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THE ASYMPTOTIC DISTRIBUTION FUNCTION OF THE 4-DIMENSIONAL SHIFTED VAN DER CORPUT SEQUENCE

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ABSTRACT. Let $\gamma_q(n)$ be the van der Corput sequence in the base q and $g(x,y,z,u)$ be an asymptotic distribution function of the 4-dimensional sequence

$$\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big),\qquad n=1,2,\dots$$

Weyl's limit relation is the equality

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big)$$
$$=\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u g(x,y,z,u).$$

In this paper we find an explicit formula for $g(x,x,x,x)$ and then as an example we find the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\max\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big)=\frac{1}{2}+\frac{3}{q}-\frac{6}{q^2}$$

for the base $q=4,5,6,\dots$ Also we find an explicit form of sth iteration $T^{(s)}(x)$ of the von Neumann-Kakutani transformation defined by $T\big(\gamma_q(n)\big)=\gamma_q(n+1)$.

1. Introduction

Let $q\in\mathbb{N}$. Then every $n\in\mathbb{N}$ has a unique representation of the form

$$n=n_kq^k+n_{k-1}q^{k-1}+\dots+n_1q+n_0,\quad\text{where}\quad n_i\in\{0,1,\dots q-1\}\quad\text{and}\quad n_k>0.\quad(1)$$

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This representation is called q-adic expansion of n. The van der Corput sequence $\gamma_q(n)$, $n=0,1,2,\dots$, is defined as

$$\gamma_q(n) = \frac{n_0}{q} + \frac{n_1}{q^2} + \dots + \frac{n_{k-1}}{q^k} + \frac{n_k}{q^{k+1}}. \tag{2}$$

In this paper we apply Weyl's limit relation (cf. [5, p. 48, Th. 6.1], [7, p. 1–61])

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(\mathbf{x}_n) = \int_{[0,1]^s} F(\mathbf{x})\,\mathrm{d}g(\mathbf{x}), \tag{3}$$

to the sequence $\mathbf{x}_n = \big(\gamma_q(n), \gamma_q(n+1), \dots, \gamma_q(n+s-1)\big)$, where $g(\mathbf{x})$ is the asymptotic distribution function (abbreviated a.d.f.) of $\mathbf{x}_n$ and for $s=4$. The case $s = 3$ is discussed in [2]. In this paper we shall extend the method in [2].[1] The paper consists of the following parts: In Section 2 we derive 2-dimensional intervals containing the sequence $\big(\gamma_q(n), \gamma_q(n+3)\big)$, $n = 1, 2, \dots$ on diagonals. In Section 3 we find 4-dimensional maximal intervals containing $\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\big)$, $n = 0, 1, 2, \dots$ on diagonals. Using this it can be computed $g(x, y, z, u)$ but after discusion of 4^4 different cases. In this paper we explicitly found only $g(x, x, x, x)$. Using this in Section 4 we compute Weyl's limit relation (3) for $F(x, y, z, u) = \max(x, y, z, u)$. We also discuss iterations of von Neumann-Kakutani transformation in Section 5. The final Section 6 contains another method of computing (3).

2. A.d.f. of the sequence $\big(\gamma_q(n), \gamma_q(n+3)\big)$, $n = 1, 2, \dots$

In order to compute the a.d.f. of the sequence $\big(\gamma_q(n), \gamma_q(n+3)\big)$, $n = 1, 2, \dots$ we investigate the following four cases:

$$1)\ n_0 < q-3, \qquad 2)\ n_0 = q-3,$$
$$3)\ n_0 = q-2, \qquad 4)\ n_0 = q-1.$$

We will start with

1) Let $n_0 < q - 3$. Then

$$n = n_k q^k + \dots + n_0,$$
$$n + 3 = n_k q^k + \dots + n_0 + 3,$$
$$\gamma_q(n) = \frac{n_0}{q} + \dots + \frac{n_k}{q^{k+1}} \le \frac{q-4}{q} + \frac{q-1}{q^2} + \frac{q-1}{q^3} + \dots = \frac{q-3}{q},$$
$$\gamma_q(n+3) = \frac{n_0+3}{q} + \dots + \frac{n_k}{q^{k+1}}.$$

[1] See also 1.12 Unsolved Problem in [4] for an exhaustive description of the problem.

Thus
$$\gamma_q(n+3)-\gamma_q(n)=\frac{3}{q},$$
and thus the points $\big(\gamma_q(n),\gamma_q(n+3)\big)$ for which n satisfies 1, lie on the line segment
$$Z=X+\frac{3}{q},\quad X\in\left[0,1-\frac{3}{q}\right] \text{ being the diagonal of } \left[0,1-\frac{3}{q}\right]\times\left[\frac{3}{q},1\right]. \tag{4}$$

2) Let $n_0=q-3$. Then

$$n=n_kq^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-1)q+q-3,$$

where $n_{i+1}<q-1$ and $i=0,1,2,\dots$,

$$n+3=n_kq^k+\cdots+(n_{i+1}+1)q^{i+1}+0q^i+0q^{i-1}+\cdots+0q+0,$$

$$\gamma_q(n)=\frac{q-3}{q}+\frac{q-1}{q^2}+\cdots+\frac{q-1}{q^i}+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+3)=\frac{0}{q}+\frac{0}{q^2}+\cdots+\frac{0}{q^i}+\frac{0}{q^{i+1}}+\frac{n_{i+1}+1}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+3)-\gamma_q(n)=\frac{1}{q^{i+2}}+\frac{2}{q}-1+\frac{1}{q^{i+1}},$$

$$-\frac{2}{q}+1-\frac{1}{q^{i+1}}=\frac{q-3}{q}+\frac{q-1}{q^2}+\cdots+\frac{q-1}{q^i}+\frac{q-1}{q^{i+1}}\le\gamma_q(n),$$

$$\gamma_q(n)\le\frac{q-3}{q}+\frac{q-1}{q^2}+\cdots+\frac{q-1}{q^i}+\frac{q-1}{q^{i+1}}+\frac{q-2}{q^{i+2}}+\cdots=1-\frac{2}{q}-\frac{1}{q^{i+2}}$$

and the points of the sequence $\big(\gamma_q(n),\gamma_q(n+3)\big)$ for which n satisfies 2) lie on the line segment

$$Z=X+\frac{1}{q^{i+2}}+\frac{2}{q}-1+\frac{1}{q^{i+1}},$$

$$X\in\left[1-\frac{2}{q}-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}-\frac{2}{q}\right] \text{ being the diagonal of}$$

$$\left[1-\frac{2}{q}-\frac{1}{q^{i+1}},1-\frac{1}{q^{i+2}}-\frac{2}{q}\right]\times\left[\frac{1}{q^{i+2}},\frac{1}{q^{i+1}}\right],\quad i=0,1,2,\dots \tag{5}$$

3) Let $n_0=q-2$. Then

$$n=n_kq^k+\cdots+n_{i+1}q^{i+1}+(q-1)q^i+(q-1)q^{i-1}+\cdots+(q-1)q+q-2,$$

where $n_{i+1}<q-1$ and $i=0,1,2,\dots$,

$$n+3=n_kq^k+\cdots+(n_{i+1}+1)q^{i+1}+0q^i+0q^{i-1}+\cdots+0q+1,$$

$$\gamma_q(n)=\frac{q-2}{q}+\frac{q-1}{q^2}+\cdots+\frac{q-1}{q^i}+\frac{q-1}{q^{i+1}}+\frac{n_{i+1}}{q^{i+2}}+\cdots+\frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+3) = \frac{1}{q} + \frac{0}{q^2} + \dots + \frac{0}{q^i} + \frac{0}{q^{i+1}} + \frac{n_{i+1}+1}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+3) - \gamma_q(n) = \frac{1}{q^{i+2}} + \frac{2}{q} - 1 + \frac{1}{q^{i+1}},$$

$$-\frac{1}{q} + 1 - \frac{1}{q^{i+1}} = \frac{q-2}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} \le \gamma_q(n),$$

$$\gamma_q(n) \le \frac{q-2}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \dots = 1 - \frac{1}{q} - \frac{1}{q^{i+2}}$$

and points $\big(\gamma_q(n), \gamma_q(n+3)\big)$, for n satisfying 3), lie on

$$Z = X + \frac{1}{q^{i+2}} + \frac{2}{q} - 1 + \frac{1}{q^{i+1}},$$

$X \in \left[1 - \frac{1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}} - \frac{1}{q}\right]$ being the diagonal of

$$\left[1 - \frac{1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}} - \frac{1}{q}\right] \times \left[\frac{1}{q^{i+2}} + \frac{1}{q}, \frac{1}{q^{i+1}} + \frac{1}{q}\right], \quad i = 0, 1, 2, \dots \qquad (6)$$

4) Let $n_0 = q - 1$. Then

$$n = n_k q^k + \dots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \dots + (q-1)q + q - 1,$$

where $n_{i+1} < q - 1$ and $i = 0, 1, 2, \dots,$

$$n + 3 = n_k q^k + \dots + (n_{i+1} + 1)q^{i+1} + 0q^i + 0q^{i-1} + \dots + 0q + 2,$$

$$\gamma_q(n) = \frac{q-1}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+3) = \frac{2}{q} + \frac{0}{q^2} + \dots + \frac{0}{q^i} + \frac{0}{q^{i+1}} + \frac{n_{i+1}+1}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+3) - \gamma_q(n) = \frac{1}{q^{i+2}} + \frac{2}{q} - 1 + \frac{1}{q^{i+1}},$$

$$1 - \frac{1}{q^{i+1}} = \frac{q-1}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} + \frac{0}{q^{i+2}} + \dots + \frac{0}{q^k} \le \gamma_q(n),$$

$$\gamma_q(n) \le \frac{q-1}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \dots = 1 - \frac{1}{q^{i+2}}$$

and points $\big(\gamma_q(n), \gamma_q(n+3)\big)$ (index n satisfies 4)) lie in

$$Z = X + \frac{1}{q^{i+2}} + \frac{2}{q} - 1 + \frac{1}{q^{i+1}},$$

$X \in \left[1 - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}}\right]$ forming the diagonal of

$$\left[1 - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}}\right] \times \left[\frac{2}{q} + \frac{1}{q^{i+2}}, \frac{2}{q} + \frac{1}{q^{i+1}}\right], \quad i = 0, 1, 2, \dots \qquad (7)$$

In the following in (5), (6) and (7) we shall reduce $(i+1) \to i$ and $i = 1, 2, \dots$
Summary:

Lemma 1. *All points* $\big(\gamma_q(n), \gamma_q(n+3)\big)$, $n = 1, 2, \dots$ *lie on the diagonals of intervals*

$$I = \left[0, 1 - \frac{3}{q}\right] \times \left[\frac{3}{q}, 1\right], \tag{8}$$

$$I^{(i)} = \left[1 - \frac{2}{q} - \frac{1}{q^i}, 1 - \frac{2}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], \qquad i = 1, 2, \dots \tag{9}$$

$$J^{(j)} = \left[1 - \frac{1}{q} - \frac{1}{q^j}, 1 - \frac{1}{q} - \frac{1}{q^{j+1}}\right] \times \left[\frac{1}{q} + \frac{1}{q^{j+1}}, \frac{1}{q} + \frac{1}{q^j}\right], \quad j = 1, 2, \dots \tag{10}$$

$$K^{(k)} = \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right] \times \left[\frac{2}{q} + \frac{1}{q^{k+1}}, \frac{2}{q} + \frac{1}{q^k}\right], \qquad k = 1, 2, \dots \tag{11}$$

2.1. von Neumann-Kakutani transformation

The continuous map $T : [0, 1] \to [0, 1]$ for which $T\big(\gamma_q(n)\big) = \gamma_q(n+1)$ is called the von Neumann-Kakutani transformation. It is known that (e.g., see [2])

$$T(x) = \begin{cases} x + \frac{1}{q} & \text{if } x \in [0, 1 - \frac{1}{q}], \\ x - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}} & \text{if } x \in [1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}], \qquad i = 1, 2, \dots \end{cases} \tag{12}$$

From (4), (5), (6) and (7) follows the third iteration

$$T^3(x) = \begin{cases} x + \frac{3}{q} & \text{if } x \in \big[0, 1 - \frac{3}{q}\big], \\ x + \frac{2}{q} - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}} & \text{if } x \in \big[1 - \frac{2}{q} - \frac{1}{q^i}, 1 - \frac{2}{q} - \frac{1}{q^{i+1}}\big], \\ & \quad \cup \big[1 - \frac{1}{q} - \frac{1}{q^i}, 1 - \frac{1}{q} - \frac{1}{q^{i+1}}\big], \\ & \quad \cup \big[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\big], \qquad i = 1, 2, \dots \end{cases} \tag{13}$$

In Section 5 there is also given the expression $T^s(x)$ for general s.

3. A.d.f. of $\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\big)$, $n = 0, 1, 2, \dots$

In this part we find 4-dimensional maximal intervals on axes (X, Y, Z, U) containing the sequence $\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\big)$, $n = 0, 1, 2, \dots$ on diagonals. We will start with 2-dimensional intervals on $(X, Y), (Y, Z), (Z, U)$ axes, respectively, containing $\big(\gamma_q(n), \gamma_q(n+1)\big)$, $n = 0, 1, 2, \dots$, on diagonals. By [2] they have intervals of the form

$$\left[0, 1 - \frac{1}{q}\right] \times \left[\frac{1}{q}, 1\right];$$

$$\left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], \qquad i = 1, 2, \dots$$

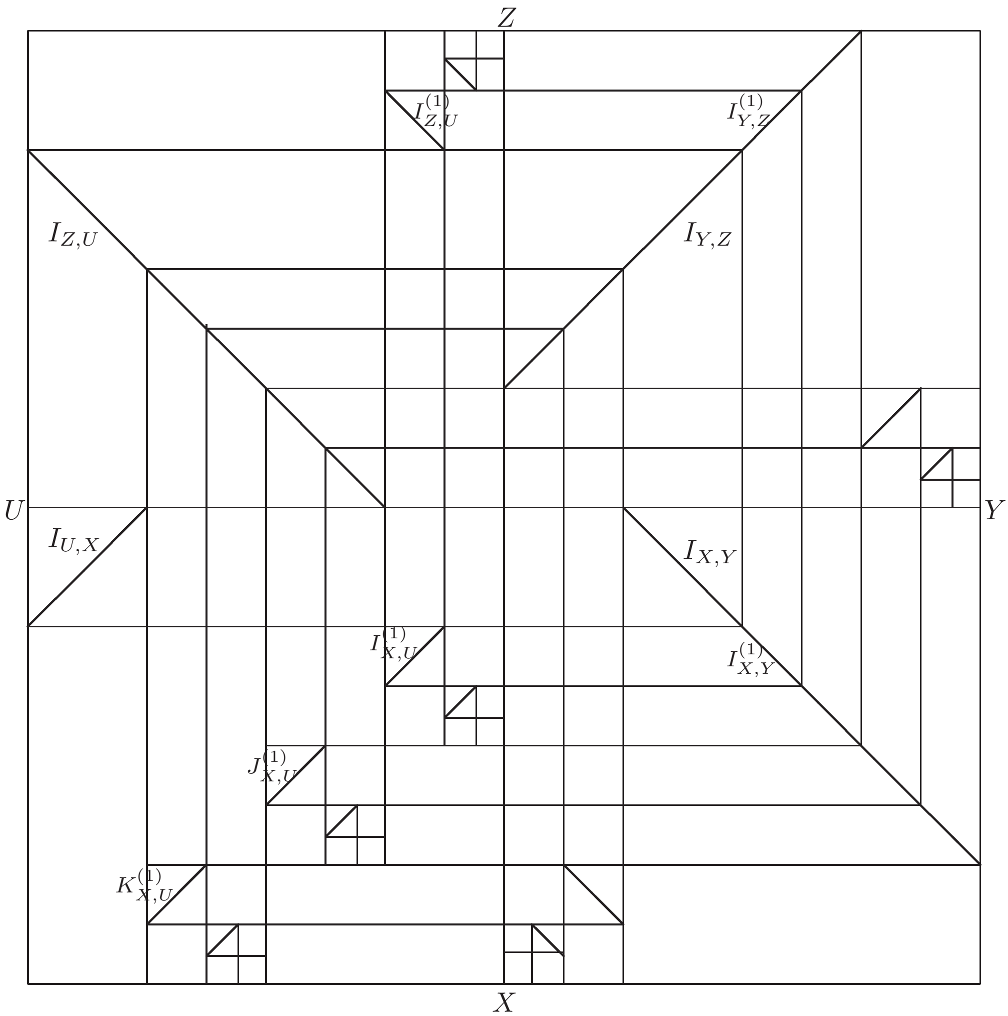

Figure 1.

By Lemma 1, put I, $I^{(i)}$, $i = 1, 2, \dots$, $J^{(j)}$, $j = 1, 2, \dots$, $K^{(k)}$, $k = 1, 2, \dots$, on (X, U) axes we have the maximal intervals containing $\big(\gamma_q(n), \gamma_q(n+3)\big)$. All these intervals are plotted in Fig. 1 on the page 80. Collecting intervals of equal length we find that

Theorem 1. *The maximal* 4*-dimensional intervals containing points*

$$\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)\big), \qquad n = 0, 1, 2, \dots$$

on diagonals are

$$I = \left[0, 1-\frac{3}{q}\right] \times \left[\frac{1}{q}, 1-\frac{2}{q}\right] \times \left[\frac{2}{q}, 1-\frac{1}{q}\right] \times \left[\frac{3}{q}, 1\right], \tag{14}$$

$$I^{(i)} = \left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right] \times \left[\frac{1}{q}+\frac{1}{q^{i+1}}, \frac{1}{q}+\frac{1}{q^i}\right] \times \left[\frac{2}{q}+\frac{1}{q^{i+1}}, \frac{2}{q}+\frac{1}{q^i}\right], \qquad i = 1, 2, \dots \tag{15}$$

$$J^{(j)} = \left[1-\frac{2}{q}-\frac{1}{q^j}, 1-\frac{2}{q}-\frac{1}{q^{j+1}}\right] \times \left[1-\frac{1}{q}-\frac{1}{q^j}, 1-\frac{1}{q}-\frac{1}{q^{j+1}}\right] \times \left[1-\frac{1}{q^j}, 1-\frac{1}{q^{j+1}}\right] \times \left[\frac{1}{q^{j+1}}, \frac{1}{q^j}\right], \quad j = 1, 2, \dots \tag{16}$$

$$K^{(k)} = \left[1-\frac{1}{q}-\frac{1}{q^k}, 1-\frac{1}{q}-\frac{1}{q^{k+1}}\right] \times \left[1-\frac{1}{q^k}, 1-\frac{1}{q^{k+1}}\right] \times \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right] \times \left[\frac{1}{q}+\frac{1}{q^{k+1}}, \frac{1}{q}+\frac{1}{q^k}\right], \qquad k = 1, 2, \dots \tag{17}$$

Now, let D be a union of diagonals of (14), (15), (16) and (17). Then

$$g(x,y,z,u) = \left|\mathrm{Project}_X\big([0,x]\times[0,y]\times[0,z]\times[0,u]\cap D\big)\right| \tag{18}$$

and it can be rewritten as

$$\begin{aligned} &g(x,y,z,u)\\ &= \min\big(|[0,x]\cap I_X|, |[0,y]\cap I_Y|, |[0,z]\cap I_Z|, |[0,u]\cap I_U|\big)\\ &\quad + \sum_{i=1}^{\infty} \min\big(|[0,x]\cap I_X^{(i)}|, |[0,y]\cap I_Y^{(i)}|, |[0,z]\cap I_Z^{(i)}|, |[0,u]\cap I_U^{(i)}|\big)\\ &\quad + \sum_{j=1}^{\infty} \min\big(|[0,x]\cap J_X^{(j)}|, |[0,y]\cap J_Y^{(j)}|, |[0,z]\cap J_Z^{(j)}|, |[0,u]\cap J_U^{(j)}|\big)\\ &\quad + \sum_{k=1}^{\infty} \min\big(|[0,x]\cap K_X^{(k)}|, |[0,y]\cap K_Y^{(k)}|, |[0,z]\cap K_Z^{(k)}|, |[0,u]\cap K_U^{(k)}|\big)\\ &= g_1(x,y,z,u) + g_2(x,y,z,u) + g_3(x,y,z,u) + g_4(x,y,z,u), \end{aligned} \tag{19}$$

respectively. To calculate (19), as a guide, we use the following Fig. 2 (here $q = 4$) for $x = y = z = u$.

Assume that $q \geq 4$. Then by Fig. 2

$$g_1(x,x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{3}{q}\right], \\ x - \frac{3}{q} & \text{if } x \in \left[\frac{3}{q}, 1\right], \end{cases} \tag{20}$$

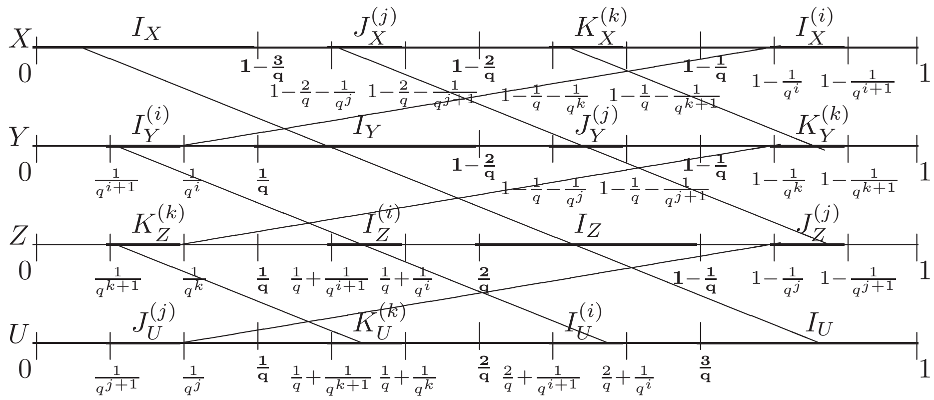

Figure 2: Projections of intervals $I, I^{(i)}, J^{(j)}, K^{(k)}$ on axes X, Y, Z, U.

and

$$g_2(x,x,x,x) = \begin{cases} 0 & \text{if } x \in \big[0, 1-\frac{1}{q}\big], \\ x - \big(1-\frac{1}{q}\big) & \text{if } x \in I_X^{(1)}, \\ x - \big(1-\frac{1}{q^2}\big) + |I_X^{(1)}| & \text{if } x \in I_X^{(2)}, \dots, \\ x - \big(1-\frac{1}{q^i}\big) + |I_X^{(1)}| + \dots + |I_X^{(i-1)}| & \text{if } x \in I_X^{(i)}, \dots \end{cases}$$

Since

$$x - \left(1 - \frac{1}{q^i}\right) + |I_X^{(1)}| + \dots + |I_X^{(i-1)}| = x - 1 + \frac{1}{q}$$

we have

$$g_2(x,x,x,x) = \begin{cases} 0 & \text{if } x \in \big[0, 1-\frac{1}{q}\big], \\ x - 1 + \frac{1}{q} & \text{if } x \in \big[1-\frac{1}{q}, 1\big]. \end{cases} \tag{21}$$

As (21) similarly holds for $g_3(x,x,x,x)$ and $g_4(x,x,x,x)$ and summing up this we have

$$g(x,x,x,x) = \begin{cases} 0 & \text{if } x \in \big[0, \frac{3}{q}\big], \\ x - \frac{3}{q} & \text{if } x \in \big[\frac{3}{q}, 1-\frac{1}{q}\big], \\ 4x - 3 & \text{if } x \in \big[1-\frac{1}{q}, 1\big]. \end{cases} \tag{22}$$

for $q \geq 4$.

Remark 1. Let $g(x,y,z)$ be the asymptotic distribution function of 3-dimensional sequence $\big(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)\big)$, $n = 0, 1, 2, \dots$ In [2] there is proved

$$g(x,x,x) = \begin{cases} 0 & \text{if } x \in \big[0, \frac{2}{q}\big], \\ x - \frac{2}{q} & \text{if } x \in \big[\frac{2}{q}, 1-\frac{1}{q}\big], \\ 3x - 2 & \text{if } x \in \big[1-\frac{1}{q}, 1\big] \end{cases} \tag{23}$$

for $q \geq 3$. As a control it can be proved that $g(x,x,x,1) = g(x,x,x)$.

4. Applications

In this part we apply Weyl's limit relation

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}F\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big)$$
$$=\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_x\,\mathrm{d}_u g(x,y,z,u), \tag{24}$$

to find the arithmetic means in the left-hand side of (24). In the right-hand side of (24) we apply integration by parts.

Assume that $F(x,y,z,u)$ is continuous on $[0,1]^4$ and $g(x,y,z,u)$ is a d.f. Then

$$\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u g(x,y,z,u) = F(1,1,1,1)$$
$$-\int_0^1 g(x,1,1,1)\,\mathrm{d}_x F(x,1,1,1) - \int_0^1 g(1,y,1,1)\,\mathrm{d}_y F(1,y,1,1)$$
$$-\int_0^1 g(1,1,z,1)\,\mathrm{d}_z F(1,1,z,1) - \int_0^1 g(1,1,1,u)\,\mathrm{d}_u F(1,1,1,u)$$
$$+\int_0^1\int_0^1 g(x,y,1,1)\,\mathrm{d}_x\,\mathrm{d}_y F(x,y,1,1) + \int_0^1\int_0^1 g(x,1,z,1)\,\mathrm{d}_x\,\mathrm{d}_z F(x,1,z,1)$$
$$+\int_0^1\int_0^1 g(1,y,z,1)\,\mathrm{d}_y\,\mathrm{d}_z F(1,y,z,1) + \int_0^1\int_0^1 g(x,1,1,u)\,\mathrm{d}_x\,\mathrm{d}_u F(x,1,1,u)$$
$$+\int_0^1\int_0^1 g(1,y,1,u)\,\mathrm{d}_y\,\mathrm{d}_u F(1,y,1,u) + \int_0^1\int_0^1 g(1,1,z,u)\,\mathrm{d}_z\,\mathrm{d}_u F(1,1,z,u)$$
$$-\int_0^1\int_0^1\int_0^1 g(x,y,z,1)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z F(x,y,z,1) - \int_0^1\int_0^1\int_0^1 g(1,y,z,u)\,\mathrm{d}_u\,\mathrm{d}_y\,\mathrm{d}_z F(1,y,z,u)$$
$$-\int_0^1\int_0^1\int_0^1 g(x,1,z,u)\,\mathrm{d}_x\,\mathrm{d}_z\,\mathrm{d}_u F(x,1,z,u) - \int_0^1\int_0^1\int_0^1 g(x,y,1,u)\,\mathrm{d}_u\,\mathrm{d}_y\,\mathrm{d}_x F(x,y,1,u)$$
$$+\int_0^1\int_0^1\int_0^1\int_0^1 g(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u F(x,y,z,u).$$

EXAMPLE 1. Put $F(x,y,z,u) = \max(x,y,z,u)$. Then

$$\mathrm{d}_x F(x,1,1,1) = \mathrm{d}_y F(1,y,1,1) = \mathrm{d}_z F(1,1,z,1) = \mathrm{d}_u F(1,1,1,u) = 0,$$

$$\begin{aligned}\mathrm{d}_x\,\mathrm{d}_y F(x,y,1,1) &= \mathrm{d}_x\,\mathrm{d}_z F(x,1,z,1) = \mathrm{d}_y\,\mathrm{d}_z F(1,y,z,1)\\ &= \mathrm{d}_x\,\mathrm{d}_u F(x,1,1,u) = \mathrm{d}_y\,\mathrm{d}_u F(1,y,1,u) = \mathrm{d}_z\,\mathrm{d}_u F(1,1,z,u) = 0,\end{aligned}$$

$$\begin{aligned}\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z F(x,y,z,1) &= \mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_u F(x,y,1,u)\\ &= \mathrm{d}_x\,\mathrm{d}_z\,\mathrm{d}_u F(x,1,z,u) = \mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u F(1,y,z,u) = 0.\end{aligned}$$

The differential $\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u F(x,y,z,u)$ is non-zero if and only if

$$x = y = z = u$$

and in this case

$$\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u F(x,y,z,u) = -\,\mathrm{d}x\,.$$

Proof. See [7, p. 1–61]: For every interval $J = [x_1^{(1)}, x_2^{(1)}] \times [x_1^{(2)}, x_2^{(2)}] \times \cdots \cdots \times [x_1^{(s)}, x_2^{(s)}] \subset [0,1]^s$ and every continuous $F(x_1, x_2, \dots, x_s)$ the differential $\Delta(F, J)$ is defined as

$$\Delta(F,J) = \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} F\left(x_{\varepsilon_1}^{(1)}, \dots, x_{\varepsilon_s}^{(s)}\right). \tag{25}$$

Putting $F(x_1, x_2, \dots, x_s) = \max(x_1, x_2, \dots, x_s)$, $x_1^{(i)} = x$, $x_2^{(i)} = x + \mathrm{d}x$, we have

$$\begin{aligned}\Delta(F,J) &= (-1)^{1+1+\cdots+1} x + \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} (x + \mathrm{d}x)\\ &= \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_s=1}^{2} (-1)^{\varepsilon_1+\cdots+\varepsilon_s} (x + \mathrm{d}x) - (-1)^{1+1+\cdots+1}\,\mathrm{d}x = (-1)^{s+1}\,\mathrm{d}x\,.\end{aligned}$$

Then

$$\begin{aligned}&\int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 F(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u g(x,y,z,u)\\ &\qquad = 1 + \int_0^1\!\!\int_0^1\!\!\int_0^1\!\!\int_0^1 g(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u F(x,y,z,u)\\ &\qquad = 1 - \int_0^1 g(x,x,x,x)\,\mathrm{d}x\,.\end{aligned} \tag{26}$$

For $q \geq 4$ and by (22) we have

$$\int_0^1 g(x,x,x,x)\,\mathrm{d}x = \int_{\frac{3}{q}}^{1-\frac{1}{q}} \left(x - \frac{3}{q}\right)\mathrm{d}x + \int_{1-\frac{1}{q}}^{1} (4x-3)\,\mathrm{d}x = \frac{1}{2} - \frac{3}{q} + \frac{6}{q^2}\,.$$

□

5. sth iteration of von Neumann-Kakutani transformation

In this part we study distribution of the sequence

$$\big(\gamma_q(n), \gamma_q(n+s)\big), \qquad n = 0, 1, 2, \dots,$$

where q is an integer, $q \geq s$. Let $n = n_k q^k + n_{k-1} q^{k-1} + \cdots + n_1 q + n_0$. In a simillar way as in Section 2 we investigate the following cases:

1) $n_0 < q - s$,
2) $n_0 = q - s$,
3) $n_0 = q - s + 1$,
...
l) $n_0 = q - s + l - 2$,
...
$(2 + s - 1)$ $n_0 = q - 1$.

In the first case 1) $n_0 < q - s$ we have

$$n = n_k q^k + \cdots + n_0,$$

$$n + s = n_k q^k + \cdots + n_0 + s,$$

$$\gamma_q(n) = \frac{n_0}{q} + \cdots + \frac{n_k}{q^{k+1}} \leq \frac{q-s-1}{q} + \frac{q-1}{q^2} + \frac{q-1}{q^3} \cdots$$

$$\cdots = \frac{-s}{q} + \frac{q-1}{q}\frac{1}{1-\frac{1}{q}} = \frac{q-s}{q},$$

$$\gamma_q(n+s) = \frac{n_0+s}{q} + \cdots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+s) - \gamma_q(n) = \frac{s}{q}.$$

Then the point $\big(\gamma_q(n), \gamma_q(n+s)\big)$ lies on the line segment

$$Z = X + \frac{s}{q}, \quad \text{where} \quad X \in \left[0, 1 - \frac{s}{q}\right]$$

and on the diagonal

$$\left[0, 1 - \frac{s}{q}\right] \times \left[\frac{s}{q}, 1\right]$$

In the general case l,

$$n_0 = q - s + l - 2, \quad l = 2, 3, \dots, 2 + s - 1$$

we have

$$n = n_k q^k + \dots + n_{i+1} q^{i+1} + (q-1)q^i + (q-1)q^{i-1} + \dots + (q-1)q + q - s + l - 2,$$

where $n_{i+1} < q - 1$ and $i = 0, 1, 2, \dots,$

$$n + s = n_k q^k + \dots + (n_{i+1} + 1)q^{i+1} + 0q^i + 0q^{i-1} + \dots + 0q + l - 2,$$

$$\gamma_q(n) = \frac{q-s+l-2}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^i} + \frac{q-1}{q^{i+1}} + \frac{n_{i+1}}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+s) = \frac{l-2}{q} + \frac{0}{q^2} + \dots + \frac{0}{q^i} + \frac{0}{q^{i+1}} + \frac{n_{i+1}+1}{q^{i+2}} + \dots + \frac{n_k}{q^{k+1}},$$

$$\gamma_q(n+s) - \gamma_q(n) = \frac{s-1}{q} + \frac{1}{q^{i+2}} + \frac{1}{q^{i+1}} - 1,$$

$$1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}} \leq \frac{q-s+l-2}{q} + \frac{q-1}{q^2} + \dots + \frac{q-1}{q^{i+1}} + \frac{0}{q^{i+2}} + \dots + \frac{1}{q^k} \leq \gamma_q(n),$$

$$\gamma_q(n) \leq \frac{q-s+l-2}{q} + \frac{q-1}{q^2} + \dots$$
$$\dots + \frac{q-1}{q^{i+1}} + \frac{q-2}{q^{i+2}} + \frac{q-1}{q^{i+3}} + \dots = 1 - \frac{s-l+1}{q} - \frac{1}{q^{i+2}}.$$

Thus, if n satisfies the case l, then the point $\big(\gamma_q(n), \gamma_q(n+s)\big)$ lies on the line segment

$$Z = X + \frac{s-1}{q} - 1 + \frac{1}{q^{i+1}} + \frac{1}{q^{i+2}},$$
$$X \in \left[1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}} - \frac{s-l+1}{q}\right] \tag{27}$$

and on the diagonal of

$$\left[1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}}, 1 - \frac{1}{q^{i+2}} - \frac{s-l+1}{q}\right]$$
$$\times \left[\frac{l-2}{q} + \frac{1}{q^{i+2}}, \frac{l-2}{q} + \frac{1}{q^{i+1}}\right], \tag{28}$$

where $i = 0, 1, 2, \dots$. In the following we shall reduce $(i+1) \to i$ and (27) gives sth iteration of von Neumann-Kakutani transformation T

$$T^s(x) = \begin{cases} x + \frac{s}{q} & \text{if } x \in \left[0, 1 - \frac{s}{q}\right], \\ x + \frac{s-1}{q} - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}} & \text{if } x \in \left[1 - \frac{s-1}{q} - \frac{1}{q^i}, 1 - \frac{s-1}{q} - \frac{1}{q^{i+1}}\right] \\ & \quad \cup \left[1 - \frac{s-2}{q} - \frac{1}{q^i}, 1 - \frac{s-2}{q} - \frac{1}{q^{i+1}}\right] \\ & \quad \dots \\ & \quad \cup \left[1 - \frac{s-l+1}{q} - \frac{1}{q^i}, 1 - \frac{s-l-1}{q} - \frac{1}{q^{i+1}}\right] \\ & \quad \dots \\ & \quad \cup \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right], \text{ where } i = 1, 2, \dots \end{cases} \tag{29}$$

Also the points $\big(\gamma_q(n), \gamma_q(n+s)\big)$, $n = 0, 1, 2, \dots$, are contained on diagonals of

$$\begin{aligned} I_0 &= \left[0, 1 - \frac{s}{q}\right] \times \left[\frac{s}{q}, 1\right], \\ I_1^{(i)} &= \left[1 - \frac{s-1}{q} - \frac{1}{q^i}, 1 - \frac{s-1}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right], & i = 1, 2, \dots, \\ I_2^{(i)} &= \left[1 - \frac{s-2}{q} - \frac{1}{q^i}, 1 - \frac{s-2}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q} + \frac{1}{q^{i+1}}, \frac{1}{q} + \frac{1}{q^i}\right], & i = 1, 2, \dots, \\ I_3^{(i)} &= \left[1 - \frac{s-3}{q} - \frac{1}{q^i}, 1 - \frac{s-3}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{2}{q} + \frac{1}{q^{i+1}}, \frac{2}{q} + \frac{1}{q^i}\right], & i = 1, 2, \dots, \\ I_4^{(i)} &= \left[1 - \frac{s-4}{q} - \frac{1}{q^i}, 1 - \frac{s-4}{q} - \frac{1}{q^{i+1}}\right] \times \left[\frac{3}{q} + \frac{1}{q^{i+1}}, \frac{3}{q} + \frac{1}{q^i}\right], & i = 1, 2, \dots, \\ &\dots \\ I_{l-1}^{(i)} &= \left[1 - \frac{s-l+1}{q} - \frac{1}{q^i}, 1 - \frac{s-l+1}{q} - \frac{1}{q^{i+1}}\right] \\ &\qquad \times \left[\frac{l-2}{q} + \frac{1}{q^{i+1}}, \frac{l-2}{q} + \frac{1}{q^i}\right], & i = 1, 2, \dots, \\ &\dots \\ I_s^{(i)} &= \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right] \times \left[\frac{s-1}{q} + \frac{1}{q^{i+1}}, \frac{s-1}{q} + \frac{1}{q^i}\right], & i = 1, 2, \dots \end{aligned}$$

6. Concluding remarks

Finding the a.d.f. of the s-dimensional sequence

$$\big(\gamma_q(n), \dots, \gamma_q(n+s-1)\big), \qquad n = 0, 1, 2, \dots, \tag{30}$$

is *Open Problem* 1.12 in [4, p. 141]. Formal solution is given by Ch. Aisleitner and M. Hofer [1]: Let T denote von Neuman-Kakutani transformation. Define an s-dimensional curve $\{\gamma(t); t \in [0, 1)\}$, where

$$\gamma(t) = \big(t, T(t), T^2(t), \dots, T^{s-1}(t)\big).$$

Then the a.d.f. (30) is

$$g(x_1, x_2, \dots, x_s) = \big|\{t \in [0, 1]; \gamma(t) \in [0, x_1] \times [0, x_2] \times \dots \times [0, x_s]\}\big|,$$

where $|X|$ is the Lebesgue measure of set X. Explicit formulas of such a.d.f.s are known for $s=2$ in [3], $s=3$ in [2] and $s=4$ in Theorem 1.

Furthermore, for an arbitrary continuous $F(x_1,x_2,\dots,x_s)$ we have

$$\int_{[0,1]^s} F(x_1,x_2,\dots,x_s)\,\mathrm{d}g(x_1,x_2,\dots,x_s) = \int_0^1 F\big(x,T(x),T^2(x),\dots,T^{s-1}(x)\big)\,\mathrm{d}x$$

since by Weyl's limit relation

$$\int_{[0,1]^s} F(x_1,x_2,\dots,x_s)\,\mathrm{d}g(x_1,x_2,\dots,x_s)$$
$$= \lim_{N\to\infty}\sum_{n=0}^{N-1} F\Big(\gamma_q(n),T\big(\gamma_q(n)\big),T^2\big(\gamma_q(n)\big),\dots,T^{s-1}\big(\gamma_q(n)\big)\Big). \quad (31)$$

The main aim of this paper is to found an explicit formula of $g(x,y,z,u)$. To do this we have found (Theorem 1) all intervals containing

$$\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big)$$

on diagonals. The second aim is to calculate the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} F\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\gamma_q(n+3)\big)$$

as the integral

$$\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,u)\,\mathrm{d}_x\,\mathrm{d}_y\,\mathrm{d}_z\,\mathrm{d}_u g(x,y,z,u).$$

Another method. An anonymous referee has sent to the authors the following method of computing (31). Denote

$$N(a,b,t)=\big\{n=a+(q-1)(q+q^2+\dots+q^{t-1})+bq^t+mq^{t+1};\ m=0,1,2,\dots\big\}. \quad (32)$$

If $n\in N(a,b,t)$ and $0\le a<q-1$, $0\le b<q-1$, $t\ge 1$, then

$$\gamma_q(n)=\frac{a}{q}+\frac{1}{q}\left(1-\frac{1}{q^{t-1}}\right)+\frac{b+\gamma_q(m)}{q^{t+1}}. \quad (33)$$

Then subsequence $\gamma(n)$, $n\in N(a,b,t)$ decompose van der Corput sequence $\gamma(n)$, $n=0,1,2,\dots$ From (33) follows

$$\gamma_q(n+j)=\frac{a+j}{q}+\frac{1}{q}\left(1-\frac{1}{q^{t-1}}\right)+\frac{b+\gamma_q(m)}{q^{t+1}}. \quad (34)$$

if $j<q-a$ and

$$\gamma_q(n+j)=\frac{a+j-q}{q}+\frac{b+1+\gamma_q(m)}{q^{t+1}} \quad (35)$$

if $j \geq q-a$ and $a+j-q<q$. Thus in s-dimensional case for $n \in N(a,b,t)$ there exists s-dimensional function $f_q\big(a,b,t,\gamma_q(m)\big)$ such that

$$\big(\gamma_q(n),\gamma_q(n+1),\dots,\gamma_q(n+s-1)\big) = f_q\big(a,b,t,\gamma_q(m)\big). \tag{36}$$

Thus

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} F\big(\gamma_q(n),\gamma_q(n+1),\gamma_q(n+2),\dots,\gamma_q(n+s-1)\big)\\
&=\sum_{a=0}^{q-1}\sum_{b=0}^{q-2}\sum_{t=1}^{\infty}\lim_{N\to\infty}\frac{M}{N}\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}F\Big(f_q\big(a,b,t,\gamma_q(m)\big)\Big)\\
&=\sum_{a=0}^{q-1}\sum_{b=0}^{q-2}\sum_{t=1}^{\infty}\frac{1}{q^{t+1}}\int_0^1 F(f_q(a,b,t,x))\,\mathrm{d}x.
\end{aligned} \tag{37}$$

Comparison. In the following we shall compare the method via (33) (denoting 1^0.) with our method via intervals (denoting 2^0), to compute, e.g., the a.d.f. $g(x,x)$ of $\big(\gamma_q(n),\gamma_q(n+1)\big)$, $n=0,1,2,\dots$ Our method [3] gives

$$g(x,x)=\begin{cases}0 & \text{if } x\in\big[0,\frac{1}{q}\big],\\ x-\frac{1}{q} & \text{if } x\in\big[\frac{1}{q},1-\frac{1}{q}\big],\\ 2x-1 & \text{if } x\in\big[1-\frac{1}{q},1\big].\end{cases} \tag{38}$$

1^0. Using (33) for $n\in N(a,b,t)$,

$$\begin{aligned} a&=0,1,2,\dots,q-2,\\ b&=0,1,2,\dots,q-2,\\ t&=1,2,\dots\end{aligned}$$

we have

$$\begin{aligned}&\big(\gamma_q(n),\gamma_q(n+1)\big)\\ &=\left(\frac{a}{q}+\frac{1}{q}\left(1-\frac{1}{q^{t-1}}\right)+\frac{b+\gamma_q(m)}{q^{t+1}},\frac{a+1}{q}+\frac{1}{q}\left(1-\frac{1}{q^{t-1}}\right)+\frac{b+\gamma_q(m)}{q^{t+1}}\right).\end{aligned} \tag{39}$$

Then $\gamma_q(n)<x$ and $\gamma_q(n+1)<x$ if and only if

$$\gamma_q(m)<q^{t+1}\left(x-\left(\frac{a+1}{q}+\frac{1}{q}\left(1-\frac{1}{q^{t-1}}\right)+\frac{b}{q^{t+1}}\right)\right). \tag{40}$$

For $a=q-1$ we have

$$\begin{aligned}&\big(\gamma_q(n),\gamma_q(n+1)\big)\\ &=\left(\frac{q-1}{q}+\frac{1}{q}\left(1-\frac{1}{q^{t-1}}\right)+\frac{b+\gamma_q(m)}{q^{t+1}},\frac{0}{q}+\frac{b+1+\gamma_q(m)}{q^{t+1}}\right).\end{aligned} \tag{41}$$

Then $\gamma_q(n) < x$ and $\gamma_q(n+1) < x$ if and only if

$$\gamma_q(m) < q^{t+1}\left(x - \left(\frac{q-1}{q} + \frac{1}{q}\left(1 - \frac{1}{q^{t-1}}\right) + \frac{b}{q^{t+1}}\right)\right). \tag{42}$$

In the following we denote by $x_q(a,b,t)$

$$x_q(a,b,t) = \frac{a}{q} + \frac{1}{q}\left(1 - \frac{1}{q^{t-1}}\right) + \frac{b}{q^{t+1}}.$$

It is the minimal van der Corput's number in (33). Then

$$\lim_{N\to\infty} \frac{1}{N}\#\{n \le N; n \in N(a,b,t), \gamma_q(n) < x, \gamma_q(n+1) < x\}$$
$$= \begin{cases} 0 & \text{if } x < x_q(a+1,b,t), \\ x - x_q(a+1,b,t) & \text{if } x_q(a+1,b,t) \le x < x_q(a+1,b+1,t), \\ \frac{1}{q^{t+1}} & \text{if } x > x_q(a+1,b+1,t) \end{cases} \tag{43}$$

Similarly,

$$\lim_{N\to\infty} \frac{1}{N}\#\{n \le N; n \in N(q-1,b,t), \gamma_q(n) < x, \gamma_q(n+1) < x\}$$
$$= \begin{cases} 0 & \text{if } x < x_q(q-1,b,t), \\ x - x_q(q-1,b,t) & \text{if } x_q(q-1,b,t) \le x < x_q(q-1,b+1,t), \\ \frac{1}{q^{t+1}} & \text{if } x > x_q(q-1,b+1,t) \end{cases} \tag{44}$$

Thus we obtain $g(x,x)$ by summing up (43) for

$$a = 0,1,2,\dots,q-2,$$
$$b = 0,1,2,\dots,q-2,$$
$$t = 1,2,\dots$$

and (44) for

$$b = 0,1,2,\dots,q-2,$$
$$t = 1,2,\dots$$

For simplicity we use it for $q=2$, $b=0$, $t=1,2,\dots$ In this case

$$x_2(1,0,t) = 1 - \frac{1}{2^t} \quad \text{and} \quad x_2(1,1,t) = 1 - \frac{1}{2^t} + \frac{1}{2^{t+1}}.$$

Let $x = \frac{1}{2}\left(2 - \frac{3}{8}\right)$. In (43):

for $t=1$ we have $\frac{1}{2^{1+1}}$,

for $t=2$ we have $x - x_2(1,0,2) = \frac{1}{16}$,

for $t=3,4,5,\dots$ we have 0.

This gives $\frac{5}{16}$. In (44) we also have $\frac{5}{16}$. Summing these results we obtain $g(x,x) = \frac{10}{16}$. The same result we have from (38).

2^0. Using our method started in [3] we have found directly the distribution function $g(x,y)$ of the sequence $\big(\gamma_q(n), \gamma_q(n+1)\big)$ by

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A, \\ 1-(1-y)-(1-x) = x+y-1 & \text{if } (x,y) \in B, \\ y - \frac{1}{q^i} & \text{if } (x,y) \in C_i, \\ x - 1 + \frac{1}{q^{i-1}} & \text{if } (x,y) \in D_i, \end{cases} \tag{45}$$

where A, B, C_i, D_i are in the Fig. 3.

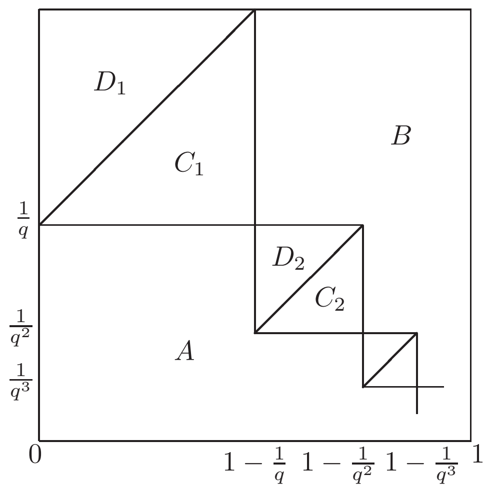

Figure 3.

P r o o f. Every point $\big(\gamma_q(n), \gamma_q(n+1)\big)$, $n = 0, 1, 2, \dots$, lie on the line segment

$$Y = X - 1 + \frac{1}{q^k} + \frac{1}{q^{k+1}}, \quad X \in \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right]$$

for $k = 0, 1, \dots$ and let T be their union. Because $\gamma_q(n)$ is u.d., then the sequence $\big(\gamma_q(n), \gamma_q(n+1)\big)$ has a.d.f. $g(x,y)$ of the form

$$g(x,y) = \left|\text{Project}_x\Big(\big([0,x) \times [0,y)\big) \cap T\Big)\right|,$$

where Project_x is a projection of a two dimensional set to the x-axis. It is a copula and $g(x,y)$ can be computed explicitly according to (45). $\square$

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