mathbb{Z}_n$.

Theorem 4.3. *For every $m, n, \delta \in \mathbb{N}$ such that $\delta \mid mn$,*

$$s_\delta(m,n) = \sum_{\substack{i \mid \gcd(m,\delta) \\ j \mid \gcd(n,\delta) \\ \delta \mid ij}} \phi\left(\frac{ij}{\delta}\right). \tag{10}$$

P r o o f. Similar to the above proof. We have

$$s_\delta(m,n) = \sum_{\substack{a \mid m \\ b \mid a}} \sum_{\substack{c \mid n \\ d \mid c}} \sum_{\substack{a/b = c/d = e \\ ad = bc = \delta}} \phi(e) = \sum_{bxe = m} \sum_{dze = n} \sum_{bde = \delta} \phi(e) = \sum_{\substack{ix = m \\ jz = n}} \sum_{\substack{be = i \\ de = j \\ bde = \delta}} \phi(e),$$

where the only term of the inner sum is obtained for $e = ij/\delta$ provided that $\delta \mid ij$, $i \mid \delta$ and $j \mid \delta$. This gives (10). □

Corollary 4.4 ([15, Th. 3.3]). *The number of subgroups of order p^c of the p-group $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ $(1 \le a \le b)$ is given by*

$$s_{p^c}(p^a, p^b) = \begin{cases} \frac{p^{c+1}-1}{p-1}, & c \le a \le b, \\ \frac{p^{a+1}-1}{p-1}, & a \le c \le b, \\ \frac{p^{a+b-c+1}-1}{p-1}, & a \le b \le c \le a+b. \end{cases} \tag{11}$$

The number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ with a given isomorphism type $\mathbb{Z}_A \times \mathbb{Z}_B$ is given by the following formula.

Theorem 4.5. *Let $m, n \in \mathbb{N}$ and let $A, B \in \mathbb{N}$ such that $A \mid B$, $AB \mid mn$. Let $A \mid \gcd(m,n)$, Then the number $N_{A,B}(m,n)$ of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, which are isomorphic to $\mathbb{Z}_A \times \mathbb{Z}_B$ is given by*

$$N_{A,B}(m,n) = \sum_{\substack{i \mid m, j \mid n \\ AB \mid ij \\ \operatorname{lcm}(i,j) = B}} \phi\left(\frac{ij}{AB}\right). \tag{12}$$

If $A \nmid \gcd(m,n)$, then $N_{A,B}(m,n) = 0$.

P r o o f. Using Proposition 3.1/ ii) we have

$$N_{A,B}(m,n) = \sum_{\substack{a \mid m \\ b \mid a}} \sum_{\substack{c \mid n \\ d \mid c}} \sum_{\substack{a/b = c/d = e \\ \gcd(b,d) = A \\ \operatorname{lcm}(a,c) = B}} \phi(e).$$

Here the condition $\gcd(b,d) = A$ implies that $A \mid m$ and $A \mid n$. In this case

$$N_{A,B}(m,n) = \sum_{\substack{bxe = m \\ dze = n}} \sum_{\substack{\gcd(b,d) = A \\ bde = AB}} \phi(e) = \sum_{\substack{ix = m \\ jz = n}} \sum_{\substack{be = i \\ de = j}} \sum_{\substack{bde = AB \\ \gcd(b,d) = A}} \phi(e),$$

where the only term of the inner sum is obtained for $e = ij/(AB)$ provided that $AB \mid ij$, $i \mid AB$ and $j \mid AB$. $\square$

Remark: It is also possible that $N_{A,B}(m,n) = 0$ even when all the conditions in the hypotheses are satisfied, e.g., see the Table in Section 5.

From the representation of the cyclic subgroups given in Proposition 3.1/ iv) we also deduce the next result.

Theorem 4.6. *For every $m, n \in \mathbb{N}$ the number $c(m,n)$ of cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ is*

$$c(m,n) = \sum_{i \mid m, j \mid n} \phi\big(\gcd(i,j)\big) \tag{13}$$

$$= \sum_{t \mid \gcd(m,n)} (\mu * \phi)(t) \tau\left(\frac{m}{t}\right) \tau\left(\frac{n}{t}\right). \tag{14}$$

P r o o f. Similar to the above proofs, using that for the cyclic subgroups one has $\gcd(b,d) = 1$. $\square$

Let $c_\delta(m,n)$ denote the number of cyclic subgroups of order δ of $\mathbb{Z}_m \times \mathbb{Z}_n$.

Theorem 4.7. *For every $m, n, \delta \in \mathbb{N}$ such that $\delta \mid mn$,*

$$c_\delta(m,n) = \sum_{\substack{i \mid m, j \mid n \\ \operatorname{lcm}(i,j) = \delta}} \phi\big(\gcd(i,j)\big).$$

P r o o f. This is a direct consequence of (12) obtained in the cases $A = 1$ and $B = \delta$. $\square$

In the paper [7] the identities (5), (6), (10), (13) and (14) were derived using another approach. The identity (13), as a special case of a formula valid for arbitrary finite abelian groups, was obtained by the author [16], [17] using different arguments. Finally, we remark that the functions $(m,n) \mapsto s(m,n)$ and $(m,n) \mapsto c(m,n)$ are multiplicative, viewed as arithmetic functions of two variables. See [7], [18] for details.

5. Table of the subgroups of $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$

To illustrate our results we describe the subgroups of the group $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$ ($m = 12$, $n = 18$). According to Proposition 4.5, there exist subgroups isomorphic to $\mathbb{Z}_A \times \mathbb{Z}_B$ $(A \mid B)$ only if $A \mid \gcd(12,18) = 6$, that is

$$A \in \{1,2,3,6\} \quad \text{and} \quad AB \mid 12 \cdot 18 = 216\,.$$

TABLE 1. Table of the subgroups of $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$.

Total number subgroups	80		
Number subgroups order 1	1	Number subgroups order 18	12
Number subgroups order 2	3	Number subgroups order 24	4
Number subgroups order 3	4	Number subgroups order 27	1
Number subgroups order 4	3	Number subgroups order 36	12
Number subgroups order 6	12	Number subgroups order 54	3
Number subgroups order 8	1	Number subgroups order 72	4
Number subgroups order 9	4	Number subgroups order 108	3
Number subgroups order 12	12	Number subgroups order 216	1
Number cyclic subgroups	48	Number noncyclic subgroups	32
Number subgroups $\simeq \mathbb{Z}_1$	1	Number subgroups $\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$	1
Number subgroups $\simeq \mathbb{Z}_2$	3	Number subgroups $\simeq \mathbb{Z}_2 \times \mathbb{Z}_4$	1
Number subgroups $\simeq \mathbb{Z}_3$	4	Number subgroups $\simeq \mathbb{Z}_2 \times \mathbb{Z}_6$	4
Number subgroups $\simeq \mathbb{Z}_4$	2	Number subgroups $\simeq \mathbb{Z}_2 \times \mathbb{Z}_{12}$	4
Number subgroups $\simeq \mathbb{Z}_6$	12	Number subgroups $\simeq \mathbb{Z}_2 \times \mathbb{Z}_{18}$	3
Number subgroups $\simeq \mathbb{Z}_9$	3	Number subgroups $\simeq \mathbb{Z}_2 \times \mathbb{Z}_{36}$	3
Number subgroups $\simeq \mathbb{Z}_{12}$	8	Number subgroups $\simeq \mathbb{Z}_3 \times \mathbb{Z}_3$	1
Number subgroups $\simeq \mathbb{Z}_{18}$	9	Number subgroups $\simeq \mathbb{Z}_3 \times \mathbb{Z}_6$	3
Number subgroups $\simeq \mathbb{Z}_{36}$	6	Number subgroups $\simeq \mathbb{Z}_3 \times \mathbb{Z}_9$	1
		Number subgroups $\simeq \mathbb{Z}_3 \times \mathbb{Z}_{12}$	2
		Number subgroups $\simeq \mathbb{Z}_3 \times \mathbb{Z}_{18}$	3
		Number subgroups $\simeq \mathbb{Z}_3 \times \mathbb{Z}_{36}$	2
		Number subgroups $\simeq \mathbb{Z}_6 \times \mathbb{Z}_6$	1
		Number subgroups $\simeq \mathbb{Z}_6 \times \mathbb{Z}_{12}$	1
		Number subgroups $\simeq \mathbb{Z}_6 \times \mathbb{Z}_{18}$	1
		Number subgroups $\simeq \mathbb{Z}_6 \times \mathbb{Z}_{36}$	1

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