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AN ERGODIC THEOREM OF ARITHMETIC TYPE

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ABSTRACT. Let $d(n)$ be the divisor function. Let $X_1, X_2, \dots$ be a sequence of centered iid random variables. We obtain strong laws of large numbers for the sums $\sum_{n=1}^{N} d(n)X_n$. Let τ be any ergodic endomorphism of the torus $\mathbf{T}$. We also establish a universal ergodic theorem, which in particular implies for any square integrable function f, that

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \pm d(n)\, f\circ\tau^n(x)}{\sum_{n=1}^{N} d(n)} = 0\,,$$

for almost all x, where the random signs are independent.

1. Introduction and results

Let $d(n) = \#\{d\colon d|n\}$ be the divisor function. Let $\mathbf{T} = [0,1[\, = \mathbb{R}/\mathbb{Z}$ be the torus equipped with the normalized Lebesgue measure m. Let also $\mathcal{X} = \{X, X_1, X_2, \dots\}$ be a sequence of centered, independent random variables. We are interested in what follows in the study of the weighted sums

$$\sum_{n=1}^{N} d(n)X_n\,, \quad \sum_{n=1}^{N} d(n)X_n T^n f\,, \tag{1}$$

$N = 1, 2\dots$, where for the second sums $f \in L^1(m)$, $Tf = f\circ\tau$, and τ is any ergodic endomorphism of $(\mathbf{T}, m)$.

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Theorem 1. *Let $\mathcal{X}$ be any iid sequence of centered, square integrable random variables. Then, for any $b > 3/2$*

$$\limsup_{N\to\infty} \frac{\sum\limits_{n=1}^{N} d(n)X_n}{\Big(\sum\limits_{n=1}^{N} d^2(n)\Big)^{1/2} \log^b\Big(\sum\limits_{n=1}^{N} d^2(n)\Big)} \stackrel{\text{a.s.}}{=} 0 .$$

This result will be deduced from a more general statement. Recall a useful notion ([KSZ] see also [We1] par. 7.4) from the theory of orthogonal series. Let $(T, \mathcal{C}, \tau)$ be some probability space and consider a sequence (f_n) of elements of $L^2(\tau)$. Let $a_{j,k} = \int_T f_j f_k \, d\tau$. A system of functions $\{f_n : n \geqslant 1\}$ such that the quadratic form defined on ℓ_2 by: $(x_n)_n \mapsto \sum_{h,k} a_{h,k} x_h x_k$ is bounded, is said *quasi-orthogonal*. Say also that a sequence $c = (c_k)_k \in \ell_2$ is *universal*, when the series $\sum_n c_n \psi_n$ converges almost everywhere for every orthonormal system of functions $(\psi_n)_n$. According to Schur's Theorem [O, p. 56], if c is universal, then the series $\sum c_n f_n$ converges almost everywhere for any quasi-orthogonal system of functions (f_n). It follows from Rademacher–Menchov Theorem, that we can choose $c_k = k^{-1/2}(\log k)^{-b}$ with $b > 3/2$.

Theorem 2. *Let $\{f_n : n \geqslant 1\}$ be a quasi-orthogonal system of functions. Then, for any $b > 3/2$*

$$\limsup_{N\to\infty} \frac{\sum\limits_{n=1}^{N} d(n)f_n}{\Big(\sum\limits_{n=1}^{N} d^2(n)\Big)^{1/2} \log^b\Big(\sum\limits_{n=1}^{N} d^2(n)\Big)} \stackrel{\text{a.s.}}{=} 0 .$$

Now, we turn to the study of the ergodic sums in (1) and are going to prove the following result.

Theorem 3. *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of iid symmetric, square integrable random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$ and let $p_n = n^s$ where s is some positive integer. Finally, let $\delta > 7$. Then, there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega \setminus N^*$, for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, any $f \in L^2(\mu)$, the series*

$$\sum_{n=1}^{\infty} \frac{d(n)X_n(\omega)}{n^{1/2}\log^{\delta/2} n} T^{p_n} f(x) ,$$

converges μ-almost surely. And, in particular

$$\lim_{N\to\infty} \frac{\sum\limits_{n=1}^{N} d(n) X_n(\omega)\, T^{p_n} f(x)}{\sum\limits_{n=1}^{N} d(n)} = 0\,,$$

μ-almost surely.

The last assertion, when applied to sequences of independent random signs, shows that there is a universal set of random signs such that for any probability space $(X, \mathcal{F}, \mu)$, any contraction T on $L^2(\mu)$, any $f \in L^2(\mu)$,

$$\lim_{N\to\infty} \frac{\sum\limits_{n=1}^{N} \pm d(n)\, T^{p_n} f(x)}{\sum\limits_{n=1}^{N} d(n)} = 0\,, \tag{2}$$

μ-almost surely.

2. Preliminary lemmas

Recall first some useful estimates concerning the divisor function.

Lemma 4. *As N tends to infinity,*

$$\sum_{n=1}^{N} d(n) \sim N \log N, \quad \sum_{n=1}^{N} d^2(n) \sim \left(\frac{N}{\pi^2}\right) \log^3 N.$$

This follows respectively from Eq. 18.2.1 p. 263 of [HW] and Eq. (B) p. 81 of [R] (see [Wi] for a proof). The next Lemma is a particular case of Corollary 1.5 from [We2].

Lemma 5. *Let $1 < p < \infty$ and consider a sequence $\{\xi_n : n \geqslant 1\}$ such that for any $1 \leqslant i \leqslant j < \infty$*

$$\mathbf{E}\left|\sum_{n=i}^{j} \xi_n\right|^p \leqslant \sum_{n=i}^{j} m_n \tag{5}$$

for some sequence of non negative reals $\{m_n : n \geqslant 1\}$ such that the series $\sum_n m_n$ diverges and note $M_N = \sum_{n=1}^{N} m_n$. If

$$\log \frac{M_N}{m_N} = \mathcal{O}(\log M_N)\,,$$

then

$$\left\| \sup_{N\geqslant 1} \frac{\left|\sum\limits_{n=1}^{N} \xi_n\right|}{M_N^{1/p}\log^b M_N} \right\|_p < \infty \qquad \text{and} \qquad \frac{1}{M_N^{1/p}\log^b M_N}\sum_{n=1}^{N} \xi_n \xrightarrow{\text{a.s.}} 0\,. \tag{6}$$

For the proof of Theorem 3, we need the following lemma which is taken from [BW] (see Theorem 3.1).

Lemma 6. *Let $(W_k)_{k\in\mathbb{N}}$ be a sequence of independent, symmetric, square integrable random variables defined on a probability space $(\Omega,\mathcal{B},\mathbf{P})$ and let $(p_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers. Suppose that there exist integers $0 := N_0 < N_1 < N_2 < \dots$ such that the following condition is satisfied:*

$$\sum_{i=0}^{\infty} \sqrt{\log(p_{N_{i+1}})}\,\mathbf{E}\left[\left(\sum_{k=N_i+1}^{N_{i+1}} \left|W_k\right|^2\right)^{\frac{1}{2}}\right] < \infty\,. \tag{7}$$

Then there exists a (universal) $\mathbf{P}$-null set $N^* \in \mathcal{B}$ such that for each $\omega \in \Omega\backslash N^*$, for any probability space $(X,\mathcal{F},\mu)$, any contraction T on $L^2(\mu)$, any $f\in L^2(\mu)$, if we define

$$\forall\omega\in\Omega\,,\ \forall x\in X\,,\ \forall n\geq 1\,, \qquad S_n(\omega,x) = \sum_{k=1}^{n} W_k(\omega)\,T^{p_k} f(x)\,, \tag{8}$$

the sequence $\big(S_{N_k}(\omega,\cdot)\big)_{k\geqslant 1}$ converges μ-almost surely.

If, in addition to condition (7), we assume

$$\sum_k \min\left[\log^2(N_{k+1}-N_k)\log p_{N_{k+1}}\left(\sum_{j=N_k+1}^{N_{k+1}} \mathbf{E}(W_j^2)\right),\ \mathbf{E}\left(\sum_{j=N_k+1}^{N_{k+1}} \left|W_j\right|\right)^2\right] < \infty\,, \tag{9}$$

then, there exists a (universal) $\mathbf{P}$-null set $N^*\in\mathcal{B}$ such that for each $\omega\in\Omega\backslash N^*$, for any probability space $(X,\mathcal{F},\mu)$, any contraction T on $L^2(\mu)$, any $f\in L^2(\mu)$, the sequence $(S_n)_{n\geqslant 1}$ converges μ-almost surely.

3. Proofs

Proof of Theorems 1 and 2. Since $\{f_n : n\geq 1\}$ is a quasi-orthogonal system, there exists a number B such that for any $1\leqslant i\leqslant j<\infty$,

$$\mathbf{E}\left|\sum_{i\leqslant n\leqslant j} d(n) f_n\right|^2 \leqslant B\sum_{i\leqslant n\leqslant j} d^2(n)\,. \tag{10}$$

Apply Lemma 5 with $p = 2$, $m_n = d^2(n)$. As $m_n \geqslant 1$, we have $\log \frac{M_N}{m_N} \leqslant \log M_N$ and the condition of Lemma 5 is satisfied. We get

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} d(n) f_n}{\left(\sum_{n=1}^{N} d^2(n)\right)^{1/2} \log^b\left(\sum_{n=1}^{N} d^2(n)\right)} \overset{\text{a. s.}}{=} 0\,, \tag{11}$$

as claimed in Theorem 2. Theorem 1 then follows as well. □

Proof of Theorem 3. We apply Lemma 6 with the choices $N_k = 2^k$, $p_k = k^s$, $W_k = \frac{d(k)X_k}{k^{1/2}\log^{\delta/2} k}$. Then,

$$\begin{aligned}
&\sum_k \log^2(N_{k+1}-N_k)\log p_{N_{k+1}}\left(\sum_{j=N_k+1}^{N_{k+1}} \mathbf{E}(W_j^2)\right)\\
&\qquad\qquad \leqslant C(\mathbf{E}X_1^2)\sum_k k^3\left(\sum_{2^k<j\leqslant 2^{k+1}} \frac{d(j)^2}{j\log^\delta j}\right)\\
&\text{(by Lemma 4)}\qquad \leqslant C(\mathbf{E}X_1^2)\sum_k k^{3-\delta}\frac{1}{2^k}\left(\sum_{j\leqslant 2^{k+1}} d(j)^2\right)\\
&\qquad\qquad \leqslant C(\mathbf{E}X_1^2)\sum_k k^{6-\delta} < \infty\,,
\end{aligned}$$

since we have assumed $\delta > 7$. Here C denotes some constant depending on s only. Hence condition (9) is satisfied. Now,

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sqrt{\log(p_{N_{k+1}})}\,\mathbf{E}\left(\sum_{j=N_k+1}^{N_{k+1}} |W_j|^2\right)^{\frac{1}{2}}\\
&\qquad\qquad \leqslant C\sum_{k=0}^{\infty} k^{1/2}\left(\sum_{2^k<j\leqslant 2^{k+1}} \mathbf{E}|W_j|^2\right)^{1/2}\\
&\qquad\qquad \leqslant C(\mathbf{E}X_1^2)^{1/2}\sum_{k=0}^{\infty} k^{1/2}\left(\sum_{2^k<j\leqslant 2^{k+1}} \frac{d(j)^2}{j\log^\delta j}\right)^{1/2}\\
&\qquad\qquad \leqslant C(\mathbf{E}X_1^2)^{1/2}\sum_{k=0}^{\infty} k^{(1-\delta)/2}\left(\frac{1}{2^k}\sum_{j\leqslant 2^{k+1}} d(j)^2\right)^{1/2}\\
&\text{(by Lemma 4)}\qquad \leqslant C(\mathbf{E}X_1^2)^{1/2}\sum_{k=0}^{\infty} k^{(4-\delta)/2} < \infty\,.
\end{aligned}$$

Condition (7) is also satisfied, and Lemma 6 thus applies. This implies that on some universal set of ω's, the series

$$\sum_{n=1}^{\infty} \frac{d(n)X_n(\omega)}{n^{1/2}\log^{\delta/2} n} T^{p_n} f(x)\,,$$

converges μ-almost surely. By the Kronecker Lemma

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} d(n)X_n(\omega)\, T^{p_n} f(x)}{N^{1/2}\log^{\delta/2} N} = 0\,,$$

μ-almost surely. And then by the first estimate in Lemma 4,

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} d(n)X_n(\omega)\, T^{p_n} f(x)}{\sum_{n=1}^{N} d(n)} = 0\,,$$

μ-almost surely. The proof is now complete. □

Concluding remark. One may ask whether a weighted strong law of large numbers, with weights $d(n)$ holds in L^1. According to Theorem 4 in [JOP] and in view of Lemma 4, this holds if and only if the following condition is fulfilled:

$$\limsup_{t\to\infty} \frac{1}{t} \#\left\{ n\colon \frac{n\log n}{t} \leqslant d(n) \right\} < \infty\,. \tag{12}$$

This question will be examined elsewhere.

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