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REMARKS ON BUCK'S MEASURE DENSITY AND A GENERALIZATION OF ASYMPTOTIC DENSITY

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ABSTRACT. In the first part the connections among $\mathcal{I}$-convergence, Buck's measure density [R. C. Buck: *The measure theoretic approach to density*, Amer. J. Math. **68** (1946), 560–580] and Alexander's weight density [R. Alexander: *Density and multiplicative structure of sets of integers*, Acta Arithm. **12** (1967), 321–332] are studied. The object of observation in the second part are some properties of Alexander's weight density.

Introduction

We recall some known notions and notations which we shall need. The fundamental notion which we shall use is $\mathcal{I}$-convergence introduced in the paper [9]. This is a generalization of statistical convergence (see [7]–[10], [12]). Let $\mathbb{N}$ be the set of positive integers. A system $\mathcal{I}$, $\emptyset \neq \mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an *ideal* if it is additive ($A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary ($B \in \mathcal{I},\ A \subseteq B \Rightarrow A \in \mathcal{I}$). The ideal $\mathcal{I}$ is called *nontrivial* if $\mathcal{I} \neq 2^{\mathbb{N}}$. If $\mathcal{I}$ is a nontrivial ideal, then $\mathcal{I}$ is called *admissible* if $\{n\} \in \mathcal{I}$ for every $n \in \mathbb{N}$.

Let $x = (x_n)$ be a sequence of real (of complex) numbers. We shall say the sequence (x_n) *$\mathcal{I}$-converges to L* (real or complex) if for every $\varepsilon > 0$

$$A(\varepsilon) = \big\{n \in \mathbb{N} \colon |x_n - L| \geqslant \varepsilon\big\} \in \mathcal{I}.$$

In the following we suppose that $\mathcal{I}$ is an admissible ideal. Then for every sequence (x_n) we have immediately that $\lim\limits_{n\to\infty} x_n = L$ implies that (x_n) also $\mathcal{I}$-converges to L. (We shall denote this by writing $\mathcal{I}\text{-}\lim\limits_{n\to\infty} x_n = L$).

The notion of $\mathcal{I}$-convergence is in certain sense equivalent to the notion of μ-statistical convergence (see [5]).

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We define also $\mathcal{I}^*$-convergence: A sequence (x_n) *$\mathcal{I}^*$-converges to L* iff there exists a set $A \in \mathcal{I}$ such that

$$\lim_{k\to\infty} x_{m_k} = L$$

where $\{m_1 < m_2 < \dots\} = \mathbb{N} \setminus A$. We shall write this fact as $\mathcal{I}^*\text{-}\lim\limits_{n\to\infty} x_n = L$. (Thus $\mathcal{I}^*$-convergence is the convergence through a "rich" set of indices.)

One can find examples for $\mathcal{I}$ and $\mathcal{I}^*$ convergence in [9]. We recall some well known types of $\mathcal{I}$- convergences.

Let $\mathcal{I}_f$ be the ideal of all finite sets. Then $\mathcal{I}_f$-convergence coincides with the usual convergence.

Let $\mathcal{I}_d = \{A \subseteq \mathbb{N}\colon d(A) = 0\}$, d is the asymptotic density of $A \subseteq \mathbb{N}$. Then $\mathcal{I}_d$-convergence is called statistical convergence. (See [1], [5], [8]–[10], [12]). Further $\mathcal{I}_\delta = \{A \subseteq \mathbb{N}\colon \delta(A) = 0\}$ where $\delta(A)$ is the logarithmic density of $A \subseteq \mathbb{N}$ (see [1]), $\mathcal{I}_u = \{A \subseteq \mathbb{N}\colon u(A) = 0\}$ where $u(A)$ is the uniform density of $A \subseteq \mathbb{N}$ (see [1], [3], [8], [9]).

For a lot of ideals $\mathcal{I}$ the notions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are equivalent. For every sequence (x_n) it holds:

$$\mathcal{I}^*\text{-}\lim_{n\to\infty} x_n = L \implies \mathcal{I}\text{-}\lim_{n\to\infty} x_n = L\,.$$

In the paper [4] of R. C. Buck, a finitely additive measure on an algebra $\mathcal{D}_\mu \subseteq 2^{\mathbb{N}}$ is constructed as follows: Let $m\mathbb{N} + a$ be an arithmetic progression with modulus m and first element a. Put for $A \subseteq \mathbb{N}$

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{k} \frac{1}{m_i}\colon A \dot{\subset} \bigcup_{i=1}^{k} m_i\mathbb{N} + a_i\,,\ k \in \mathbb{N}\right\},$$

where the symbol $A \dot{\subset} B$ denotes that at most a finite number of elements of A do not belong to B. Then the system of sets $\mathcal{D}_\mu = \{A \subseteq \mathbb{N}\colon \mu^*(A) + \mu^*(\mathbb{N}\setminus A) = 1\}$ is an algebra of sets and the restriction $\mu = \mu^*|\mathcal{D}_\mu$ is a finitely additive measure (see [4]).

Put $\mathcal{I}_\mu = \{A \subseteq \mathbb{N}\colon \mu(A) = 0\}$. Then I_μ is an ideal and $\mathcal{I}_\mu \subset \mathcal{I}_d$. This yields that

$$\mathcal{I}_\mu\text{-}\lim_{n\to\infty} x_n = L \implies \mathcal{I}_d\text{-}\lim_{n\to\infty} x_n = L\,.$$

In this paper we construct a sequence which is $\mathcal{I}_\mu$- convergent, but which does not converge through any set of positive measure.

We conclude this section recalling the notion of Alexander's density (see [1]): Let $c = (c_n)$ be a sequence of positive numbers, such that $\sum\limits_{n=1}^{\infty} c_n = +\infty$. Put

$$S(n) = \sum_{k=1}^{n} c_k$$

and

$$S(A,n) = \sum_{k=1}^{n} \chi_A(k) c_k \,,$$

where $A \subseteq \mathbb{N}$ and χ_A is the characteristic function of A.

Put

$$h_n(A) = \frac{S(A,n)}{S(n)} \,.$$

The value $\underline{h}(A) = \liminf\limits_{n\to\infty} h_n(A) \big(= \underline{h}_c(A)\big)$ is called the *lower Alexander's density* and the value $\overline{h}(A) = \limsup\limits_{n\to\infty} h_n(A) \big(= \overline{h}_c(A)\big)$ is called the *upper Alexander's density* of the set A. If $\overline{h}(A) = \underline{h}(A)$, then this value is called the *Alexander's density* of the set A and we denote this value by $h(A)$ (or $h_c(A)$ because this value depends on the sequence $c = (c_n)$) (see [1], [10]).

Finally we introduce two assertions which will be useful in our further considerations.

THEOREM A. (Stolz). *Let $y = (y_n)$ be a strictly increasing sequence of real numbers and $\lim\limits_{n\to\infty} y_n = +\infty$. If there exists $\lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = l$ (finite or infinite), then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = l$.*

THEOREM B. *Let $(m,r) = 1$ and $l \in \{0, 1, \dots, r-1\}$. Then there exists a unique non negative integer $q_l < m$ such that for each $j \in \mathbb{N}$ holds*

$$mj \equiv l \pmod{r} \iff mj = r(mq + q_l) + l \,, \tag{A}$$

where $q \in \mathbb{N}$.

Proof. As $(m,r) = 1$ there exists $\overline{l} < r$ such that $mj \equiv l \pmod{r} \Leftrightarrow j \equiv \overline{l} \pmod{r}$ and so $j = rq + \overline{l}$. Therefore $mj = mrq + m\overline{l}$. Clearly $m\overline{l} = rq_l + l$, $q_l < m$ and so $mj = r(mq + q_l) + l$. $\square$

This work has two parts. The object of the first part is the ideal convergence. In the second part we shall study some properties of $\overline{h}$.

1. Ideal convergence

We recall that the Buck's measure density is a finitely additive measure on the algebra $\mathcal{D}_\mu \subseteq 2^{\mathbb{N}}$ and

$$\mathcal{I}_\mu = \big\{ A \subseteq \mathbb{N} \colon \mu(A) = 0 \big\}$$

is an admissible ideal. Now we construct a sequence which has very different properties from the point of view of $\mathcal{I}_\mu$-convergence and $\mathcal{I}_\mu^*$-convergence. Consider for $k = 1, 2, \dots$ the sets

$$A_k = \{n! + k \colon n = 1, 2, \dots\}.$$

We define the sequence $f(n)$ as follows: $f(n) = 0$ if $n \notin \bigcup_{k=1}^{\infty} A_k$. For $n \in \bigcup_{k=1}^{\infty} A_k$ we put $f(n) = \frac{1}{l}$, where $l = \min\{k \colon n \in A_k\}$.

THEOREM 1.1. *For the sequence $\big(f(n)\big)$ it holds:*

(a) $I_\mu\text{-}\lim\limits_{n\to\infty} f(n) = 0$,

(b) *If $A \in \mathcal{D}_\mu$ and $\lim\limits_{\substack{n\to\infty\\ n\in A}} f(n) = 0$ then $\mu(A) = 0$.*

Proof. For arbitrary m we have $n! + k \in m\mathbb{N} + k$ for $n \geqslant m$. Thus $A_k \dot\subset m\mathbb{N} + k$ and so from the definition of μ^* we have $\mu^*(A_k) \leqslant \frac{1}{m}$ for $m = 1, 2, \dots$ and this implies $\mu(A_k) = 0$, $k = 1, 2, \dots$

If $\varepsilon > 0$, then

$$\big\{n \colon f(n) > \varepsilon\big\} \subseteq A_1 \cup \dots \cup A_{k_0},$$

where k_0 is the biggest positive integer $\leqslant \frac{1}{\varepsilon}$. From the additivity of μ we have $\mu\big(\{n \colon f(n) > \varepsilon\}\big) = 0$ and (a) follows.

Let $A \in \mathcal{D}_\mu$ and $\mu(A) > 0$. Then from the definition of $\mathcal{D}_\mu$ we have that

$$m\mathbb{N} + k \dot\subset A$$

for some arithmetic progression, $m, k \in \mathbb{N}$. If we consider $m\mathbb{N} + k \dot\subset A$, then $A_k \dot\subset A$, and so for infinitely many $n \in A$ we have $f(n) \geqslant \frac{1}{k}$. Therefore the relation $\lim\limits_{\substack{n\to\infty\\ n\in A}} f(n) = 0$ is impossible. ☐

Let $\mathcal{D}$ be the class of the sets $A \subseteq \mathbb{N}$ which have asymptotic density. It holds $\mathcal{D}_\mu \subseteq \mathcal{D}$ (see [4]). Put for $A \subseteq \mathbb{N}$

$$\varrho(A) = \sum_{k=1}^{\infty} \chi_A(k) \cdot 2^{-k},$$

the dyadic number of the set A. Then we consider the sets $\varrho(\mathcal{D}) = \big\{\varrho(A) \colon A \in \mathcal{D}\big\}$, $\varrho(\mathcal{D}_\mu) = \big\{\varrho(A) \colon A \in \mathcal{D}_\mu\big\}$. These sets are subsets of $[0,1]$. In [14] it is proved that $\lambda\big(\varrho(\mathcal{D}_\mu)\big) = 0$, but it is well known that $\lambda\big(\varrho(\mathcal{D})\big) = 1$ ($\lambda(\cdot)$ is the Lebesgue measure). Thus from this point of view the class $\mathcal{D}$ is essentially richer. In the paper [9] the following characterization of the equivalence of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence is given:

Let $\mathcal{I}$ be on admissible ideal. Then $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence are equivalent if and only if the following condition (AP) holds:

Let (B_k) be a sequence of sets from $\mathcal{I}$. Then there exists a set $B \in \mathcal{I}$ such that $B_k \dot{\subset} B$, for $k = 1, 2, \dots$

We remark that this condition is a very weak form of σ-ideal. Typical example of an ideal satisfying this condition is the ideal $\mathcal{I}_d$ (see [12]). From Theorem 1.1 we have that the ideal $\mathcal{I}_\mu$ of the sets with Buck-measure density does not satisfy this condition. Let $u(A)$ be the uniform density of $A \subseteq \mathbb{N}$ (see [2], [3], [5], [7], [8]). Then the ideal

$$\mathcal{I}_u = \{A \subseteq \mathbb{N}\colon u(A) = 0\}$$

also does not satisfy the condition (AP)(see [9]). In [12] is proved that the ideal $\mathcal{I}_d$ satisfies (AP). Ideas from this proof lead to:

Theorem 1.2. *Let $c = (c_n)$, $c_n > 0$, $\sum\limits_{n=1}^{\infty} c_n = +\infty$. Put*

$$\mathcal{I}_c = \{A \subseteq \mathbb{N}\colon h_c(A) = 0\}.$$

Then $\mathcal{I}_c$ is an admissible ideal and $\mathcal{I}_c$ satisfies (AP).

P r o o f. The fact that $\mathcal{I}_c$ is admissible follows immediately from $\sum\limits_{n=1}^{\infty} c_n = +\infty$.

Let $B_1, B_2, \dots \subseteq \mathbb{N}$ and $h_c(B_k) = 0$, $k = 1, 2, \dots$ Put $D_k = B_1 \cup B_2 \dots \cup B_k$, $k = 1, 2, \dots$ Then $D_1 \subset D_2 \subset \dots \subset D_k \subset \dots$ Let $S(N)$ and $S(A, N)$ have the same sense as in the introduction. Clearly $h_c(D_k) = 0$, $k = 1, 2, \dots$ and so there exists a sequence of indices $N_1 < N_2 < \dots < N_k < \dots$ such that

$$\frac{S(D_k, N)}{S(N)} < \frac{1}{k+1}, \qquad N \geqslant N_k,\ k = 1, 2, \dots \tag{1}$$

Put $D_k' = D_k \setminus \{a \in D_k\colon a \leqslant N_k\}$ and $B = \bigcup\limits_{k=1}^{\infty} D_k'$. Then $B_k \subset D_k' \dot{\subset} D_k \subset B$. Thus $B_k \dot{\subset} B$, $k = 1, 2, \dots$ Let $N > N_k$. Thus $N_j < N \leqslant N_{j+1}$ for some $j \geqslant k$. Thus if $n \in \mathbb{N}$, then $n \notin D_h'$ for $h \geqslant j+1$ and so if $n \leqslant N$ and $n \in B$, then $n \in D_l'$ for some $l \leqslant j$. From the monotonicity of (D_r) we have now

$$n \in B \wedge n \leqslant N \implies n \in D_j.$$

This yields $S(B, N) \leqslant S(D_j, N)$. Therefore

$$\frac{S(B, N)}{S(N)} \leqslant \frac{S(D_j, N)}{S(N)}.$$

But $N \geqslant N_j$ and so from (1) we obtain

$$\frac{S(B, N)}{S(N)} \leqslant \frac{1}{j+1} \leqslant \frac{1}{k+1}.$$

This implies $h_c(B) = 0$. $\square$

We have:

Corollary 1.1. *Let $c = (c_n)$, $c_n > 0$, $\sum_{n=1}^{\infty} c_n = +\infty$. Then $\mathcal{I}_c$-convergence and $\mathcal{I}_c^*$-convergence are equivalent.*

Corollary 1.2. *Let (s_n) be a sequence of nonnegative numbers such that*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} c_k s_k}{S(n)} = 0 .$$

Then there exists a set $A \subseteq \mathbb{N}$ such that $h_c(A) = 1$ and $\lim_{\substack{n\to\infty\\ n\in A}} s_n = 0$.

Proof. We prove that $\mathcal{I}_c\text{-}\lim_{n\to\infty} n s_n = 0$. Suppose it is not. Then for some $\varepsilon > 0$, and some sequence of indexes (n_j) it holds

$$\frac{S(B(\varepsilon), n_j)}{S(n_j)} > q > 0 ,$$

where $B(\varepsilon) = \{k\colon s_k > \varepsilon\}$. This yields

$$\sum_{k=1}^{n_j} c_k s_k \geqslant \sum_{\substack{k\leqslant n\\ k\in B(\varepsilon)}} c_k s_k \geqslant \varepsilon \sum_{\substack{k\leqslant n\\ k\in B(\varepsilon)}} c_k \geqslant \varepsilon \cdot S(B(\varepsilon), n_j)$$

and so

$$\frac{\sum_{k=1}^{n_j} c_k s_k}{S(n_j)} \geqslant \varepsilon \cdot q ,$$

which is a contradiction. The assertion follows now from Corollary 1.1 $\square$

2. Some properties of Alexander's density

In 1986 R. Estrada and R. P. Kanwal proved (see [6]), that if $\sum_{k=1}^{\infty} t_k$ is a divergent series with nonnegative terms, then there exists a set $A \subseteq \mathbb{N}$ with zero asymptotic density such that $\sum_{k\in A} t_k = +\infty$. Thus if a series converges through all sets of zero asymptotic density, then this series converges. This fact led to the notion of full class (see [15], [13]).

The class $\Phi \subseteq 2^{\mathbb{N}}$ is called *full class*, if $\bigcup \Phi = \mathbb{N}$, Φ is hereditary and fulfills this condition: for every sequence $(t_k), t_k \geqslant 0$ it holds:

$$\forall M \in \Phi, \quad \sum_{k \in M} t_k < +\infty \implies \sum_{k=1}^{\infty} t_k < +\infty \, .$$

In the paper [6] it is proved, that the ideal $\mathcal{I}_d$ is a full class. This result was improved in [13] for the ideal $\mathcal{I}_u$, where u is the uniform density. It was proved also that $\mathcal{I}_u \subsetneq \mathcal{I}_d$ (see [13]).

In the paper [11] the notion of compact submeasure is introduced. The set function $m\colon 2^{\mathbb{N}} \to [0, +\infty)$ is called *submeasure* if it is monotone ($A \subseteq B \Rightarrow m(A) \leqslant m(B)$) and subadditive ($m(A \cup B) \leqslant m(A) + m(B)$). The submeasure m is called *compact* if $m(\{k\}) = 0$ for $k \in \mathbb{N}$ and fulfils the following condition: for every $\varepsilon > 0$ there exists a decomposition $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_s$ of $\mathbb{N}$ such that $m(A_j) < \varepsilon$ for each $j = 1, 2, \dots s$. In [11] it is proved that in this case the class

$$\Phi = \bigl\{A \subseteq \mathbb{N}\colon m(A) = 0\bigr\}$$

is a full class. Remark that if we consider that for each $m \in \mathbb{N}$

$$\mathbb{N} = (m\mathbb{N}) \cup (m\mathbb{N} + 1) \cup \dots \cup \bigl(m\mathbb{N} + (m-1)\bigr)$$

we obtain that asymptotic density is a compact submeasure ($d(m\mathbb{N} + j) = \frac{1}{m}$), uniform density is a compact submeasure ($u(m\mathbb{N} + j) = \frac{1}{m}$) and for the same reason also Buck's measure density is a compact submeasure.

Now we give two conditions which provide that the Alexander's density is a compact submeasure.

Theorem 2.1. *Let the sequence $c = (c_n)$ satisfy the condition $\sum\limits_{n=1}^{\infty} c_n = +\infty$ and*

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1 \, . \tag{2}$$

Then

(a) *for every $A \subseteq \mathbb{N}$ having the Alexander's density h_c we have*

$$h_c(A+1) = h_c(A) \, .$$

(*where $A + 1 = \{a + 1\colon a \in A\}$*).

(b) *for $m \in \mathbb{N}$, $r = 0, 1, 2, \dots$ we have*

$$h_c(m\mathbb{N} + r) = \frac{1}{m} \, .$$

(c) *Suppose that condition (2) is replaced by*

$$\lim_{n\to\infty} \frac{c_{n+r}}{c_n} = 1 \tag{3}$$

for some $r \in \mathbb{N}$ *and* $\sum\limits_{n \equiv l \,(\mathrm{mod}\ r)} c_n = +\infty$ *for* $l \in \{0, 1, \dots, r-1\}$. *Then for* $m \in \mathbb{N}$ *and* $(m, r) = 1$ *it holds*

$$h_c(m\mathbb{N} + a) = \frac{1}{m},$$

where $a \in \mathbb{N}$.

Proof.

(a) If $\sum\limits_{k \in A} c_k < \infty$, then (2) yields $\sum\limits_{k \in A} c_{k+1} < \infty$ and $h_c(A) = 0 = h_c(A+1)$. Suppose $\sum\limits_{k \in A} c_k = \infty$. Let $A = \{k_1 < k_2 < \dots\}$ Then $S(A+1, n) = \sum\limits_{k_j + 1 \leqslant n} c_{k_j+1} = \sum\limits_{k_j \leqslant n-1} c_{k_j+1}$. Thus from (2) and the Theorem A we have $\frac{S(A+1,n)}{S(A,n-1)} \to 1$ as $n \to \infty$. Clearly

$$\frac{S(A+1,n)}{S(n)} = \frac{S(A,n-1)}{S(n-1)} \cdot \frac{S(n-1)}{S(n)} \cdot \frac{S(A+1,n)}{S(A,n-1)}.$$

It can be verified easily that $\frac{S(n-1)}{S(n)} \to 1$, $n \to \infty$. Thus from the last equality we obtain the assertion. (b)

(b) Using (a) it suffices to prove that $h_c(m\mathbb{N}) = \frac{1}{m}$. From (2) we have that for each $k \in \mathbb{N}$ it holds

$$\lim_{n \to \infty} \frac{c_{n+k}}{c_n} = 1. \tag{4}$$

Denote $y_j = c_{(j-1)\cdot m+1} + \dots + c_{j \cdot m}$, $j = 1, 2, \dots$ The term y_j has m summands and so from (4) we obtain

$$\lim_{n \to \infty} \frac{c_{j \cdot m}}{y_j} = \frac{1}{m}. \tag{5}$$

Clearly $S(j \cdot m) = y_1 + \dots + y_j$ and $S(m\mathbb{N}, j \cdot m) = c_m + \dots + c_{j \cdot m}$. Thus from (5) and Stolz's Theorem we have

$$\lim_{n \to \infty} \frac{S(m\mathbb{N}, j \cdot m)}{S(j \cdot m)} = \frac{1}{m}. \tag{6}$$

Arbitrary $n \in \mathbb{N}$ can be expressed in the form $n = m \cdot j_n + k_n$, where $j_n \to \infty$ as $n \to \infty$ and $k_n < m$. Then we have

$$S(n) - S(m \cdot j_n) \leqslant c_n + \dots + c_{n-m} \tag{7}$$

and

$$S(m\mathbb{N}, n) - S(m\mathbb{N}, m \cdot j_n) \leqslant c_n + \dots + c_{n-m}. \tag{8}$$

The condition (2) guarantees that

$$\lim_{n\to\infty} \frac{c_{n-k}}{c_1+\cdots+c_n} = 0$$

is fulfilled for arbitrary k. Thus (7) yields

$$\lim_{n\to\infty} \frac{S(m\cdot j_n)}{S(n)} = 1\,. \tag{9}$$

From (8) we have

$$\lim_{n\to\infty} \frac{S(m\mathbb{N},n) - S(m\mathbb{N}, m\cdot j_n)}{S(n)} = 0$$

and the assertion follows from (6) and (9).

(c) Put $c_n^{(l)} = c_{r\cdot n+l}$, $n=1,2,\dots$, $l=0,1,\dots,r-1$ and $c^{(l)} = (c_n^{(l)})$. From (3) we obtain that $c^{(l)}$ fulfils (2), thus $h_{c^{(l)}}(m\mathbb{N}+a) = \frac{1}{m}$ for $m\in\mathbb{N}$. Put now $S_l(N) = \sum_{n\leqslant N} c_n^{(l)}$, $S_l(A,N) = \sum_{n\leqslant N} \chi_A(n)c_n^{(l)}$ for $l=0,1,\dots,r-1$. For the sum $S(k\cdot r\cdot m)$, $k\in\mathbb{N}$, trivially we get

$$S(k\cdot r\cdot m) = S_0(k\cdot m) + S_1(k\cdot m-1) + \cdots + S_{r-1}(k\cdot m-1)\,.$$

For the expression $S(m\mathbb{N}, k\cdot m\cdot r) = \sum_{l=0}^{r-1} \sum_{\substack{j\cdot m\equiv l\\ j\cdot m\leqslant m\cdot k\cdot r}} c_{j\cdot m}$. But from (A) we obtain for $l\neq 0$

$$\begin{aligned}\sum_{\substack{j\cdot m\equiv l\\ j\cdot m\leqslant m\cdot k\cdot r}} c_{j\cdot m} &= \sum_{r\cdot(m\cdot q+q_l)+l\leqslant k\cdot r\cdot m} c_{r\cdot(m\cdot q+q_l)+l}\\ &= \sum_{q\leqslant k-\frac{q_l}{m}-\frac{l}{m\cdot r}} c^{(l)}_{m\cdot q+q_l} = S_l(m\mathbb{N}+q_l, k\cdot m-1)\,.\end{aligned}$$

If $l=0$ we can calculate that $\sum_{\substack{j\cdot m\equiv 0\\ j\cdot m\leqslant m\cdot k\cdot r}} c_{j\cdot m} = S_0(m\mathbb{N}+q_0, k\cdot m)$. Clearly $\dfrac{S_l(m\mathbb{N}+q_l,k\cdot m-1)}{S_l(k\cdot m-1)} \to \dfrac{1}{m}$ as $k\to\infty$ and so

$$\lim_{n\to\infty} k\frac{S(m\mathbb{N},k\cdot m\cdot r)}{S(k\cdot m\cdot r)} = \frac{1}{m}\,.$$

If we consider that $\dfrac{c_{n-j}}{c_1+\cdots+c_n} \leqslant \dfrac{c_{n-j}}{\sum_{\substack{s\equiv n-j\\ s\leqslant n}} c_s}$ we obtain also in this case $\dfrac{c_{n-j}}{c_1+\cdots+c_n} \to 0$ as $n\to\infty$ and so similarly as in (b) we can prove $h_c(m\mathbb{N}) =$

$\frac{1}{m}$. Similarly as in (a) it can be proved that (3) implies $h_c(A+r) = h_c(A)$. But $r, 2r, \dots, mr$ is a complete remainder system modulo m and (c) follows. □

Note that the condition $(m,r)=1$ in part c) of previous theorem cannot be omitted. For example, if we choose $m = r = 2$, $c_{2n} = 2n$, $c_{2n+1} = \frac{1}{2n+1}$, $n = 1, 2 \dots$, then the conditions in Theorem 2.1 (c) are fulfilled, but $(m,r) \neq 1$ and $h_c(2\mathbb{N}) = 1$, $h_c(2\mathbb{N}+1) = 0$.

If we consider the decomposition $\mathbb{N} = \bigcup_{j=0}^{m-1} m\mathbb{N} + j$ we obtain:

COROLLARY 2.1. *If* $\lim_{n\to\infty} n\frac{c_{n+1}}{c_n} = 1$, *then* $\overline{h}_c(A)$ *is a compact submeasure.*

THEOREM 2.2. *Let* $c = (c_n)$ *be a sequence of nonnegative numbers such that* $\sum_{n=1}^{\infty} c_n = +\infty$ *and* $\sum_{n=1}^{\infty} |c_n - c_{n+1}| = M < +\infty$. *Then for every arithmetic progression* $m\mathbb{N}+r$, $m \in \mathbb{N}$, $r = 0, 1, 2, \dots$, *we have*

$$h_c(m\mathbb{N}+r) = \frac{1}{m}.$$

P r o o f. Let $r \in \{0, 1, \dots, m-2\}$. Then

$$\left|h_n(m\mathbb{N}+r) - h_n(m\mathbb{N}+r+1)\right| \leqslant \frac{1}{S(n)} \sum_{k=0}^{k_n} \left|c_{k\cdot m+r} - c_{k\cdot m+r+1}\right| + \frac{\delta_n}{S(n)},$$

where $k_n \in \mathbb{N}$ and $\delta_n = 0$ or δ_n is a term of (c_n). Therefore

$$\left|h_n(m\mathbb{N}+r) - h_n(m\mathbb{N}+r+1)\right| \leqslant \frac{M}{S(n)} + \frac{\delta_n}{S(n)}$$

and thus

$$\lim_{n\to\infty} n\left|h_n(m\mathbb{N}+r) - h_n(m\mathbb{N}+r+1)\right| = 0.$$

By induction we obtain

$$\lim_{n\to\infty} n\big(h_n(m\mathbb{N}+r) - h_n(m\mathbb{N}+q)\big) = 0$$

for arbitrary two remainders r, q. This yields

$$h_n(m\mathbb{N}+r) = \nu_n + \varepsilon_n^r, \qquad n = 1, 2, \dots, \tag{10}$$

and $r \in \{0, 1, \dots, m-1\}$, where $\varepsilon_n^r \to 0$ as $n \to \infty$. The arithmetic progression $m\mathbb{N}+r$, $r = 0, 1, \dots, m-1$, form a disjoint decomposition and

$$\sum_{r=0}^{m-1} h_n(m\mathbb{N}+r) = 1.$$

From (10) we have now

$$m\nu_n + \sum_{r=0}^{m-1} \varepsilon_n^r = 1 .$$

This implies $\nu_n \to \frac{1}{m}$ as $n \to \infty$. The condition $\sum_{n=1}^{\infty} c_n = \infty$ guarantees that $h_c(A) = 0$ if A is a finite set. For an arbitrary non negative integer r we have $m\mathbb{N} + r = (m\mathbb{N} + r') \setminus A$ where $r' < m$, $r \equiv r' \pmod{m}$ and A is a finite set. The assertion follows. ☐

Corollary 2.2. *If $c = (c_n)$ fulfils the condition of Theorem 2.2, then $\overline{h}_c$ is a compact submeasure.*

Corollary 2.3. *If $c = (c_n)$ fulfils the conditions $(a), (b)$ of Theorem 2.1 or Theorem 2.2, then for each set $A \in \mathcal{D}_\mu$ (measurable in the sense of Buck) it holds $h_c(A) = \mu(A)$.*

Proof. The assertion follows immediately from the fact that if $A \in \mathcal{D}_\mu$, then for arbitrary $\varepsilon > 0$ the arithmetic progressions $a_1 + m\mathbb{N}, \dots, a_j + m\mathbb{N}$ are disjoint, $b_1 + m\mathbb{N}, \dots, b_k + m\mathbb{N}$ are also disjoint such that

$$\bigcup_{i=1}^{j} a_i + m\mathbb{N} \subseteq A \subseteq \bigcup_{i=1}^{k} b_i + m\mathbb{N}$$

and $\dfrac{k}{m} - \dfrac{j}{m} < \varepsilon$. ☐

Remark that $c_n = n^\alpha$ is an example of $c = (c_n)$ which satisfies the conditions of Theorem 2.1. If $\alpha \geqslant 1$ then this sequence does not satisfy the conditions of Theorem 2.2. Further put $c_m = \frac{1}{m}$ for $m \neq 3^n$ and $c_m = \frac{1}{2m}$ for $m = 3^n$. Then the sequence (c_m) satisfies the conditions in Theorem 2.2 but does not satisfy the condition in Theorem 2.1.

Now we give a necessary and sufficient condition for $\overline{h}_c$ to be a compact submeasure:

Theorem 2.3. *Let $\sum_{n=1}^{\infty} c_n = +\infty$. Then the Alexander's upper density $\overline{h}_c$ is a compact submeasure if and only if*

$$\lim_{n\to\infty} n \frac{c_n}{S(n)} = 0 . \tag{11}$$

Before the proof we recall what is the Darboux property. We say that a set function m defined on a class $\mathcal{Z}$ has the *Darboux property* if and only if for every set $B \in \mathcal{Z}$ and for every $x \in [0, m(B)]$ there exists a subset $A \subseteq B$, $A \in \mathcal{Z}$, such that $m(A) = x$.

P r o o f o f T h e o r e m 2.3. In [10] it is proved that h_c has Darboux property if and only if (11) holds. If (11) holds then $\mathbb{N}$ can be decomposed into $\mathbb{N} = \mathbb{N}_1^{(1)} \cup \mathbb{N}_2^{(2)}$ such that $h_c(\mathbb{N}_1^{(1)}) = \frac{1}{2} = h_c(\mathbb{N}_2^{(2)})$. In this way we can construct by induction a decomposition $\mathbb{N} = \mathbb{N}_1^{(k)} \cup \dots \cup \mathbb{N}_{2^k}^{(k)}$ such that $h_c(\mathbb{N}_j^{(k)}) = \frac{1}{2^k}$, $j = 1, 2, \dots 2^k$, $k = 1, 2, \dots$, and this implies that h_c is a compact submeasure.

Let (11) do not hold. Then there exists an infinite sequence (n_k) and $\alpha > 0$ such that

$$\frac{c_{n_k}}{S(n_k)} > \alpha \,, \tag{12}$$

$k = 1, 2, \dots$ Consider a decomposition $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_s$. This decomposition is finite, therefore one of the sets $A_1, \dots, A_s$ must contain infinitely many elements of (n_k). Let A_m contain $\{n_{k(1)}, n_{k(2)}, \dots\}$. Then

$$h_{n_{k(j)}}(A_m) \geqslant \frac{c_{n_{k(j)}}}{S(n_{k(j)})} > \alpha$$

and so $\overline{h}_c(A_m) \geqslant \alpha$. Therefore $\overline{h}_c$ is not a compact submeasure. □

COROLLARY 2.4. *Let* $\sum\limits_{n=1}^{\infty} c_n = +\infty$. *Then* $\overline{h}_c$ *has the Darboux property if and only if* $\overline{h}_c$ *is a compact submeasure.*

In the paper [15] there are studied some classes $\mathcal{L} \subseteq 2^{\mathbb{N}}$ which are full classes. The set $A \subseteq \mathbb{N}$ is called *lacunary* if $A = \{a_1 < a_2 < \dots\}$ and $\lim\limits_{n\to\infty} n(a_{n+1} - a_n) = +\infty$. The class of lacunary sets is a full class. In the following theorem we give a sufficient condition for having $\overline{h}_c(A) = 0$ for each lacunary set A.

THEOREM 2.4. *Let* $c_k > 0$, $k = 1, 2, \dots$, *and* $\sum\limits_{k=1}^{\infty} c_k = +\infty$. *If*

$$\frac{\max\limits_{1\leqslant k\leqslant n} c_k}{S(n)} = O\left(\frac{1}{n}\right) \tag{13}$$

then $h_c(A) = 0$ *for each lacunary set* A.

COROLLARY 2.5. *If* (13) *holds, then the system*

$$\mathcal{S}_{h_c}^0 = \{A \subseteq \mathbb{N} \colon h_c(A) = 0\}$$

is a full class.

P r o o f o f T h e o r e m 2.4. Let $A = \{a_1 < a_2 < \dots\}$ be a lacunary set. Then for every $M > 0$ there exists k_0 that for $k \geqslant k_0$ it holds

$$a_{k+1} - a_k > M \,. \tag{14}$$

Then we have

$$h_n(A) = \frac{\sum\limits_{j=1}^{n} \chi_A(j)c_j}{S(n)} = \frac{\sum\limits_{j=1}^{k_0} \chi_A(j)c_j}{S(n)} + \frac{\sum\limits_{j=k_0+1}^{n} \chi_A(j)c_j}{S(n)} =: \nu_n + P_n\,.$$

Clearly $\nu_n \to 0$ as $n \to \infty$ and for P_n we have a trivial bound

$$P_n \leqslant \frac{\max\limits_{k \leqslant n} c_k \cdot A(k_0+1, n)}{S(n)}\,, \tag{15}$$

where $A(k_0+1, n)$ is the number of elements of A in $[k_0+1, n)$. Let $p = A(k_0+1, n)$. Then $a_{j+1}, a_{j+2}, \dots, a_{j+p} \in [k_0+1, n)$ for some $j \geqslant k_0$. From (14) we obtain $a_{j+p} > a_{j+1} + M \cdot p$. This yields $a_{j+1} + M \cdot p < n$ and thus $p < \frac{n}{M}$. From (15) we obtain now

$$P_n \leqslant \frac{\max\limits_{k \leqslant n} c_k}{S(n)} \cdot \frac{n}{M}\,.$$

This yields

$$\overline{h}_c(A) = O\left(\frac{1}{M}\right).$$

If we let $M \to \infty$, we obtain $h_c(A) = 0$. □

Example. Let the sequence (c_n) satisfy the following condition:

$$0 < a \leqslant c_n \leqslant b < \infty\,,$$

a, b fixed, $n = 1, 2, \dots$ Then $S(n) \geqslant n \cdot a$ and so

$$\frac{\max\limits_{k \leqslant n} c_k}{S(n)} \leqslant \frac{b}{n \cdot a} = O\left(\frac{1}{n}\right).$$

The condition (13) does not hold for $c_n = \frac{1}{n^\alpha}$, $\alpha \in (0, 1]$. If $\alpha < 1$, then $S(n) \sim \int\limits_1^n \frac{dt}{t^\alpha} \sim \frac{n^{-\alpha+1}}{1-\alpha}$. Therefore

$$\frac{\max\limits_{k \leqslant n} c_k}{S(n)} = \frac{c_1}{S(n)} \sim c_1 \frac{1-\alpha}{n^{1-\alpha}} \neq O\left(\frac{1}{n}\right).$$

For $\alpha = 1$ we have $S(n) \sim \log n$.

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