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APPLICATIONS OF UNIFORM DISTRIBUTION THEORY TO THE RIEMANN ZETA-FUNCTION

Selin Selen Özbek — Jörn Steuding

ABSTRACT. We give two applications of uniform distribution theory to the Riemann zeta-function. We show that the values of the argument of $\zeta\big(\frac{1}{2}+iP(n)\big)$ are uniformly distributed modulo $\frac{\pi}{2}$, where $P(n)$ denotes the values of a polynomial with real coefficients evaluated at the positive integers. Moreover, we study the distribution of $\arg\zeta'\big(\frac{1}{2}+i\gamma_n\big)$ modulo π, where γ_n is the nth ordinate of a zeta zero in the upper half-plane (in ascending order).

1. Uniform distribution of the argument of the zeta-function on the critical line

In a recent paper [14] the authors studied the distribution of the argument of the Riemann zeta-function $\zeta(s)$ on arithmetic progressions on the critical line $\frac{1}{2}+i\mathbb{R}$. Among other things it was shown that the argument of the Riemann zeta-function on an arbitrary infinite arithmetic progression on the critical line, $\arg\zeta\big(\frac{1}{2}+i(\tau+n\delta)\big)$ for $n=1,2,\ldots$, is uniformly distributed modulo $\frac{\pi}{2}$ (and even modulo π under a certain condition). In this note we shall consider the more general situation where the arithmetic progression is replaced by the values of an arbitrary real polynomial P of positive degree at the positive integers.

Theorem 1. *Let P be a polynomial with real coefficients of positive degree. Then the sequence of the arguments $\arg\zeta\big(\frac{1}{2}+iP(n)\big)$ for $n=1,2,\ldots$ is uniformly distributed modulo $\frac{\pi}{2}$. If the number $m(N)$ of zeros of $\zeta\big(\frac{1}{2}+iP(n)\big)$ for $n=1,2,\ldots,N$ satisfies $m(N)=o(N)$ as $N\to\infty$, then $\arg\zeta\big(\frac{1}{2}+iP(n)\big)$ is uniformly distributed modulo π.*

Recall that a sequence of real numbers x_n is said to be uniformly distributed modulo μ, where μ is a fixed positive real number, if for all α,β with $0\le\alpha<\beta\le\mu$

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the proportion of the fractional parts of the x_n modulo μ in the interval $[\alpha, \beta)$ corresponds to its length in the following sense:

$$\lim_{N\to\infty} \frac{1}{N} \sharp\big\{1 \leq n \leq N : x_n \bmod \mu \in [\alpha, \beta)\big\} = \frac{\beta - \alpha}{\mu}.$$

Here $x_n \bmod \mu$ is defined by $x_n \bmod \mu := x_n - \lfloor \frac{x_n}{\mu} \rfloor \mu$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. This notion has been introduced by H. Weyl [20] in the case $\mu = 1$; however, for our purpose a modulus related to the geometry of the complex plane (i.e., a modulus μ such that $\frac{2\pi}{\mu}$ is a positive integer) is more natural.

The method of proof is rather similar to the one in [14]. Our reasoning starts with the functional equation

$$\zeta(s) = \Delta(s)\zeta(1 - s), \tag{1}$$

where

$$\Delta(s) = 2^s \pi^{s-1} \sin \tfrac{\pi s}{2} \Gamma(1 - s),$$

and Γ denotes Euler's gamma-function. (For this and further details about the zeta-function we refer to H. E. Edwards' monography [3].) In view of the identity $\zeta(\overline{s}) = \overline{\zeta(s)}$ (a consequence of the reflection principle) it follows that the zeros are symmetrically distributed with respect to the real axis and the critical line (which also explains that we discuss only zeros in the upper half-plane). Recall that $\zeta(s)$ has so-called trivial zeros at $s = -2n$ for $n \in \mathbb{N}$; all other zeros are called nontrivial and they are known to be located in the critical strip $0 < \mathrm{Re}\, s < 1$ but not on the real axis. The yet unsolved Riemann Hypothesis claims that all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ lie on the so-called critical line $\mathrm{Re}\, s = \frac{1}{2}$. The number $N(T)$ of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$ (counting multiplicities) is asymptotically given by the Riemann-von Mangoldt formula:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{2}$$

Now let

$$S(t) := \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + it \right);$$

here the argument of the zeta-function on the critical line is defined as follows. In view of the multi-valued complex logarithm we may fix the value of the logarithm $\log \zeta$ at $\frac{1}{2} + it$ by continuous variation along the polygon with vertices $2, 2+it, \frac{1}{2}+it$, provided t is not equal to an ordinate of a nontrivial zero $\rho = \beta+i\gamma$; otherwise, when $t = \gamma$, we define $S(\gamma)$ by

$$S(\gamma) = \tfrac{1}{2} \lim_{\epsilon\to 0+} \big(S(\gamma + \epsilon) + S(\gamma - \epsilon)\big). \tag{3}$$

Since $\zeta(2)$ is a positive real number, we may choose $\log\zeta(s)$ as the principal branch of the logarithm on the subinterval $(1,\infty)$ of the real axis. Notice that $S(t)$ is a continuous function for t different from any zeta ordinate γ. The argument is linked with the zero counting function by

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(T^{-1}\right).$$

(Actually, the estimate of $S(T) = O(\log T)$ implies the Riemann-von Magoldt formula (2) from above.) Moreover, the latter formula implies that $\pi S(t)$ jumps at each ordinate by an integer multiple of π (according to the multiplicity of the zero). Thus, for $\mu = \frac{\pi}{2}$, $\arg\zeta\big(\frac{1}{2}+it\big)$ is continuous modulo $\frac{\pi}{2}$. For the case $\mu=\pi$, however, we have to use the assumption that there is only a negligible number of zeros in the sequence $\frac{1}{2}+iP(n)$ (which matches what is widely expected).

In order to prove uniform distribution we shall use H. Weyl's criterion [20], resp. its variation from [14] which claims that $\arg\zeta\big(\frac{1}{2}+iP(n)\big)$ is uniformly distributed modulo μ if, and only if, for all integers $m\neq 0$

$$\sum_{M<n\leq M+N} \exp\Big(\tfrac{2\pi}{\mu} im \arg\zeta\left(\tfrac{1}{2}+iP(n)\right)\Big) = o(N), \qquad \text{as } N\to\infty, \quad \text{where } M\in\mathbb{N}. \tag{4}$$

Without loss of generality we may assume that for sufficiently large n the values $P(n)$ are positive, bounded below by M, and strictly increasing, i.e.,

$$P(n+1) > P(n) \gg M \qquad \text{for all} \quad n > M.$$

Moreover, taking into account $m(N) = o(N)$ we may assume that

$$\zeta\left(\tfrac{1}{2}+iP(n)\right)\neq 0.$$

Hence, there exist real numbers $r_n>0$ and $\phi_n\in[0,2\pi)$ such that

$$\zeta\left(\tfrac{1}{2}+iP(n)\right) = r_n\exp(i\phi_n),$$

and

$$\arg\zeta\left(\tfrac{1}{2}+iP(n)\right)\equiv\phi_n \bmod 2\pi.$$

In view of the functional equation (1) and the reflection principle, $\overline{\zeta(s)}=\zeta(\overline{s})$, we find

$$\exp(2i\phi_n) = \Delta\left(\tfrac{1}{2}+iP(n)\right).$$

(Indeed, (1) implies that $\Delta(s)$ is of absolute value one for $s\in\frac{1}{2}+i\mathbb{R}$.) Next we shall make use of

$$\Delta(\sigma+it) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it}\exp\big(i(t+\tfrac{\pi}{4})\big)\Big(1+O\left(t^{-1}\right)\Big) \tag{5}$$

(see E. C. Titchmarsh [17, § 7.4]).

Thus, we need to show that the values $\frac{1}{2}P(n)\log\frac{P(n)}{2\pi e}$ are uniformly distributed modulo μ which is, by Weyl's criterion,

$$\sum_{M<n\leq M+N}\exp\left(\frac{\pi}{\mu}imP(n)\log\frac{P(n)}{2\pi e}\right)=o(N). \tag{6}$$

Recall J. van der Corput's difference theorem: *if (x_n) is a sequence of real numbers such that for every positive integer h the sequence $(x_{n+h}-x_n)$ is uniformly distributed modulo one, then (x_n) is also uniformly distributed modulo one* [19] (resp. the monography [12] of L. Kuipers and H. Nierderreiter). This result might be influenced by Weyl's reasoning for proving the uniform distribution modulo one of polynomials with at least one irrational coefficient [20]. For our purpose we need the difference theorem for the moduli $\frac{\pi}{2}$ and π in place of 1, however, the proof of van der Corput's theorem easily extends to an arbitrary modulus. Consequently, in order to establish (5) we may prove

$$\sum_{M<n\leq M+N}\exp\left(\frac{\pi}{\mu}imQ_h(n)\log\frac{Q_h(n)}{2\pi e}\right)=o(N)$$

$$\text{with}\ \ Q_h(n)=P(n+h)-P(n),\quad \text{where}\ \ h\in\mathbb{N}.$$

Since Q_h is a polynomial in n of degree $\deg Q_h=\deg P-1$, the latter estimates follow by induction on $\deg P$ from our previous result [14], where we have shown the uniform distribution result for arbitrary linear polynomials. This reasoning would also apply for other moduli provided that a corresponding uniform distribution result holds for linear polynomials.

2. Shanks' conjecture

D. Shanks [16] conjectured that the curve $\mathcal{C}: t\mapsto\zeta\big(\frac{1}{2}+it\big)$ approaches the origin most of the times from the third or fourth quadrant; this has been proved by A. Fujii [7] and T. S. Trudgian [18]. In view of

$$\frac{\partial}{\partial t}\zeta\left(\tfrac{1}{2}+it\right)=i\zeta'\left(\tfrac{1}{2}+it\right)$$

the values of the derivative $\zeta'\big(\frac{1}{2}+i\gamma\big)$ at zeta zeros are thus positive real on average. Moreover, we observe that the argument of $\zeta'\big(\frac{1}{2}+i\gamma\big)$ minus $\frac{\pi}{2}$ coincides modulo 2π with the direction of the slope of the curve $\mathcal{C}$ at time $t=\gamma$ (as long as $\frac{1}{2}+i\gamma$ is not a multiple zero). Therefore, it seems unlikely that the argument of the first derivative of the zeta-function at its nontrivial zeros is uniformly distributed modulo 2π. However, we believe that the picture with the module 2π replaced by the smaller module π changes the situation completely.

For our discussion of this case we have to assume the truth of two open conjectures, namely the Riemann Hypothesis and the Essential Simplicity Hypothesis on the zeros of zeta-function. Recall the Essential Symplicity Hypothesis which states that almost all zeros are simple; more precisely, the latter conjecture can be formulated such that the limit

$$\lim_{T\to\infty} \frac{1}{N(T)} \sharp\Big\{\rho = \beta + i\gamma : \zeta'(\rho) \neq 0\Big\}$$

exists and equals one. In order to obtain a uniform distribution result for $\arg\zeta'(\frac{1}{2} + i\gamma_n)$ we even have to assume some hypothetical bounds for certain exponential sums which seem to be out of reach with present day methods:

Theorem 2. *Assume the Riemann Hypothesis and the Essential Simplicity Hypothesis. If additionally the following estimates for exponential sums hold,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{M<n\leq M+N} \exp\left(im\gamma_n \log\frac{\gamma_n}{2\pi e}\right) = 0 \qquad \textit{for all} \quad m \in \mathbb{Z}\setminus\{0\}, \tag{7}$$

then the sequence $\arg\zeta'(\frac{1}{2} + i\gamma_n)$ *is uniformly distributed modulo* π*, where* γ_n *denotes the ordinate of the* n*th nontrivial zero in the upper half-plane in ascending order.*

For the proof we differentiate the functional equation (1) which leads to

$$\zeta'(s) = \Delta'(s)\zeta(1-s) - \Delta(s)\zeta'(1-s).$$

Since $\zeta(1-s)$ vanishes for a zeta zero $s = \rho = \frac{1}{2} + i\gamma$ too, it follows that

$$\zeta'\Big(\tfrac{1}{2} + i\gamma\Big) = -\Delta\Big(\tfrac{1}{2} + i\gamma\Big)\zeta'\Big(\tfrac{1}{2} - i\gamma\Big). \tag{8}$$

Similarly as for the zeta-function we have

$$\zeta'\Big(\tfrac{1}{2} - i\gamma\Big) = \overline{\zeta'\Big(\tfrac{1}{2} + i\gamma\Big)}$$

for its derivative. Assuming the simplicity of the zeta zero $\frac{1}{2} + i\gamma$, we may write

$$\zeta'\Big(\tfrac{1}{2} + i\gamma_n\Big) = r_n \exp(i\phi_n) \tag{9}$$

with real numbers $r_n > 0$ and $\phi_n \in [0, 2\pi)$. Notice that the argument of the first derivative of the zeta-function can be defined by continuous variation in a rather analogous way as the argument of the zeta-function in Section 1. In view of (4), (8) and (9) we find

$$\arg\zeta'\Big(\tfrac{1}{2} + i\gamma_n\Big) \equiv \phi_n \bmod 2\pi$$

and

$$\exp(2i\phi_n) = -\Delta\Big(\tfrac{1}{2} + i\gamma_n\Big).$$

Taking into account (5) we need to show that, for every integer $m \neq 0$,

$$\sum_{M<n\leq M+N} \exp\Big(im\gamma_n \log \frac{\gamma_n}{2\pi e}\Big) = o(N),$$

which is just a reformulation of (7). Unfortunately, such estimates are not known to hold for general $m \in \mathbb{Z} \setminus \{0\}$. Assuming (7), however, the assertion of Theorem 2 follows. The non-vanishing of $\zeta'\big(\frac{1}{2} + i\gamma\big)$ for a zeta zero $\frac{1}{2} + i\gamma$ appears relevant only once. It turns out that a *small* number of multiple zeta zeros would not damage our reasoning. Therefore we only have to assume the Essential Simplicity Hypothesis.

The history of the exponential sums in (7) is rather long. It was E. Landau [13] who obtained an asymptotic formula for sums of the form $\sum_\rho x^\rho$, where the summation is over all nontrivial zeros. His explicit formula has been extended and generalized by several authors; see, for example, K. Ford, K. Soundararajan and A. Zaharescu [5]. This has found many applications, for instance in the celebrated proof of the uniform distribution modulo one of the ordinates of the nontrivial zeros by H. Rademacher [15], P. Elliott [4], and E. Hlawka [11]. The slightly different exponential sum (7) has first been studied by G. H. Hardy and J. E. Littlewood [10] and later by A. Fujii in a series of papers, e.g., [6], [8] (to mention only two). It appears that only for small $|m|$ an estimate of the form (7) is available by present day methods. Recently, J. Arias de Reyna [1] investigated the uniform distribution modulo one of the normalized nontrivial zeros (which is of interest since the average spacing then is equal to one). He provides the limit

$$\lim_{T\to\infty} \frac{1}{N(T)} \sum_{0<\gamma\leq T} \exp\Big(2i\kappa\gamma \log \frac{\gamma}{2\pi e}\Big) = 0, \quad \text{valid for } \kappa \in \Big(0, \frac{6}{5}\Big).$$

It is an open conjecture that the values of the Riemann zeta-function $\zeta(s)$ on the critical line $\frac{1}{2} + i\mathbb{R}$ lie dense in the complex plane (in some literature attributed to K. Ramachandra). It may be noticed that H. Bohr and R. Courant [2] proved that the set

$$\big\{\zeta(\sigma + it) : t \in \mathbb{R}\big\}$$

is dense in $\mathbb{C}$ for every fixed $\sigma \in (\frac{1}{2}, 1]$. However, R. Garunkštis and J. Steuding [9] showed under assumption of the truth of the Riemann Hypothesis that

$$\big\{\zeta(\sigma + it) : t \in \mathbb{R}\big\}$$

is not dense in $\mathbb{C}$ for every fixed $\sigma < \frac{1}{2}$; the graphics in Figure 1 are taken from their paper. It seems that Theorem 2 alone does not imply the existence of a neighbourhood of the origin in which the curve $t \mapsto \zeta\big(\frac{1}{2} + it\big)$ is dense.

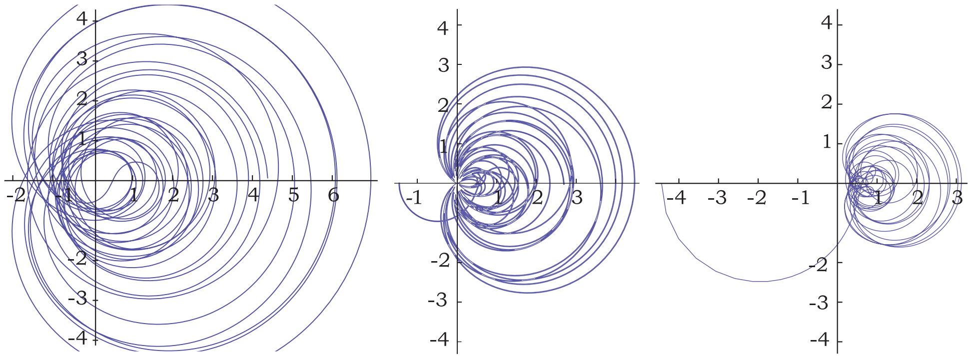

Figure 1: The curves $t \mapsto \zeta(\sigma + it)$ for $\sigma = \frac{1}{5}, \frac{1}{2}$, and $\frac{4}{5}$ from left to right, all for $t \in [0, 100]$. The curve on the right is known to be dense in the complex plane, the curve on the left is not dense if Riemann's Hypothesis is true, and for the curve in the middle this question is open.

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Selin Selen Özbek
Akdeniz University
Department of Mathematics
TR–07-058 Antalya
TURKEY
E-mail: s.selenozbek@gmail.com

Department of Mathematics
Würzburg University
Emil-Fischer-Str. 40
D–97-074 Würzburg
GERMANY
E-mail: selen.oezbek@mathematik.uni-wuerzburg.de

Jörn Steuding
Department of Mathematics
Würzburg University
Emil-Fischer-Str. 40
D–97-074 Würzburg
GERMANY
E-mail: steuding@mathematik.uni-wuerzburg.de