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DENSITIES AND SOCIAL CHOICE TRADE-OFFS

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ABSTRACT. In this paper we pursue a twofold goal. Firstly we survey results by Campbell and Kelly in which they use the notion of asymptotic density in order to solve a social choice problem with countably many alternatives. The second goal is to discuss the extent to which other notions of density (logarithmic, uniform) could play some role in treating social choice questions of this type, observing that from a certain point of view (invariance of the asymptotic density under the Lévy group), the Campbell–Kelly theorem is the best possible.

0. Introduction

We consider social choice trade-offs when there are countably many consumer's alternatives, which are linearly ordered in a sequence. This linear order of alternatives allows us to formulate trade-offs between the weakening of the Pareto criterion of the social preferences and the extent we can move away from dictatorship. We will survey the results of D. E. Campbell and J. S. Kelly in the case of various density concepts on the countable set of alternatives. In Section 1 and 2 we introduce classical definitions and notations used in social choice literature. In Section 3 we present the Campbell–Kelly approach. In Section 4 we make several observations on the possibility of replacing the asymptotic density by other density concepts.

1. Some definitions and notations

We consider a finite society of agents $S = \{1, 2, \dots, s\}$ each one having to express a preference over the same set of alternatives X. Until further notice, X is an arbitrary set. Every agent $i \in S$ has a preference relation denoted

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$\succsim_i$, which is a complete and transitive binary relation on X. If $x \succsim_i y$ we say that the alternative x is *at least as good as* the alternative y for the i's agent. If $Y \subseteq X$ let $\mathcal{P}(Y)$ denote the set of all preference relations on Y. From a preference relation $\succsim$ on X we can derive several other useful relations.

The restriction of $\succsim$ to a subset $Y \subset X$ is defined by

$$\succsim|_Y := \succsim \cap\, Y \times Y .$$

The inverse of a preference relation $\succsim$ on X is also a preference relation on X and we denote it $\precsim$ i.e.,

$$x \precsim y \iff y \succsim x .$$

Another relation derived from a preference relation $\succsim$ is the *strict preference relation* defined by

$$x \succ y \iff x \succsim y \;\;\&\;\; y \not\succsim x$$

and we say that x is *preferred to* y. The strict preference relation $\succ$ is the biggest asymmetric part of the preference relation $\succsim$ since $\succ = \succsim \setminus \precsim$.

The biggest symmetric part of $\succsim$ is called *indifference relation* derived from $\succsim$ and is defined by

$$x \sim y \iff x \succsim y \;\;\&\;\; y \succsim x$$

and we say that x is *indifferent with* y. Indifference relation derived from any preference relation on a set X is an equivalence relation on X and consequently it contains the diagonal $\Delta_X := \{(x,x) \colon x \in X\}$, the smallest equivalence relation on X. Obviously we have

$$\succsim = \succ \cup \sim$$

which explains our choice of the notation of preference relation. A preference relation with derived indifference relation $\sim = \Delta_X$ is called *a linear order* on X and usually is denoted by $\succeq$ or simply by $\geqslant$. $\mathcal{L}(X)$, or just $\mathcal{L}$ when X is fixed, denotes the set of all linear orders on the set X. Observe that $\mathcal{L} \subset \mathcal{P}$ and consequently $\mathcal{L}^s \subset \mathcal{P}^s$—the set of all *preference profiles* on X of s individuals.

2. Aggregation of preferences

Definition 1. A *social welfare function* [s.w.f.] defined on a given set of profiles $\mathcal{D} \subseteq \mathcal{P}^s$ is a mapping $f \colon \mathcal{D} \to \mathcal{P}$ that assigns to any profile $r := (\succsim_1, \dots, \succsim_s) \in \mathcal{D}$ a preference relation $f(r)$ interpreted as the social preference relation. We denote by ${}^a f(r)$ the asymmetric part of the preference relation $f(r)$, i.e.,

$$(x,y) \in {}^a f(r) \iff (x,y) \in f(r) \;\;\&\;\; (y,x) \notin f(r) .$$

We read $x\ f(r)\ y$ as x is *socially at least as good as* y and $x\ {}^{a}f(r)\ y$ as x is *socially preferred to* y.

The minimal condition demanded from aggregated preference relation $f(\succsim_1, \dots, \succsim_s)$ is that it respects unanimous individual preferences.

Definition 2. A social welfare function $f \colon \mathcal{D} \to \mathcal{P}$ is called *Paretian* if for any profile $r := (\succsim_1, \dots, \succsim_s) \in \mathcal{D}$ and any pair of alternatives $\{x, y\} \subseteq X$ we have

$$[\forall i \leqslant s \colon x \succ_i y] \quad \Longrightarrow \quad x\ {}^{a}f(r)\ y\,,$$

i.e., x is socially preferred to y if every agent prefers x to y.

A very special case of a Paretian s.w.f. is the dictatorial s.w.f.

Definition 3. We say that the s.w.f. $f \colon \mathcal{D} \to \mathcal{P}$ is *directly dictatorial* if there is an agent $k \in S$ such that for any profile $r := (\succsim_1, \dots, \succsim_s) \in \mathcal{D}$ and any pair of alternatives $\{x, y\} \subseteq X$ we have

$$x \succ_k y \quad \Longrightarrow \quad x\ {}^{a}f(r)\ y\,.$$

We say that f is *inversely dictatorial* if

$$x \succ_k y \quad \Longrightarrow \quad y\ {}^{a}f(r)\ x\,.$$

We say that f is *dictatorial* if it is directly or inversely dictatorial.

The dictatorial property is not desirable in the course of collective decision making. So it was a big surprise when Arrow [1] proved that for a large class of social welfare functions (so called Arrowian s.w.f.) it is impossible to avoid dictatorship. Before we sketch the proof of the Arrow impossibility theorem we define an essential property needed in the formulation of this theorem.

Definition 4. A social welfare function $f \colon \mathcal{D} \to \mathcal{P}$ is called *Arrowian* when it satisfies the *independence of irrelevant alternatives* condition: for any pair of profiles $r := (\succsim_1, \dots, \succsim_s) \in \mathcal{D}$, $r' := (\succsim'_1, \dots, \succsim'_s) \in \mathcal{D}$ and any pair of alternatives $\{x, y\} \subseteq X$ the following implication is true:

$$\Big[\forall i \leqslant s \colon \succsim_i\big|_{\{x,y\}} = \succsim'_i\big|_{\{x,y\}}\Big] \quad \Longrightarrow \quad f(r)\big|_{\{x,y\}} = f(r')\big|_{\{x,y\}}\,. \tag{IIA}$$

The (IIA) condition says that the social preferences between any two alternatives depend only on the individual preferences between the same two alternatives. It enormously facilitates the task of making social (collective) decisions because the determination of social ranking on a subset of alternatives does not need any information on individual preferences over alternatives outside this subset.

THEOREM 1. (Arrow's impossibility theorem [1].) *If X contains at least three alternatives and the domain of admissible individual profiles is either $\mathcal{D} = \mathcal{P}^s$ or $\mathcal{D} = \mathcal{L}^s$ then every Arrowian social welfare function $f\colon \mathcal{D} \to \mathcal{P}$ that is Paretian, is directly dictatorial.*

Sketch of the proof. Essentially we follow the proof given in [7]. We proceed by investigating the structure of subsets of agents who are decisive for social preference over alternatives. In what follows, when we refer to pairs of alternatives we always mean distinct alternatives. We distinguish four types of decisive subsets. A subset of agents $A \subseteq S$ is

(i) *decisive for x over y* if whenever every agent in A prefers x to y and every agent not in A prefers y to x, x is socially preferred to y;

(ii) *decisive* if A is decisive for x over y for any pair of alternatives $\{x, y\} \subseteq X$;

(iii) *completely decisive for x over y* if whenever every agent in A prefers x to y, x is socially preferred to y;

(iv) *completely decisive* if A is completely decisive for x over y for any pair of alternatives $\{x, y\} \subseteq X$.

It can be shown (see [7]) that if a set of agents is decisive for some pair of alternatives then it is decisive and any decisive set is completely decisive. It is straightforward that the family of all decisive subsets of agents is an ultrafilter on the set of all agents S. The set S being finite, any ultrafilter on S must be trivial, i.e., contains only sets all of which contain a specific element $h \in S$ (see [5]). In particular the singleton $\{h\}$ is decisive set. It is obvious that the agent h is dictator. $\square$

3. Trade-off between unanimity and dictatorship

The conclusion of Arrow's impossibility theorem is somewhat disturbing, but it would be too facile to conclude from it that the "democracy is impossible". What it shows is something else—that we should not expect a collectivity of individuals to behave with the same kind of coherence that we may hope from an individual. In practice collective decisions are made so Arrow's theorem tells in essence that the institutional details and procedures of the decision process cannot be neglected. It was explored extensively to what extent we can escape the dictatorship conclusion if we relax some of the assumptions imposed by Arrow's theorem, e.g., relaxing rationality of social preference relation or aggregation rule defined on a restricted domain. Campbell and Kelly in [3] made another kind of relaxation namely they keep (IIA) condition and determine some bounds on the trade-off between violation of the Pareto condition and violation of non-dictatorship. To formulate this trade-off we need some more technical notions and notations.

For any set of alternatives $Y \subseteq X$ and preference profile $r := (\succsim_1, \dots, \succsim_s)$ (an element of $\mathcal{P}(X)^s$) we denote

$$r|_Y := (\succsim_1|_Y, \dots, \succsim_s|_Y) \in \mathcal{P}(Y)^s .$$

If $\mathcal{D} \subseteq \mathcal{P}(X)^s$ we set $\mathcal{D}(Y) := \{r|_Y : r \in \mathcal{D}\}$.

If $f \colon \mathcal{D} \to \mathcal{P}$ satisfies Arrow's independence of irrelevant alternatives condition, we can define the restriction of f to Y in the following way

$$f|_Y \colon \mathcal{D}(Y) \to \mathcal{P}(Y), \quad q \mapsto f(r)|_Y ,$$

where r is any profile in $\mathcal{D}$ satisfying $r|_Y = q$.

We say that $f \colon \mathcal{D} \to \mathcal{P}$ is a *null social welfare function* if ${}^a f(r) = \emptyset$ for any $r \in \mathcal{D}$.

We say that a pair $\{x, y\}$ is *fixed by s.w.f.* $f \colon \mathcal{D} \to \mathcal{P}$ if $x \neq y$ and at least one of the following conditions is satisfied:

(i) $f|_{\{x,y\}}$ is null, i.e., $\forall q \in \mathcal{D}(\{x, y\}) : {}^a f(q) = \emptyset$;

(ii) $(x, y) \in {}^a f(r)$ for all $r \in \mathcal{D}$;

(iii) $(y, x) \in {}^a f(r)$ for all $r \in \mathcal{D}$.

This means that pairs of distinct alternatives are socially ranked independently of everyone's preferences. We also say that $f|_{\{x,y\}}$ is a *constant s.w.f.*

We say that a domain has the *free triple property* if either

(i) $\mathcal{D}(\{x, y, z\}) = \mathcal{P}(\{x, y, z\})^s$ for every $\{x, y, z\} \subseteq X$;

or

(ii) $\mathcal{D}(\{x, y, z\}) = \mathcal{L}(\{x, y, z\})^s$ for every $\{x, y, z\} \subseteq X$.

The following lemma proved by Wilson in [10] is essential in the proof of the main theorem about the trade-off.

Lemma 1. (Wilson's partition lemma.) *If $f \colon \mathcal{D} \to \mathcal{P}$ is an Arrowian social welfare function whose domain $\mathcal{D}$ has the free triple property then there is a unique partition $\mathcal{C}$ of X and a linear order $>$ on $\mathcal{C}$ such that statements* (1) *and* (2) *are true for each distinct $Y, Z \in \mathcal{C}$:*

(1) *$f|_Y$ is dictatorial or null or Y contains exactly two elements;*

(2) *$Y > Z$ if and only if $\forall y \in Y\ \forall z \in Z\ \forall r \in \mathcal{D}$: $y\ {}^a f(r)\ z$.*

The pair $(\mathcal{C}, >)$ is referred to as a component structure for f.

We say that $Y \in \mathcal{C}$ is a *null* (*dictatorial*) *component* if $f|_Y$ is null (dictatorial).

We can now formulate the result of Donald E. Campbell and Jerry S. Kelly who have quantified the trade-offs when relaxing the Pareto criterion and the extent to which it is possible to move away from dictatorship.

THEOREM 2. ([4]) *If f is an Arrowian social welfare function that has no dictatorial components of measure greater than t, then the product measure of the set of ordered pairs of alternatives that are fixed by f is at least $1-t$.*

In the paper [3] they consider the set of alternatives $X \subseteq \mathbb{R}^n$ and use the Lebesgue measure to quantify the trade-off. Some qualifications are needed assuring the measurability of the Wilson components and the set of fixed pairs. When the set X of the alternatives is finite with m members, they take for measure of any subset of X its cardinality divided by m. Moreover the condition $mt \geqslant 2$ must be added. In the paper [4] they consider countably infinite alternatives set and use the asymptotic density to measure the extent of dictatorship and imposition. They prove the above theorem assuming a finite number of Wilson components having natural density.

4. Discussion

The asymptotic density is not the only density type measure on a countably infinite set. We will consider several other densities and will discuss the social choice trade-offs with respect to such measures. We can follow a general scheme by introducing a density measure on the set of positive integers $\mathbb{N}$ and then, given a countably infinite set of alternatives X, we consider an one-to-one correspondence with $\mathbb{N}$ which allows to define the density notion on the set X. So in what follows we are going to introduce some densities on $\mathbb{N}$ and we leave to the reader the task to transfer them to the alternatives set X. We start with recalling the *lower* and *upper asymptotic density* of a set $A \subseteq \mathbb{N}$. They are defined by

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k), \quad \overline{d}(A) := \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k),$$

where χ_A is the characteristic function of the set A. If $\underline{d}(A) = \overline{d}(A) =: d(A)$ the set A is said to have the *asymptotic density* $d(A)$.

The *lower* and *upper logarithmic density* of a set $A \subseteq \mathbb{N}$ are defined as:

$$\underline{\delta}(A) := \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_A(k)\frac{1}{k}}{\log n}, \quad \overline{\delta}(A) := \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_A(k)\frac{1}{k}}{\log n}.$$

If $\underline{\delta}(A) = \overline{\delta}(A) =: \delta(A)$ the number $\delta(A)$ is called the *logarithmic density* of the set A.

Another density concept is the uniform density. For any $A \subseteq \mathbb{N}$, $t \geqslant 0$, $s \geqslant 1$ denote by $A(t+1, t+s)$ the number of elements of the set $A \cap [t+1, t+s]$. Put

$$\alpha_s := \liminf_{t\to\infty} A(t+1, t+s), \quad \alpha^s := \limsup_{t\to\infty} A(t+1, t+s).$$

Then there exist the limits (c.f. [9])

$$\underline{u}(A) := \lim_{s\to\infty} \frac{\alpha_s}{s}, \quad \overline{u}(A) := \lim_{s\to\infty} \frac{\alpha^s}{s}$$

called the *lower* and *upper uniform density* of A, respectively. If $\underline{u}(A) = \overline{u}(A) =: u(A)$ then the common value $u(A)$ is called the *uniform density* of A (cf. [2]).

The following relations between these densities can be verified (cf. [2], [6]):

$$0 \leqslant \underline{u}(A) \leqslant \underline{d}(A) \leqslant \underline{\delta}(A) \leqslant \overline{\delta}(A) \leqslant \overline{d}(A) \leqslant \overline{u}(A) \leqslant 1.$$

These relationships allow us to discuss the validity of Campbell and Kelly trade-off theorem for the three concepts of density. If we suppose that the Wilson components have asymptotic density, then they also have logarithmic density, and both densities are equal. Consequently the trade-off theorem is true also for the logarithmic density. Of course we could admit only the existence of logarithmic density of Wilson components which would not imply that they have asymptotic density and so we can not conclude about the asymptotic trade-off. But still it could be that the logarithmic trade-off can be proved but actually we have no proof of such a theorem. Also we do not know whether stronger assumption about uniform density of Wilson components allows to prove a stronger result about uniform trade-off.

Instead of looking for the proofs of the trade-offs with respect to different densities we show that in a sense asymptotic density is the most reasonable density concept applicable to a countable infinite set of alternatives. We start with the observation about a linear order on the alternatives. This order is often based on practical considerations of importance of different alternatives. Since time is omnipresent in the economic considerations (e.g., in infinite horizon issues) the most natural order of alternatives is that one based on time accessibility of various alternatives. So when measuring a set of alternatives by a density function with real values the decision maker prefers the alternatives accessible in the near future to the alternatives in the very distant future. It means that it is quite reasonable to suppose that the density function gives the same values to two sets of alternatives not too much reshuffled. We can formalize this intuitive notion by the Lévy permutation group.

Definition 5. Let $\operatorname{aut}(\mathbb{N})$ be the group of all permutations of $\mathbb{N}$. The set

$$\mathcal{G} := \Big\{ g \in \operatorname{aut}(\mathbb{N})\colon \lim_{N\to\infty} \frac{1}{N} \big|\{n \leqslant N : g(n) > N\}\big| = 0 \Big\}$$

forms a subgroup of aut$(\mathbb{N})$ and is called the *Lévy group.*

It can be shown that the asymptotic density is invariant under the Lévy group. What is more important for us is that, among densities, the asymptotic is the only one which is invariant under the Lévy group. This was proved by Nobuaki Obata in [8].

THEOREM 3. *Let γ be a nonnegative finitely additive set function defined on the system $\mathcal{F}$ of all sets having asymptotic density. If γ is invariant under the Lévy group and $\gamma(\mathbb{N}) = 1$ then γ coincides with the asymptotic density d.*

So if we would like to use a finitely additive measure on countably infinite set of alternatives X for measuring subsets of alternatives having asymptotic density (for a linear order defined by a sequence $(X) = (x_1, x_2, \dots)$), and we suppose that the measure of a subset will not change after having reshuffled not too much the elements of the sequence (X), then the finitely additive measure on X is necessarily the natural (i.e., asymptotic) density function. Under this constraint the Campbell–Kelly result is the best possible.

REFERENCES

[1] ARROW, K. J.: *Social Choice and Individual Values*, 2nd ed., Wiley, New York, 1963.

[2] BROWN, T. C.—FREEDMAN, A. R.: *The uniform density of sets of integers and the Fermat's last theorem*, C. R. Math. Rep. Acad. Sci. Canada **12** (1990), 1–6.

[3] CAMPBELL, D. E.—KELLY, J. S.: *Lebesgue measure and social choice trade-offs*, Econometric Theory **5** (1995), 445–459.

[4] CAMPBELL, E. E.—KELLY, J. S.: *Asymptotic density and social choice trade-offs*, Math. Social Sci., **29** (1995), 181–194.

[5] COMFORT, W. W.—NEGREPONTIS, S.: *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974.

[6] HALBERSTAM, H.—ROTH, K. F.: *Sequences I*, Oxford University Press, Oxford, 1966.

[7] MASS-COLEL, A.—WHINSTON, M. D.—GREEN, J. R.: *Microeconomic Theory*, Oxford University Press, Oxford, 1995.

[8] OBATA, N.: *Density of natural numbers and the Lévy group*, J. Number Theory **30** (1988), 288–297.

[9] ŠALÁT, T.—TOMA, V.: *A classical Olivier's theorem and statistical convergence*, Ann. Math. Blaise Pascal **10** (2003), 305–313.

[10] WILSON, R. B.: *Social choice without the Pareto principle*, J. Econom. Theory **5** (1972), 14–20.

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