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ON ESTIMATIONS OF DISPERSIONS OF CERTAIN DENSE BLOCK SEQUENCES

Ferdinánd Filip — János T. Tóth

ABSTRACT. Properties of dispersion of block sequences were investigated by [Tóth, J. T., Mišík, L., Filip, F.: *On some properties of dispersion of sequences of positive integers*, Math. Slovaca, **54** (2004), 453–464]. The present paper is a continuation of the study of relations between the density of a block sequence and the so called dispersion of a block sequence.

Preliminaries

In this part we recall some basic definitions. Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For $X \subset \mathbb{N}$ let $X(n) = \operatorname{card}\{x \in X : x \leqslant n\}$. Define

$$\underline{d}(X) = \liminf_{n\to\infty} \frac{X(n)}{n}, \quad \overline{d}(X) = \limsup_{n\to\infty} \frac{X(n)}{n}, \quad d(X) = \lim_{n\to\infty} \frac{X(n)}{n}$$

as the *lower asymptotic density, upper asymptotic density*, and *asymptotic density* (if they exist), respectively.

In the whole paper we will assume that X is infinite. Denote by $R(X) = \left\{\frac{x}{y} : x \in X,\ y \in X\right\}$ the *ratio set of* X and say that a set X is (R)-dense if $R(X)$ is (topologically) dense in the set $\mathbb{R}^+$. Let us notice that the concept of (R)-density was defined and first studied in papers [Š1] and [Š2].

Now let $X = \{x_1, x_2, \dots\}$ where $x_n < x_{n+1}$ are positive integers. The following sequence of finite sequences derived from X

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

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is called *a block sequence* of the sequence X. Thus the block sequence is formed by blocks $X_1, X_2, \dots, X_n, \dots$, where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right); \quad n = 1, 2, \dots$$

is called the *nth block*. This kind of block sequences were studied in the paper [S–T 2]. For every $n \in \mathbb{N}$ let

$$D(X_n) = \max\left\{\frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \dots, \frac{x_{i+1} - x_i}{x_n}, \dots, \frac{x_n - x_{n-1}}{x_n}\right\},$$

the maximum distance between two consecutive terms in the nth block. In this paper we will consider the following characteristic, called the *dispersion* of the sequence X

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n),$$

and its relations to the previously mentioned (R)-density. Notice that the (R)-density of the set X is equivalent to the density of its block sequence in the interval $(0, 1)$.

When calculating the value $\underline{D}(X)$ the following theorems are often useful.

(A1) *Let*

$$X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N},$$

where $x_n < x_{n+1}$ *and let* $c_n < d_n < c_{n+1}$, *for* $n \in \mathbb{N}$, *be positive integers. Then*

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i \colon i = 1, 2, \dots, n\}}{d_{n+1}}.$$

(See [TMF], Theorem 1.)

(A2) *Let* X *be of the same form as in Theorem* (A1). *Suppose that there exists a positive integer* n_0 *such that for all integers* $n > n_0$

$$c_{n+1} - d_n \leqslant c_{n+2} - d_{n+1}.$$

Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}}.$$

(See [TMF], Corollary 1.)

Results

The following relations between asymptotic density and (R)-density are known.

(B1) *If* $\underline{d}(X) + \overline{d}(X) \geqslant 1$ *then* X *is* (R)*-dense*, (see [S-T1], Corollary.)

(B2) *Let* $X = \{x_1, x_2, \dots, x_n, \dots\} = \bigcup_{n=1}^{\infty} \langle c_n, d_n\rangle \cap \mathbb{N}$ *be an* (R)*-dense set where* $c_n < d_n < c_{n+1}$ *for all* $n \in \mathbb{N}$*. Then*

$$\limsup_{n\to\infty} \frac{d_n}{c_n} \geqslant \liminf_{n\to\infty} \frac{c_{n+1}}{d_n}\,.$$

(See [TMF], Lemma 1.)

It is natural to ask whether the set X is (R)-dense if the following inequality

$$\lim_{n\to\infty} \frac{d_n}{c_n} \geqslant \lim_{n\to\infty} \frac{c_{n+1}}{d_n}$$

holds. The answer is positive as the following theorem shows. Moreover the theorem states that $\underline{d}(X) + \overline{d}(X) \geqslant 1$ what is a more general property then the (R)-density of X (see (B1)).

Theorem 1. *Let* $X = \bigcup_{n=1}^{\infty} \langle c_n, d_n\rangle \cap \mathbb{N}$, *where* $c_n < d_n < c_{n+1}$ *are positive integers for all* $n \in \mathbb{N}$*. If there exist finite limits*

$$b = \lim_{n\to\infty} \frac{c_{n+1}}{d_n}\,, \qquad a = \lim_{n\to\infty} \frac{d_n}{c_n}\,, \tag{2}$$

and if we have

$$a \geqslant b \quad \text{and} \quad a > 1\,,$$

then the block sequence (1) *of the set* X *is dense in* $(0,1)$, *and, moreover,*

$$\underline{d}(X) + \overline{d}(X) = 1 + \frac{a-b}{ab-1} \geqslant 1\,.$$

P r o o f. Let ε be a positive real number such that

$$(a-\varepsilon)(b-\varepsilon) > 1\,. \tag{3}$$

From (2) it follows that there exists a positive integer n_0 that for every $n > n_0$ the following inequalities are valid

$$a - \varepsilon < \frac{d_n}{c_n} < a + \varepsilon \qquad \text{and} \qquad b - \varepsilon < \frac{c_{n+1}}{d_n} < b + \varepsilon\,. \tag{4}$$

Let m_0 denote the least positive integer such that $m_0 > n_0$ and

$$\sum_{i=1}^{n_0} \frac{d_i - c_i}{d_{m_0}} < \varepsilon . \tag{5}$$

Now let m be an integer where $m > m_0$, then

$$\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} = \frac{\sum_{i=n_0+1}^{m}(d_i - c_i)}{d_m} + \frac{\sum_{i=1}^{n_0}(d_i - c_i)}{d_m} .$$

From this, (4) and (5) we obtain that

$$\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \leqslant \max_{n_0 < i \leqslant m} \left\{ \frac{d_i}{c_i} - 1 \right\} \sum_{i=n_0+1}^{m} \frac{c_i}{d_m} + \varepsilon \leqslant (a + \varepsilon - 1) \sum_{i=n_0+1}^{m} \frac{c_i}{d_m} + \varepsilon . \tag{6}$$

Furthermore, (4) yields that

$$\frac{c_i}{d_m} = \left(\frac{c_i}{d_i} \frac{d_i}{c_{i+1}} \right) \dots \left(\frac{c_{m-1}}{d_{m-1}} \frac{d_{m-1}}{c_m} \right) \frac{c_m}{d_m} \leqslant \frac{1}{a - \varepsilon} \left(\frac{1}{(a - \varepsilon)(b - \varepsilon)} \right)^{m-i} \tag{7}$$

for every i, $n_0 < i \leqslant m$. Then, using the previously derived relations and taking into account (6), we have

$$\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \leqslant (a + \varepsilon - 1) \sum_{i=n_0+1}^{m} \frac{1}{a - \varepsilon} \left(\frac{1}{(a - \varepsilon)(b - \varepsilon)} \right)^{m-i} + \varepsilon . \tag{8}$$

Then (3) and (8) yield that

$$\begin{aligned} \frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} &< (a + \varepsilon - 1) \sum_{j=0}^{\infty} \frac{1}{a - \varepsilon} \left(\frac{1}{(a - \varepsilon)(b - \varepsilon)} \right)^{j} + \varepsilon \\ &= \frac{(a + \varepsilon - 1)(b - \varepsilon)}{(a - \varepsilon)(b - \varepsilon) - 1} + \varepsilon . \end{aligned} \tag{9}$$

Since (9) holds for every sufficiently small $\varepsilon > 0$, letting $\varepsilon \to 0$, we have

$$\limsup_{m \to \infty} \frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \leqslant \frac{(a-1)b}{ab - 1} . \tag{10}$$

On the other hand,

$$\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \geqslant \frac{\sum_{i=n_0}^{m}(d_i - c_i)}{d_m} \geqslant \min_{n_0 < i \leqslant m} \left\{ \frac{d_i}{c_i} - 1 \right\} \sum_{i=n_0+1}^{m} \frac{c_i}{d_m} . \tag{11}$$

Further, (4) yields that

$$\frac{c_i}{d_m} = \left(\frac{c_i}{d_i}\frac{d_i}{c_{i+1}}\right)\cdots\left(\frac{c_{m-1}}{d_{m-1}}\frac{d_{m-1}}{c_m}\right)\frac{c_m}{d_m} \geqslant \frac{1}{a+\varepsilon}\left(\frac{1}{(a+\varepsilon)(b+\varepsilon)}\right)^{m-i}$$

for every i, $n_0 < i \leqslant m$. That is why we can obtain from (4) and (11) that

$$\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \geqslant (a-\varepsilon-1)\sum_{i=n_0+1}^{m}\frac{1}{a+\varepsilon}\left(\frac{1}{(a+\varepsilon)(b+\varepsilon)}\right)^{m-i}.$$

This implies

$$\liminf_{m\to\infty}\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \geqslant \frac{(a-\varepsilon-1)(b+\varepsilon)}{(a+\varepsilon)(b+\varepsilon)-1}.$$

Letting $\varepsilon \to 0$ in the previous inequality, we obtain

$$\liminf_{m\to\infty}\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} \geqslant \frac{(a-1)b}{ab-1}. \tag{12}$$

The relations (10) and (12) yield that there exists the limit

$$\lim_{m\to\infty}\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} = \frac{(a-1)b}{ab-1}. \tag{13}$$

For $\overline{d}(X)$ the following equality holds

$$\overline{d}(X) = \limsup_{m\to\infty}\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} = \lim_{m\to\infty}\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} = \frac{(a-1)b}{ab-1}. \tag{14}$$

From the definition of $\underline{d}(X)$, (2) and (13) we also have

$$\underline{d}(X) = \liminf_{m\to\infty}\frac{\sum_{i=1}^{m}(d_i - c_i)}{c_{m+1}} = \lim_{m\to\infty}\frac{d_m}{c_{m+1}}\frac{\sum_{i=1}^{m}(d_i - c_i)}{d_m} = \frac{a-1}{ab-1}. \tag{15}$$

The fact that $a \geqslant b$, $a > 1$, thus (14) and (15) yield that

$$\underline{d}(X) + \overline{d}(X) = \frac{a-1}{ab-1} + \frac{(a-1)b}{ab-1} = \frac{(a-1)(b+1)}{ab-1} = 1 + \frac{a-b}{ab-1} \geqslant 1.$$

$\square$

The basic properties of the dispersion $\underline{D}(X)$ and the relations between dispersion and (R)-density are investigated in the paper [TMF]. The next theorem states the upper bound for dispersions $\underline{D}(X)$ of (R)-dense sets where $1 \leqslant a = \lim_{n\to\infty}\frac{d_n}{c_n} < \infty$.

THEOREM ([TMF, Theorem 12.]). *Let* $X = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle \cap \mathbb{N}$ *be an* (R)*-dense set, where* $c_n < d_n < c_{n+1}$ *for all* $n \in \mathbb{N}$*, and suppose that the limit* $\lim_{n\to\infty} \frac{d_n}{c_n} = a$ *exists. Then*

$$\underline{D}(X) \leqslant \min\left\{\frac{1}{a+1}, \max\left\{\frac{a-1}{a^2}, \frac{1}{a^2}\right\}\right\},$$

i.e.,

$$\underline{D}(X) \leqslant \begin{cases} \dfrac{1}{a+1} & \text{if } a \in \left\langle 1, \dfrac{1+\sqrt{5}}{2}\right\rangle, \\ \dfrac{1}{a^2} & \text{if } a \in \left\langle \dfrac{1+\sqrt{5}}{2}, 2\right\rangle, \\ \dfrac{a-1}{a^2} & \text{if } a \in \langle 2, \infty). \end{cases}$$

The following theorem shows that in the third case (if $a \geqslant 2$), the dispersion $\underline{D}(X)$ can be any number in the interval $\langle 0, \frac{a-1}{a^2}\rangle$, where $X = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle \cap \mathbb{N}$ is (R)-dense and $\lim_{n\to\infty} \frac{d_n}{c_n} = a$. Thus the upper bound for $\underline{D}(X)$ is the best possible in the case $a \geqslant 2$. We have:

THEOREM 2. *Let* $a \geqslant 1$ *be a real number and* k *be an arbitrary natural number. Then for every* $\alpha \in \langle 0, \frac{a^k-1}{a^{2k}}\rangle$ *there exists an* (R)*-dense set*

$$X = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle \cap \mathbb{N},$$

where $c_n < d_n < c_{n+1}$ *are positive integers for every* $n \in \mathbb{N}$*, such that* $\lim_{n\to\infty} \frac{d_n}{c_n} = a$ *and* $\underline{D}(X) = \alpha$.

Proof. There are two possible cases: $\alpha > 0$ or $\alpha = 0$.

First, let us consider the case $\alpha > 0$. One can easily see that in this case $a > 1$, and there exists β, $1 \leqslant \beta < a^k$, such that

$$\alpha = \frac{a^k - \beta}{a^{2k}}.$$

Let us define the set

$$Y = \bigcup_{n=1}^{\infty} \langle a_n, b_n \rangle \cap \mathbb{N}$$

as follows:

Let $a_1 = 1$, and for every $n \in \mathbb{N}$ we put

$$b_n = [a^k a_n] + 1 \qquad \text{and} \qquad a_{n+1} = \left[\frac{a^k}{\beta} b_n\right] + 1 . \tag{16}$$

It can be easily seen that $a_n < b_n < a_{n+1}$ for every $n \in \mathbb{N}$ because $a^k > 1$ and $\frac{a^k}{\beta} > 1$.

Also, as $a_n \to \infty$ and $b_n \to \infty$, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\left[\frac{a^k}{\beta} b_n\right]}{b_n} = \frac{a^k}{\beta} \tag{17}$$

and

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{a^k a_n}{a_n} = a^k. \tag{18}$$

Let us apply Theorem 1. We have $\underline{d}(Y) + \overline{d}(Y) \geqslant 1$ because $\frac{a^k}{\beta} \leqslant a^k$ and $a^k > 1$. Thus Y is (R)-dense.

Now we will to show that for every appropriately large n

$$a_{n+2} - b_{n+1} > a_{n+1} - b_n . \tag{19}$$

Since (17) holds and $b_n \to \infty$, $\frac{a^k}{\beta} > 1$, so that

$$\lim_{n\to\infty} (a_{n+1} - b_n) = \lim_{n\to\infty} b_n \left(\frac{a_{n+1}}{b_n} - 1\right) = \lim_{n\to\infty} b_n \left(\frac{a^k}{\beta} - 1\right) = \infty . \tag{20}$$

Then

$$\begin{aligned} a_{n+2} - b_{n+1} &= \left[\frac{a^k}{\beta} b_{n+1}\right] - [a^k a_{n+1}] > \frac{a^k}{\beta} b_{n+1} - 1 - a^k a_{n+1} \\ &= \frac{a^k}{\beta}\Big([a^k a_{n+1}] + 1\Big) - a^k\left(\left[\frac{a^k}{\beta} b_n\right] + 1\right) - 1 \\ &> \frac{a^{2k}}{\beta} a_{n+1} - \left(\frac{a^{2k}}{\beta} b_n + a^k\right) - 1 \\ &= a_{n+1} - b_n + \left(\frac{a^{2k}}{\beta} - 1\right)(a_{n+1} - b_n) - (a^k + 1) . \end{aligned} \tag{21}$$

As $\frac{a^{2k}}{\beta} > 1$, the relations (20) and (21) imply that (19) is true for every sufficiently large n.

According to (19) and Theorem (A2) we have

$$\underline{D}(Y) = \liminf_{n\to\infty} \frac{a_{n+1} - b_n}{b_{n+1}} = \liminf_{n\to\infty} \frac{b_n}{a_{n+1}} \frac{\frac{a_{n+1}}{b_n} - 1}{\frac{b_{n+1}}{a_{n+1}}} .$$

By (17) and (18) we have

$$\underline{D}(Y) = \frac{\beta}{a^k}\frac{\frac{a^k}{\beta}-1}{a^k} = \frac{a^k-\beta}{a^{2k}}\,.$$

Now, let us define $X \subseteq \mathbb{N}$ in the following way:

$$X = Y \setminus Z\,,$$

where

$$Z = \bigcup_{n=1}^{\infty}\left\{\left[a_n\left(\frac{b_n}{a_n}\right)^{\frac{l}{k}}\right] : l = 1,2,\dots,k-1\right\}\,. \tag{22}$$

Now let $s \in \mathbb{N}$ be a natural number such that $b_n \leqslant s < b_{n+1}$. Then

$$0 \leqslant \frac{Z(s)}{s} \leqslant \frac{(n+1)(k-1)}{b_n}\,. \tag{23}$$

From the definition of b_n we can see that $\lim\limits_{n\to\infty}\frac{n}{b_n} = 0$. From (23) it is evident that $d(Z) = 0$. From this we have $\underline{d}(X) = \underline{d}(Y)$ and $\overline{d}(X) = \overline{d}(Y)$. Then $\underline{d}(X) + \overline{d}(X) \geqslant 1$, so the set X is (R)-dense.

Then, if the set X is given in the form

$$X = \bigcup_{n=1}^{\infty}(c_n, d_n\rangle \cap \mathbb{N}\,,$$

where $c_n < d_n < c_{n+1}$ are positive integers for every $n \in \mathbb{N}$, then trivially

$$\frac{d_n}{c_n} = \frac{\left[a_n\left(\frac{b_n}{a_n}\right)^{\frac{l}{k}}\right]+1}{\left[a_n\left(\frac{b_n}{a_n}\right)^{\frac{l-1}{k}}\right]}$$

for some $l = 1,2,\dots,k-1$. That is why

$$\lim_{n\to\infty}\frac{d_n}{c_n} = \lim_{n\to\infty}\frac{a_n\left(\frac{b_n}{a_n}\right)^{\frac{l}{k}}}{a_n\left(\frac{b_n}{a_n}\right)^{\frac{l-1}{k}}} = \lim_{n\to\infty}\left(\frac{b_n}{a_n}\right)^{\frac{1}{k}} = (a^k)^{\frac{1}{k}} = a\,.$$

On the other hand, from the definition of X it can be seen that for a sufficiently large n

$$c_{n+1} - d_n = \max\{c_{i+1} - d_i : i = 1,2,\dots,n\}$$

if and only if

$$c_{n+1} = a_{l+1} \quad \text{and} \quad d_n = b_l$$

for some l.

We will show that $\underline{D}(X) = \underline{D}(Y)$. Let us define the set X in the following way:

$$X = \{x_1 < x_2 < \dots < x_k < \dots\}.$$

Now if $a_n < x_k \leqslant b_n = x_l$ for a sufficiently large n, then by the previous conditions we have

$$D(X_k) = \frac{a_n - b_{n-1}}{x_k}.$$

From this it can be seen that

$$D(X_k) \geqslant D(X_l).$$

Thus

$$\underline{D}(X) = \liminf_{k\to\infty} D(X_k) = \liminf_{n\to\infty} \frac{a_{n+1} - b_n}{b_{n+1}} = \underline{D}(Y).$$

If $\alpha = 0$, then we define the set X in the following way: $c_1 = 2$ and $d_n = [ac_n] + 1$, $c_{n+1} = d_n + 1$ for all $n \in \mathbb{N}$. Then trivially we have $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = a$.

If the set X is given in the form

$$X = \{x_1 < x_2 < \dots < x_n < \dots\},$$

then

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n) = \liminf_{n\to\infty} \frac{2}{x_n} = 0,$$

and $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ which means that the set X is (R)-dense. □

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Ferdinánd Filip
Department of Mathematics
Pedagogical Faculty
University of J. Selye
Roľníckej školy 1519
SK–945 01, Komárno
SLOVAKIA
E-mail: filip.ferdinand@selyeuni.sk

János T. Tóth
Department of Mathematics
Faculty of Science
University of Ostrava
30. dubna 22
CZ–701 03 Ostrava
CZECH REPUBLIC

Institute for Research and Applications of Fuzzy Modeling
University of Ostrava
30. dubna 22
CZ–701 03 Ostrava
CZECH REPUBLIC
E-mail: janos.toth@osu.cz