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IMPROVEMENT ON THE DISCREPANCY OF $(t, \mathbf{e}, s)$-SEQUENCES

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ABSTRACT. Recently, a notion of $(t, \mathbf{e}, s)$-sequences in base b was introduced, where $\mathbf{e} = (e_1, \dots, e_s)$ is a positive integer vector, and their discrepancy bounds were obtained based on the signed splitting method. In this paper, we first propose a general framework of $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences, and present that it includes $(\mathbf{T}, s)$-sequences and $(t, \mathbf{e}, s)$-sequences as special cases. Next, we show that a careful analysis leads to an asymptotic improvement on the discrepancy bound of a $(t, \mathbf{e}, s)$-sequence in an even base b. It follows that the constant in the leading term of the star discrepancy bound is given by

$$c_s^* = \frac{b^t}{s!} \prod_{i=1}^{s} \frac{b^{e_i} - 1}{2e_i \log b}.$$

1. Introduction

Low-discrepancy sequences form a mainstay of quasi-Monte Carlo methods in scientific computing. All current constructions of s-dimensional low-discrepancy sequences yield a bound on the discrepancy, low-discrepancy sequences, signed splitting method, $(t, \mathbf{e}, s)$-sequences discrepancy D_N^* of the form

$$D_N^* \le c_s^* \frac{(\log N)^s}{N} + O\left(\frac{(\log N)^{s-1}}{N}\right) \tag{1}$$

for all $N > 1$. Then it is the aim to obtain constructions and/or methods for bounding discrepancy that achieve a constant c_s^* as small as possible. At present, we have two types of constructions: Halton sequences and (t, s)-sequences. (To be precise, Kronecker sequences (see, e.g., [2]) are known to satisfy the definition (1) only for the one-dimensional case $s = 1$.) Recently, the author [16] introduced a generalization of the theory of (t, s)-sequences, namely the concept of $(t, \mathbf{e}, s)$-sequences with $\mathbf{e} = (e_1, \dots, e_s)$ being an s-tuple of positive integers, where

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the special case of $\mathbf{e} = (e_1, \ldots, e_s) = (1, \ldots, 1)$ corresponds to that of classical (t, s)-sequences. Then, the author applied this concept to prove that a generalized Niederreiter sequence [14], [15] is a $(t, \mathbf{e}, s)$-sequence with $t = 0$, where e_i is the degree of the ith base polynomial in the construction, and obtained much better discrepancy bounds for generalized Niederreiter sequences by using the signed splitting method described below. Subsequently, under the framework of $(t, \mathbf{e}, s)$-sequences, Hofer and Niederreiter [10] and Niederreiter and Yeo [12] proposed new constructions of low-discrepancy sequences with better discrepancy bounds based on global function fields.

There are three methods of obtaining discrepancy bounds depending on the particular constructions.

- (Chinese remainder theorem) In 1960, Halton [9] became the first who obtained a construction of low-discrepancy sequences which satisfy the definition (1). Today, his construction is called the Halton sequence. He employed the Chinese remainder theorem to analyze the discrepancy of his sequences. Although some improvements in this direction have been done (see, e.g., Faure [4]), the constant c_s^* in the discrepancy bound still grows super-exponentially in the dimension.
- (Double recursion method) In 1967, Sobol' [13] invented the double recursion method to analyze the discrepancy of what is today called (t, s)-sequences in base 2. However, the constant c_s^* obtained for the Sobol' sequence, which is a special case of (t, s)-sequences in base 2, still super-exponentially increases in the dimension. In 1982, Faure [4] applied this method to the so-called Faure sequence, which is a special case of $(0, s)$-sequences in a prime base b, and showed that the constant c_s^* converges to zero as the dimension goes to infinity, provided that the base b is chosen to be the least prime with $b \geq s$. In 1987, Niederreiter [11] introduced a notion of (t, s)-sequences in base b for an arbitrary integer $b \geq 2$, and established a general framework of what is today called the net theory of digital sequences [3].
- (Signed splitting method) In 2004, Atanassov [1] proposed the signed splitting method for the discrepancy analysis of generalized Halton sequences, and showed that the constant c_s^* for the Halton sequence converges to zero as the dimension goes to infinity. Recently, the author [16] applied this method to $(t, \mathbf{e}, s)$-sequences, and showed that the constant c_s^* for the Niederreiter sequence in base 2, which is a special case of (t, s)-sequences in base 2, converges to zero as the dimension goes to infinity.

In this paper, we present recent results on $(t, \mathbf{e}, s)$-sequences. In Section 2, we introduce a general concept of $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences in base b, and show that $(\mathbf{T}, s)$-sequences and $(t, \mathbf{e}, s)$-sequences are special cases of $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences.

In Section 3, we first overview the signed splitting method for $(t, \mathbf{e}, s)$-sequences in base b, as well as the previous results on the discrepancy of these sequences. Then, we improve the discrepancy bound for $(t, \mathbf{e}, s)$-sequences in even bases, which yields the smaller constant c_s^* than the previous one [16]. In the final section, we discuss interesting open questions for future research.

2. $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences and $(t, \mathbf{e}, s)$-sequences

First, we introduce the definition of discrepancy. For a point set $P_N = \{X_0, X_1, \dots, X_{N-1}\}$ of N points in $[0,1]^s$ and an interval $J \subseteq [0,1]^s$, we define $A_N(J)$ as the number of n, $0 \le n \le N-1$, with $X_n \in J$ and $\mu(J)$ is the volume of J. Then the star discrepancy of P_N is defined by

$$D_N^* = \sup_J \left| \frac{A_N(J)}{N} - \mu(J) \right|,$$

where the supremum is taken over all intervals J of the form $\prod_{i=1}^s [0, \alpha_i)$ for $0 < \alpha_i \le 1$. The (unanchored) discrepancy D_N is obtained when the supremum is taken over all intervals J of the form $\prod_{i=1}^s [\alpha_i, \beta_i)$ for $0 \le \alpha_i < \beta_i \le 1$.

Let $b \ge 2$ be an integer. An elementary interval in base b, which is a key concept of the net theory, is an interval of the form

$$E(\mathbf{l}; \mathbf{a}) = \prod_{i=1}^s \left[\frac{a_i}{b^{l_i}}, \frac{a_i + 1}{b^{l_i}} \right),$$

where a_i and l_i are integers with $0 \le a_i < b^{l_i}$ and $l_i \ge 0$ for $i = 1, \dots, s$. Denote a subset of nonnegative integer vectors by $\mathcal{E} \subseteq \mathbb{N}_0^s$, where $\operatorname{card}(\mathcal{E}) = \infty$. Define a set of elementary intervals as follows:

$$\mathbf{E}(\mathcal{E}) = \bigcup_{\mathbf{l} \in \mathcal{E}} E(\mathbf{l}),$$

where

$$E(\mathbf{l}) = \big\{ E(\mathbf{l}; \mathbf{a}) \,|\, 0 \le a_i < b^{l_i},\ (1 \le i \le s) \big\}.$$

Define $|\mathbf{l}| = l_1 + \dots + l_s$ for a nonnegative integer vector $\mathbf{l} = (l_1, \dots, l_s)$. Denote a set of nonnegative integers by $\mathbb{N}(\mathcal{E}) = \big\{ |\mathbf{l}| \mid \mathbf{l} = (l_1, \dots, l_s) \in \mathcal{E} \big\}$. Remark that $\operatorname{card}\big(\mathbb{N}(\mathcal{E})\big) = \infty$ because $\operatorname{card}(\mathcal{E}) = \infty$. We first give the definition of $(t, m, \mathcal{E}, s)$-nets as follows:

Definition 1. Let t and m be integers with $0 \le t \le m$ such that $m - t \in \mathbb{N}(\mathcal{E})$. A $(t, m, \mathcal{E}, s)$-net in base b is a point set of b^m points in $[0,1]^s$ such that $A_{b^m}(E) = b^t$ for every elementary interval $E \in \mathbf{E}(\mathcal{E})$ with $\mu(E) = b^{t-m}$.

Let $\mathbf{T}_{\mathcal{E}}$ be a mapping from $\mathbb{N}_0$ to $\mathbb{N}_0$, where $0 \le \mathbf{T}_{\mathcal{E}}(m) \le m$, such that there are infinitely many m satisfying $m - \mathbf{T}_{\mathcal{E}}(m) \in \mathbb{N}(\mathcal{E})$. Then $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences are defined as follows:

Definition 2. A $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequence in base b is an infinite sequence, $X = (X_n)_{n\geq 0}$, of points in $[0,1]^s$ such that for all integers $k \geq 0$ and all $m \geq \mathbf{T}_{\mathcal{E}}(m)$ satisfying $m - \mathbf{T}_{\mathcal{E}}(m) \in \mathbb{N}(\mathcal{E})$, the point set $\big\{[X_{kb^m}]_{b,m}, \ldots, [X_{(k+1)b^m-1}]_{b,m}\big\}$ is a $(\mathbf{T}_{\mathcal{E}}(m), m, \mathcal{E}, s)$-net, where $[X_n]_{b,m}$ means the coordinate-wise b-ary m-digit truncation of a point X_n.

It is easy to obtain the following propositions.

Proposition 1. *When $\mathcal{E} = \mathbb{N}_0^s$, a $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequence in base b is identical to a $(\mathbf{T}, s)$-sequence in base b.*

Proposition 2. *Let $\mathbf{e} = (e_1, \ldots, e_s)$ be a positive integer vector. When the mapping $\mathbf{T}_{\mathcal{E}}$ is constant, i.e., $\mathbf{T}_{\mathcal{E}} \equiv t$, and $\mathcal{E} = \big\{\mathbf{l} \,|\, e_i$ divides l_i $(1 \leq i \leq s)\big\}$, then a $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequence in base b is identical to a $(t, \mathbf{e}, s)$-sequence in base b.*

The example below gives two $(0, \mathbf{e}, 1)$-sequences in base 2 with $\mathbf{e} = (3)$.

Example 1. The first case is the following generator matrix of a strict $(2,1)$--sequence in base 2.

$$G_1 = \begin{pmatrix} J & O & O \\ O & J & O \\ O & O & \ddots \end{pmatrix},$$

where J is a 3×3 matrix defined as

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this case, we observe that

$$\mathbf{T}(m) = \begin{cases} 0 & \text{if } m = 0 \pmod 3, \\ 1 & \text{if } m = 1 \pmod 3, \\ 2 & \text{if } m = 2 \pmod 3. \end{cases}$$

Thus, this is a $(0, \mathbf{e}, 1)$-sequence in base 2 with $\mathbf{e} = (3)$, equivalently, a $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$--sequence with $\mathcal{E} = \big\{l \,|\, l = 0 \pmod 3\big\}$ and $\mathbf{T}_{\mathcal{E}} \equiv 0$.

The second generator matrix of a strict $(2,1)$-sequence in base 2 is as follows:

$$G_2 = \begin{pmatrix} J & O \\ O & I \end{pmatrix},$$

where I is an infinite identity matrix. In this case, we observe that

$$\mathbf{T}(m) = \begin{cases} 0 & \text{if } m = 0 \text{ or } m \geq 3, \\ 1 & \text{if } m = 1, \\ 2 & \text{if } m = 2. \end{cases}$$

Thus, this is also a $(0, \mathbf{e}, 1)$-sequence in base 2 with $\mathbf{e} = (3)$.

3. Signed splitting method for (t, e, s)-sequences

3.1. Overview of previous results

We first overview the notion of signed splitting introduced by A t a n a s s o v [1] and its relevant lemma.

Definition 3. Consider an interval $J \subseteq [0,1]^s$. We call a signed splitting of J any collection of intervals $J_1, \ldots, J_n$ and respective signs $\epsilon_1, \ldots, \epsilon_n$ equal to ± 1, such that for any finitely additive function ν on the intervals in $[0,1]^s$, we have $\nu(J) = \sum_{i=1}^{n} \epsilon_i \nu(J_i)$.

Lemma 1. *Let $J = \prod_{i=1}^{s} [0, z^{(i)})$ be an interval in $[0,1]^s$ and let $n_i \geq 0$ be a given integer for $i = 1, \ldots, s$. Set $z_0^{(i)} = 0, z_{n_i+1} = z^{(i)}$ and, if $n_i \geq 1$, let $z_{j_i}^{(i)} \in [0,1]$ be arbitrary numbers for $j_i = 1, \ldots, n_i$. Then the collection of intervals*

$$I(j_1, \ldots, j_s) = \prod_{i=1}^{s} \Big[\min\Big(z_{j_i}^{(i)}, z_{j_i+1}^{(i)} \Big), \max\Big(z_{j_i}^{(i)}, z_{j_i+1}^{(i)} \Big) \Big)$$

with signs $\epsilon(j_1, \ldots, j_s) = \prod_{i=1}^{s} \operatorname{sgn}\big(z_{j_i+1}^{(i)} - z_{j_i}^{(i)} \big)$ for $j_i = 0, 1, \ldots, n_i$ $(1 \leq i \leq s)$, is a signed splitting of the interval J.

Take any $\mathbf{z} = \big(z^{(1)}, \ldots, z^{(s)} \big) \in [0,1]^s$. Each $z^{(i)}$ is expanded as $\sum_{j=0}^{\infty} a_j^{(i)} b^{-e_i j}$, with $|a_0^{(i)}| \leq 1$, $|a_j^{(i)}| \leq \lfloor b^{e_i}/2 \rfloor$, and $|a_j^{(i)}| + |a_{j+1}^{(i)}| \leq b^{e_i} - 1$, for $j \geq 1$. (The existence of such an expansion is proved in A t a n a s s o v [1].) Let $n_i = \lfloor \frac{\log_b N - t}{e_i} \rfloor + 1$ and define $z_0^{(i)} = 0$ and $z_{n_i+1}^{(i)} = z^{(i)}$. Consider the numbers $z_k^{(i)} = \sum_{j=0}^{k-1} a_j^{(i)} b^{-e_i j}$ for $k = 1, \ldots, n_i$. By using the additivity of the local discrepancy, and applying the above lemma, we have

$$A_N(J) - N\mu(J) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_s=0}^{n_s} \epsilon(\mathbf{j}) \Big(A_N\big(I(\mathbf{j})\big) - N\mu\big(I(\mathbf{j})\big) \Big) = \sum\nolimits_1 + \sum\nolimits_2 ,$$

where in $\sum_1$ we put all vectors $\mathbf{j}$ such that $b^{(\mathbf{e}, \mathbf{j})+t} \leq N$, and in $\sum_2$ the rest.

In order to estimate the two sums, $\sum_1$ and $\sum_2$, the following two lemmas [16] are needed.

Lemma 2. *Let $b \geq 2$ be an arbitrary integer, $\mathbf{e} = (e_1, \ldots, e_s)$ and $\mathbf{j} = (j_1, \ldots, j_s)$ be integer vectors with $e_i \geq 1$ and $j_i \geq 0$ for $i = 1, \ldots, s$. Let I be an interval given by*

$$I = \prod_{i=1}^{s} \Big[\frac{a_i}{b^{e_i j_i}}, \frac{c_i}{b^{e_i j_i}} \Big),$$

where a_i and c_i are integers with $0 \leq a_i < c_i \leq b^{e_i j_i}$ for $i = 1, \ldots, s$. Then, for the first N points of the truncated version of a $(t, \mathbf{e}, s)$-sequence in base b, *we have*

$$|A_N(I) - N\mu(I)| \le b^t \prod_{i=1}^{s}(c_i - a_i)$$

for every positive integer N, and $A_N(I) \le b^t \prod_{i=1}^{s}(c_i - a_i)$ if $N < b^{(\mathbf{e},\,\mathbf{j})+t}$, where the truncation size is taken to be large enough depending on $\mathbf{j}$.

By using the fact that the number of positive integer vectors $\mathbf{j} = (j_1, \dots, j_s)$ satisfying $(\mathbf{e}, \mathbf{j}) \le \alpha$ for $\alpha > 0$ is bounded by $\frac{1}{s!}\prod_{i=1}^{s}\frac{\alpha}{e_i}$, we obtain the following

Lemma 3. *Let $e_1, \dots, e_s$ be positive integers and $\alpha > 0$. Let some numbers $g_j^{(i)} \ge 0$ be given for $j \ge 0$ and $i = 1, \dots, s$ satisfying $g_0^{(i)} \le 1$ and $g_j^{(i)} \le f_i(e_i)$ for $j \ge 1$, where $f_1(e_1), \dots, f_s(e_s)$ are some numbers. Then*

$$\sum_{\substack{(j_1,\dots,j_s)\\ (\mathbf{e},\,\mathbf{j})\le\alpha}} \prod_{i=1}^{s} g_{j_i}^{(i)} \le \frac{1}{s!}\prod_{i=1}^{s}\left(f_i(e_i)\frac{\alpha}{e_i} + s\right).$$

Based on the above lemmas, the next theorem was obtained [16].

Theorem 1. *Let $b \ge 2$ be an arbitrary integer. The star discrepancy for the first $N > b^t$ points of a $(t, \mathbf{e}, s)$-sequence in base b is bounded as follows:*

$$ND_N^* \le \frac{b^t}{s!}\prod_{i=1}^{s}\left(\frac{\lfloor b^{e_i}/2\rfloor}{e_i}(\log_b N - t) + s\right) + \sum_{k=0}^{s-1}\frac{b^{t+e_{k+1}}}{k!}\prod_{i=1}^{k}\left(\frac{\lfloor b^{e_i}/2\rfloor}{e_i}(\log_b N - t) + k\right). \tag{2}$$

In the above theorem, the first term of the righthand side of (2) is the upperbound of $|\sum_1|$ and the second term is that of $|\sum_2|$. Since $|\sum_2| = O\big((\log_b N)^{s-1}\big)$, the leading constant is given as follows:

$$c_s^* = \frac{b^t}{s!}\prod_{i=1}^{s}\frac{\lfloor b^{e_i}/2\rfloor}{e_i \log b}. \tag{3}$$

3.2. Improvement of the leading constant

We now give the main result of this section. The key idea of the proof, which exploits the property $|a_j^{(i)}| + |a_{j+1}^{(i)}| \le b^{e_i} - 1$ to improve the leading constant, is due to Atanassov [1]. Faure and Lemieux [6] applied his idea in a slightly modified form to (t, s)-sequences. However, their proof contains a serious error (see the corrigendum [7] for the detail). The proof given below can be viewed as a generalized and corrected version, because (t, s)-sequences are a special case of $(t, \mathbf{e}, s)$-sequences.

Theorem 2. *Let $b \geq 2$ be an arbitrary integer. The star discrepancy for the first $N > b^t$ points of a $(t, \mathbf{e}, s)$-sequence in base b is bounded as follows:*

$$\begin{aligned} ND_N^* \leq \frac{b^t}{s!} \prod_{i=1}^{s} \left(\frac{b^{e_i} - 1}{2e_i} (\log_b N - t) + s \right) \\ + \frac{b^{t+e_s} |\mathbf{e}|}{2} \prod_{i=1}^{s-1} \left(\frac{\lfloor b^{e_i}/2 \rfloor}{e_i} (\log_b N - t) + \lfloor b^{e_i}/2 \rfloor \right) \\ + \sum_{k=0}^{s-1} \frac{b^{t+e_{k+1}}}{k!} \prod_{i=1}^{k} \left(\frac{\lfloor b^{e_i}/2 \rfloor}{e_i} (\log_b N - t) + k \right). \end{aligned} \tag{4}$$

P r o o f. First, we divide the first sum $\sum_1$ as follows:

$$\textstyle\sum_1 = \sum_{1A} + \sum_{1B},$$

where we take $\sum_{1A}$ over all $\mathbf{j}$ with $b^{(\mathbf{e},\mathbf{j})+t} \leq Nb^{-|\mathbf{e}|}$, and $\sum_{1B}$ over all $\mathbf{j}$ with $Nb^{-|\mathbf{e}|} < b^{(\mathbf{e},\mathbf{j})+t} \leq N$. First, the sum $\sum_{1A}$ is considered. Define $S = \{\mathbf{j} \,|\, (\mathbf{e}, \mathbf{j}) \leq \log_b N - |\mathbf{e}| - t\}$, $\bar{S} = \{(2\lfloor (j_1+1)/2 \rfloor - \delta_1, \ldots, 2\lfloor (j_s+1)/2 \rfloor - \delta_s) \,|\, (j_1, \ldots, j_s) \in S$ and $(\delta_1, \ldots, \delta_s) \in \{0,1\}^s\}$, and $\bar{S}' = \{(\lfloor (j_1+1)/2 \rfloor, \ldots, \lfloor (j_s+1)/2 \rfloor) \,|\, (j_1, \ldots, j_s) \in \bar{S}\}$. We define integers $c_h^{(i)} = |a_{2h-1}^{(i)}| + |a_{2h}^{(i)}|$ for $h \geq 1$ and $c_0^{(i)} = 1$. Note that $c_0^{(i)} \geq |a_0^{(i)}|$ because $a_0^{(i)} = 0$ or 1. Then, we observe

$$\sum_{\delta_1=0}^{1} \cdots \sum_{\delta_s=0}^{1} \prod_{i=1}^{s} |a_{2h_i - \delta_i}^{(i)}| = \prod_{i=1}^{s} \left(|a_{2h_i-1}^{(i)}| + |a_{2h_i}^{(i)}| \right) = \prod_{i=1}^{s} c_{h_i}^{(i)},$$

in other words,

$$\sum_{\substack{(j_1,\ldots,j_s) \\ j_i' = h_i (1 \leq i \leq s)}} \prod_{i=1}^{s} |a_{j_i}^{(i)}| = \prod_{i=1}^{s} c_{j_i'}^{(i)},$$

for any positive vector $\mathbf{j}$, where $j_i' = j_i/2$ if j_i is even; otherwise $(j_i + 1)/2$. Remark that if there is some i with $j_i = 0$, we have

$$\sum_{\substack{(j_1,\ldots,j_s) \\ j_i' = h_i (1 \leq i \leq s)}} \prod_{i=1}^{s} |a_{j_i}^{(i)}| \leq \prod_{i=1}^{s} c_{j_i'}^{(i)}.$$

Let $\tilde{b} = b^2$. For any $\mathbf{j} \in \bar{S}$, there exists one $\mathbf{j}' \in \bar{S}'$. Thus, we have

$$\begin{aligned} \tilde{b}^{e_1 j_1' + \cdots + e_s j_s'} &= b^{2(\sum_{even\ j_i} e_i j_i/2 \ + \ \sum_{odd\ j_i} e_i (j_i+1)/2)} \\ &\leq b^{(\mathbf{e},\mathbf{j})} b^{|\mathbf{e}|} \leq \frac{Nb^{-t}}{b^{|\mathbf{e}|}} b^{|\mathbf{e}|} = Nb^{-t}. \end{aligned}$$

Since $c_j^{(i)} = |a_{2j-1}^{(i)}| + |a_{2j}^{(i)}| \le b^{e_i} - 1$ and $\log_b \tilde{b} = 2$, Lemmas 2 and 3 give us

$$\left|\sum_{1A}\right| \le b^t \sum_{\mathbf{j}\in S} \prod_{i=1}^{s} |a_{j_i}^{(i)}| \le b^t \sum_{\mathbf{j}\in \bar{S}} \prod_{i=1}^{s} |a_{j_i}^{(i)}| \le b^t \sum_{\mathbf{j}'\in \bar{S}'} \prod_{i=1}^{s} c_{j_i'}^{(i)}$$
$$\le \frac{b^t}{s!} \prod_{i=1}^{s} \left(\frac{b^{e_i} - 1}{2e_i} (\log_b N - t) + s \right).$$

Next, we consider the sum $\sum_{1B}$. Taking the logarithm of the condition, we have

$$\log_b N - t - |\mathbf{e}| < (\mathbf{e}, \mathbf{j}) \le \log_b N - t.$$

This means that we have linear relations $(\mathbf{e}, \mathbf{j}) = m$ in $(j_1, \dots, j_s)$ for $\log_b N - t - |\mathbf{e}| < m \le \log_b N - t$. Thus, the number of nonnegative integer vectors $(j_1, \dots, j_s)$ satisfying such relations is at most $|\mathbf{e}| \prod_{i=1}^{s-1} \big(\lfloor (\log_b N - t)/e_i \rfloor + 1 \big)$. Taking into consideration that each $\mathbf{j}$ contributes at most $b^t \prod_{i=1}^{s} \lfloor b^{e_i}/2 \rfloor$ to the estimate of $\sum_{1B}$, we obtain

$$\left|\sum_{1B}\right| \le \frac{b^{t+e_s}|\mathbf{e}|}{2} \prod_{i=1}^{s-1} \left(\frac{\lfloor b^{e_i}/2 \rfloor}{e_i} (\log_b N - t) + \lfloor b^{e_i}/2 \rfloor \right).$$

Remark that the third term of the righthand side of (4) is the upper bound of $|\sum_2|$, which remains the same as that obtained in Theorem 1. Since both bounds on the sums, $\sum_{1A}$ and $\sum_{1B}$, are independent of each interval $J = \prod_{i=1}^{s} [0, z^{(i)})$, the discrepancy bound for the truncated version is obtained. As it is shown in [16], the discrepancy bound for the untruncated version remains the same as the truncated version. The proof is complete. □

Since we have $|\sum_{1B}| = O\big((\log_b N)^{s-1}\big)$ in the above theorem, the leading constant for $(t, \mathbf{e}, s)$-sequences in base b is given as

$$c_s^*(new) = \frac{b^t}{s!} \prod_{i=1}^{s} \frac{b^{e_i} - 1}{2e_i \log b}.$$

In comparison with the previous constant of (3), the new constant yields an improvement for the case of even bases. We should notice that for any odd base b the bound in (4) has no improvement on the previous bound in (2).

The leading constant currently known as the best for (t, s)-sequences in base b, which was recently obtained by Faure and Kritzer [5] based on an improvement of the double recursion method, is given as

$$c_s^*(t, s) = \begin{cases} \frac{b^2}{2(b^2-1)} \frac{b^t}{s!} \left(\frac{b-1}{2\log\ b} \right)^s & \text{if } b \text{ is even,} \\ \frac{1}{2} \frac{b^t}{s!} \left(\frac{b-1}{2\log\ b} \right)^s & \text{otherwise.} \end{cases}$$

Since (t, s)-sequences in base b are equivalent to $(t, \mathbf{e}, s)$-sequences in base b with $\mathbf{e} = (1, \ldots, 1)$, we can compare the above constants to conclude that the new constant is slightly bigger than $c_s^*(t, s)$ by a factor of at most 2. In the most practical case of $b = 2$, the factor is 1.5 for the new constant, while it is 1.5×2^s for the previous constant of (3). Therefore, our improvement is significant in particular for large dimensions s.

Remark 1. The new paper of Faure-Lemieux [8] uses a different approach from the one in this paper to obtain the same leading constant as $c_s^*(new)$. If we look at their Theorem 2 and its preceding paragraph, we can easily find what is the main difference between their new bound and the bound in (2) of this paper, namely the term $\log_b N - t$ in (2) is replaced by $\log_b N + \sum e_i$ in their result.

As for any odd base b, it is obvious that their new upper bound is always bigger than the upper bound in (2). Even for any even base b, whereas the term b^{e_i} is reduced to $b^{e_i} - 1$, the same situation happens at least if $N < b^{bt}$.

Furthermore, in the paragraph just after the proof of Lemma 7 of their paper, the significance of their new finding "$N \geq 1$" in Theorem 2 is emphasized. However, this paragraph is completely wrong and misleading. First, we should notice that their upper bound in Theorem 2 is bigger than b^t. Second, all of us know the trivial upper bound of the discrepancy, i.e., $D^*(N, X) \leq N$. Combining these two facts, we find that their Theorem 2 is meaningless unless $N > b^t$.

In conclusion, their "new bound" gives almost no improvement compared to the previous bound of (2).

4. Discussions

Generalized Niederreiter sequences [14], [15] include Sobol' sequences, Niederreiter sequences, generalized Faure sequences, polynomial Halton sequences, etc. The generator matrices of this class of sequences are constructed by using rational functions, which consist of numerators and denominators. In the construction, numerators are commonly called direction numbers, and denominators are called base polynomials. When we apply the discrepancy bounds (2) and (4) to generalized Niederreiter sequences, we set the parameter e_i to be the degree of the ith base polynomial for $i = 1, \ldots, s$.

Figure 1 shows numerical results of the leading constants, $2^s c_s^*(new)$, of the unanchored discrepancy for the Sobol' sequence and the Niederreiter sequence in base $b = 2$, up to 360 dimensions. The difference between the two sequences is as follows: the Sobol' sequence uses primitive polynomials over $GF(2)$ for the base polynomials, except for the first base polynomal which is $p(z) = z$. On the other hand, the Niederreiter sequence uses irreducible polynomials for the base polynomials. In both cases, the degrees of the base polynomials are

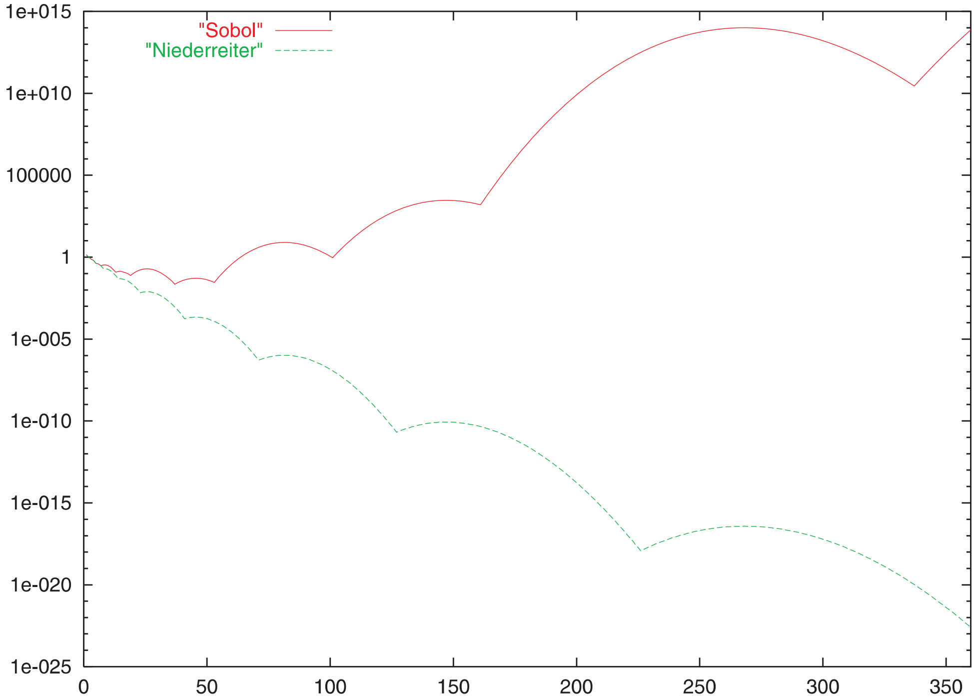

FIGURE 1. The leading constants of the unanchored discrepancy for the Sobol' sequence and the Niederreiter sequence in base $b = 2$, up to 360 dimensions.

sorted in a nondecreasing order. Surprisingly enough, the figure shows that the Sobol' constant looks going to infinity, while the Niederreiter constant looks converging to zero. Although the difference between the primitivity and the irreducibility is small, the constants based on $(t, \mathbf{e}, s)$-sequences clearly distinguish them. We should note that the constants obtained for Sobol' and Niederreiter sequences based on (t, s)-sequences cannot do, because both of them, whichever star or unanchored discrepancy, super-exponentially go to infinity. Further theoretical investigation into this phenomenon will be interesting.

It is well known that direction numbers are very important parameters for obtaining good practical performance in real world applications. As easily seen, discrepancy bounds of generalized Niederreiter sequences based on $(t, \mathbf{e}, s)$--sequences as well as (t, s)-sequences do not contain any information about direction numbers. A general framework of $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences is capable of dealing with such information by an appropriate choice of the set $\mathcal{E}$. For example, getting back to Example 1, if we choose $\mathcal{E} = \{l \mid l \geq 3\}$, the generator G_2 has $\mathbf{T}_{\mathcal{E}} \equiv 0$, while the generator G_1 has $\mathbf{T}_{\mathcal{E}} \not\equiv 0$. In order to employ the signed splitting method for analyzing $(\mathbf{T}_{\mathcal{E}}, \mathcal{E}, s)$-sequences, the set $\mathcal{E}$ must be a direct product, i.e.,

$$\mathcal{E} = \mathcal{E}^{(1)} \times \cdots \times \mathcal{E}^{(s)},$$

with

$$\mathcal{E}^{(i)} = \left\{ l_1^{(i)}, l_2^{(i)}, \ldots \right\}, \qquad 1 \le i \le s,$$

where $0 \le l_1^{(i)} < l_2^{(i)} < \cdots$ are an increasing sequence of integers for $i = 1, \ldots, s$. This direction of research must be interesting.

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