DE GRUYTER
OPEN

TATRA
MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2015-0046
Tatra Mt. Math. Publ. **64** (2015), 127–131

ON ONE TYPE OF COMPACTIFICATION OF POSITIVE INTEGERS

Milan Paštéka

ABSTRACT. The object of observation is a compact metric ring containing positive integers as dense subset. It is proved that this ring is isomorphic with a ring of reminder classes of polyadic integers.

Let $\mathbb{N}$ be the set of positive integers. A mapping $|| \cdot || \, : \, \mathbb{N} \to < 0, \infty)$ will be called norm if and only if the following conditions are satisfied for $a, b \in \mathbb{N}$

$$||a|| = 0 \Leftrightarrow a = 0, \quad ||a + b|| \leq ||a|| + ||b||, \quad ||ab|| \leq ||b||.$$

There are various examples of norms on $\mathbb{N}$. One of these is polyadic norm defined in [N], [N1]. We start by a generalization of polyadic norm. Denote by $a+(m)$ the arithmetic progression with difference m which contains a. Instead of $0 + (m)$ we write only (m).

A subset $A \subset \mathbb{N}$ we call *closed to divisibility* or shortly CD-set if and only if

$$1 \in A, \quad m \in A, d|m \Rightarrow d \in A, \quad m_1, m_2 \in A \Rightarrow [m_1, m_2] \in A,$$

for $d, m, m_1, m_2 \in \mathbb{N}$.

Suppose that A is infinite CD-set and $\{B_n\}$ is such sequence elements of A that for every $d \in A$ there exists n_0 that $d|B_n$ for $n > n_0$. It is easy to see that the mapping

$$||a||_A = \sum_{n=1}^{\infty} \frac{h_n(a)}{2^n}$$

for $a \in \mathbb{N}$, where $h_n(a) = 1 - \mathcal{X}_{(B_n)}$, is a norm. This norm will be called generalized polyadic norm and the completion with respect the metric given by this norm will be called the ring of generalized polyadic integers.

2010 Mathematics Subject Classification: 11K06.
Keywords: density, polyadic integers.
Research was supported by grant VEGA 2/0146/14.

If $A = \mathbb{N}$ and $B_n = n!$, we get polyadic norm and the completion will be the ring of polyadic integers. In the case $A = \{p^n; n = 0, ., 2, ..\}$ and $B_n = p^n$ for a given prime p we a obtain p-adic norm and the completion will be the ring of p-adic integers.

In the following text we shall assume that there is given a compact metric space (Ω, ρ) containing $\mathbb{N}$ as a dense subset. We suppose that the operations addition and multiplication on $\mathbb{N}$ are continuous and are extended to whole Ω to continuous operations. Thus $(\Omega, +, \cdot)$ is a topological commutative semiring.

Since Ω is compact, we can suppose that there exists an increasing sequence of positive integers $\{x_n\}$ convergent to an element of Ω. Put

$$a_n = x_{2n} - x_n, \qquad n = 1, 2, \dots$$

Then

$$a_n \geq n \quad \text{and} \quad a_n \to 0 \tag{1}$$

in the topology of Ω.

For $\beta \in \Omega$ and $b_n \to \beta$, $b_n \in \mathbb{N}$ we can consider the sequence of positive integer $\{a_{k_n} - b_n\}$, for a suitable increasing sequence $\{k_n\}$, such that

$$a_{k_n} - b_n \to \beta', \quad \text{where} \quad \beta + \beta' = 0.$$

We see that $(\Omega, +)$ is a compact group.

Clearly, for every $m \in \mathbb{N}$ there holds $cl\big(r + (m)\big) = r + m\Omega$, where $m\Omega$ is the principal ideal in the ring $(\Omega, +, \cdot)$ generated by m. This yields

$$\Omega = m\Omega \cup (1 + m\Omega) \cup \cdots \cup (m - 1 + m\Omega). \tag{2}$$

Since the divisibility by m in $\mathbb{N}$ is not necessary equivalent with the divisibility by m in Ω, it is not assumed that the last decomposition is disjoint.

Lemma 1. *Let $m \in \mathbb{N}$ be such positive integer that it is also the minimal generator of the ideal $m\Omega$. Then every positive integer is divisible by m in $\mathbb{N}$ if and only if it is divisible by m in Ω.*

Proof. One implication is trivial. Suppose now that some positive integer a is divisible by m in Ω. Thus $a \in m\Omega$. Put $d = (a, m)$—the greatest common divisor in $\mathbb{N}$. Then $d = ax + my$ for certain integers x, y. This yields $d \in m\Omega$. We get $d\Omega = m\Omega$ and the minimality of m implies $m = d$. $\square$

For every $n \in \mathbb{N}$ we can define $g(n)$ as the minimal positive generator of $n\Omega$. Put $\mathcal{A} = \{g(n); n \in \mathbb{N}\}$.

The set $r + m\Omega$ is closed and so from (2) we see that also open, which we refer as *clopen* set.

It is easy to check that the set $\mathcal{A}$ is a CD-set.

Let $\{a_n\}$ be the sequence of positive integers given in (1). Clearly,

$$\bigcap_{n=1}^{\infty} a_n\Omega = \{0\},$$

this yields

$$\bigcap_{m\in\mathcal{A}} m\Omega = \{0\}. \tag{3}$$

So we obtain that the set $\mathcal{A}$ is infinite. Since $m\Omega$ is open for $m \in \mathcal{A}$, equality (3) implies that for each sequence $\{\alpha_n\}$ there holds

$$\alpha_n \to 0 \Longleftrightarrow \forall m \in \mathcal{A} \exists n_0; \quad n \geq n_0 \Longrightarrow m|\alpha_n.$$

If we define the congruence by the natural manner: $\alpha \equiv \beta \pmod{\gamma}$ if and only if γ divides $\alpha - \beta$, for $\alpha, \beta, \gamma \in \Omega$, then there holds:

$$\alpha_n \to \beta \Longleftrightarrow \forall m \in \mathcal{A} \exists n_0; \quad n \geq n_0 \Longrightarrow \alpha_n \equiv \beta \pmod{m}.$$

Thus the convergence can be metrised by the generalized polyadic norm. Let

$$\mathcal{A} = \{m_n, n = 1, 2, \ldots\} \quad \text{and} \quad M_n = [m_1, \ldots, m_n], \ \ n = 1, 2, \ldots,$$

then

$$||\alpha||_{\mathcal{A}} = \sum_{n=1}^{\infty} \frac{1 - \mathcal{X}_{M_n\Omega}(\alpha)}{2^{-n}}.$$

We get the following

Theorem 1. *The metric ρ is equivalent with the metric $\rho_{\mathcal{A}}$, where*

$$\rho_{\mathcal{A}}(\alpha, \beta) = \|\alpha - \beta\|_{\mathcal{A}} \quad \textit{for } \ \alpha, \beta \in \Omega.$$

So for every set $S \subset \Omega$ we have

$$cl(S) = \bigcap_{n=1}^{\infty} (S + M_n\Omega). \tag{4}$$

Denote the Haar probability measure defined on $(\Omega, +)$ by P. For $m \in \mathcal{A}$ the decomposition (2) is disjoint and so $P(r + m\Omega) = \frac{1}{m}$. If we define the submeasure ν^* on the system of subsets of $\mathbb{N}$ as $\nu^*(S) = P\big(cl(S)\big)$, we get from (4) and upper semicontinuity of measure that for each S

$$\nu^*(S) = \lim_{n\to\infty} \frac{R(S : M_n)}{M_n},$$

where $R(S : M_n)$ the number of elements of S incongruent modulo M_n. Thus ν^* is the covering density defined in [P].

Theorem 2. *Let $\alpha, \beta \in \Omega$. There exist $\alpha_1, \beta_1 \in \Omega$ such that the element $\delta = \alpha_1\alpha + \beta_1\beta$ divides α and β.*

P r o o f. Let $\{a_n\}, \{b_n\}$ be the sequences of positive integers that $a_n \to \to \alpha, b_n \to \beta$. Let d_n the greatest common divisor of $a_n, b_n, n = 1, 2, \dots$ Then $d_n = v_n a_n + u_n b_n$ for some u_n, v_n-integers. The compactness of Ω provides that $u_{k_n} \to \alpha_1$ and $v_{k_n} \to \beta_1$ for a suitable increasing sequence $\{k_n\}$. Put $\delta = \alpha_1\alpha + \beta_1\beta$. We see that $d_{k_n} \to \delta$. For $n = 1, 2, \dots$ we have $a_{k_n} = c_n d_{k_n}$. Since $\{c_n\}$ contains a convergent subsequence, we get that δ divides α. Analogously, it can be derived that δ divides β. □

The element δ from Theorem 2 will be called the *greatest common divisor* of α, β and we shall write $\delta \sim (\alpha, \beta)$.

Corollary 1. *If $p \in \mathcal{A}$ is a prime then for every $\alpha \in \Omega$ there holds p divides α or $(\alpha, p) \sim 1$.*

P r o o f. If p does not divide α then $\alpha \in \Omega \setminus p\Omega$. Consider a sequence of positive integers $\{a_n\}$ which converges to α. The set $\Omega \setminus p\Omega$ is open, thus we can suppose that $(a_n, p) = 1$. This yields $\ell_n a_n + s_n p = 1$ for suitable integers ℓ_n, s_n. Since Ω is a compact space there exists an increasing sequence $\{k_n\}$ that $\ell_{k_n} \to \lambda, s_{k_n} \to \sigma$. And so $\lambda\alpha + \sigma p = 1$. □

Corollary 2 can be proved analogously

Corollary 2. *An element $\alpha \in \Omega$ is invertible if and only if $(\alpha, p) \sim 1$ for every prime $p \in \mathcal{A}$.*

Lemma 2. *Each closed ideal in Ω is principal ideal.*

P r o o f. Let $I \subset \Omega$ be closed ideal. Let $\alpha \in I$. Denote by I_α the set of all divisors of α belonging to I. The compactness of Ω yields that I_α is a closed set. From Lemma 2 we get that for every $\alpha, \beta \in I$ there exists $\delta \in I$ so that $I_\delta \subset I_\alpha \cap I_\beta$. And so it can be proved by induction that $I_\alpha, \alpha \in I$ is a centered system of closed sets. Thus its intersection is non empty, and contains an element γ. Then $I = \gamma\Omega$. □

In the sequel we denote Ω the ring of polyadic integers, thus completion of $\mathbb{N}$ with respect to norm $||\cdot||_{\mathbb{N}}$ and we suppose that an infinite CD-set A is given. The completion of $\mathbb{N}$ with respect to the norm $||\cdot||_A$ we denote as Ω_A.

Lemma 2 provides that $\cap_{a\in A} a\Omega = \alpha\Omega$ for suitable $\alpha \in \Omega$.

We will prove the folowing

Theorem 3. *The ring Ω_A is isomorphic with the factor ring $\Omega/\alpha\Omega$.*

P r o o f. If a sequence of positive integers is Cauchy's with respect to $||\cdot||_{\mathbb{N}}$, then it is Cauchy's with respect to $||\cdot||_A$ as well.

If $\{a_n\}, \{b_n\}$ are sequences of positive integer, then

$$|||a_n - b_n|||_{\mathbb{N}} \to 0 \Longrightarrow |||a_n - b_n|||_A \to 0.$$

Therefore we can define a mapping $F : \Omega \to \Omega_A$ in the following way. If $\beta \in \Omega, b_n \to \beta$ with respect to $||\cdot||_{\mathbb{N}}$, then $F(\beta)$ is the limit of $\{b_n\}$ with respect to $||\cdot||_A$. Clearly, F is a surjective morphism with kernel $\cap_{a\in A} a\Omega$ and the assertion follows. $\square$

Theorem 4. *The ring Ω_A is an integrity domain if and only if $A = \{p^n; n = 0, 1, 2 \ldots\}$, where p is a prime number.*

P r o o f. Suppose that $A = \{p^n; n = 0, 1, 2 \ldots\}$. It suffices to prove that $\cap_{a\in A} a\Omega$ is a prime ideal. Let α, β do not belong to $\cap_{a\in A} a\Omega$. Then $\alpha = p^k \alpha_1$, $\beta = p^j \beta_1$, where $(p, \alpha_1) \sim 1, (p, \beta_1) \sim 1$, $j, k < \infty$. Thus

$$\alpha\beta = p^{k+j} \alpha_1 \beta_1 \notin \cap_{a\in A} a\Omega.$$

Assume that A contains at least two different primes. The elements of the sequence $\{M_k\}$ can be decomposed into $M_k = d_k c_k$ such that

$$(d_k, c_k) = 1, \quad d_k > 1, \quad c_k > 1, \qquad k = 1, 2, \ldots$$

Let $\{k_n\}$ be a subsequence such that $d_{n_k} \to \delta, c_{n_k} \to \gamma$. Then $\gamma, \delta \notin \cap_{a\in A} a\Omega$ and $\gamma\delta \in \cap_{a\in A} a\Omega$, thus $\cap_{a\in A} a\Omega$ is not a prime ideal. $\square$

REFERENCES

[N] NOVOSELOV, E. V.: *Topological theory of polyadic numbers*, Trudy Tbilis. Mat. Inst. **27** (1960), 61–69. (In Russian)

[N1] NOVOSELOV, E. V.: *New method in the probability number theory*, Doklady Akademii Nauk. Ser. Mat. **28** (1964) no. 2, 307–364. (In Russian)

[P] PAŠTÉKA, M.: *Covering densities*, Math. Slovaca **42** (1992), 593–614.

[Po] POSTNIKOV, R.G.: *Introduction to Analytic Number Theory*. Moscov, Nauka, 1971. (In Russian); Amer. Math. Soc., Providence RI, 1981 (English translation).

[P-P] PAŠTÉKA, M.—PORUBSKÝ, Š.: *On the distribution on the sequences of integers*, Math. Slovaca **43** (1993), 521–639.

Received October 12, 2015

Department of Mathematics and Computer Science
Faculty of Education
University of Trnava
Priemyselná 4
SK–918-43 Trnava
SLOVAKIA
E-mail: milan.pasteka@mat.savba.sk