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ON THE REGULARITY OF DENSITY SETS

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ABSTRACT. The density set of a sequence of natural integers A is the two-dimensional set $\mathbf{S}(A) = \{(\overline{\mathrm{d}}B, \underline{\mathrm{d}}B) : B \subseteq A\}$. A set $\mathbf{S}$ is said to be admissible if there exists a set of natural integers A such that $\mathbf{S} = \mathbf{S}(A)$. Answering to a question by S. Pichorides, it is shown that for any given admissible set $\mathbf{S}$, there exists a set A of natural integers such that $\mathbf{S}(A) = \mathbf{S}$, and for any admissible subset $\mathbf{S}'$ of $\mathbf{S}$, there exists a subset A' of A with $\mathbf{S}(A') = \mathbf{S}'$.

1. Introduction

Let $A \subseteq \mathbb{N} = \{1, 2, \dots\}$ be a set of natural integers. For any positive integer n, let $A(n) = \bigl|A \cap [1, n]\bigr|$ the number of elements of A up to n. We denote by $\mathrm{D}(A) = (\overline{\mathrm{d}}A, \underline{\mathrm{d}}A)$ the density point of A, where

$$\overline{\mathrm{d}}A = \limsup_{n \to +\infty} \frac{A(n)}{n},$$

and

$$\underline{\mathrm{d}}A = \liminf_{n \to +\infty} \frac{A(n)}{n}.$$

Since $\underline{\mathrm{d}}A \leqslant \overline{\mathrm{d}}A$, we plainly have that $D(A)$ belongs to the triangle $\mathbf{T} := \bigl\{(x, y) \in \mathbb{R}^2 \text{ such that } 0 \leqslant y \leqslant x \leqslant 1\bigr\}$. The two-dimensional set

$$\mathbf{S}(A) := \bigl\{D(B) : B \subseteq A\bigr\}$$

is called the density set of A. It is entirely included in the triangle $\mathbf{T}$. Moreover $\mathbf{S}(\mathbb{N}) = \mathbf{T}$ and $\mathbf{S}(A) = \bigl\{(0, 0)\bigr\}$ whenever $\overline{\mathrm{d}}A = 0$.

For any pair of real numbers $1 \geqslant u \geqslant v \geqslant 0$, we define the triangle

$$\mathbf{T}_{u,v} = \bigl\{(x, y) \in [0, u] \times [0, v] \text{ such that } uy \leqslant vx\bigr\}$$

and the trapezium

$$\mathbf{Z}_{u,v} = \bigl\{(x, y) \in [0, u] \times [0, v] \text{ such that } x \geqslant y\bigr\}.$$

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With these notations, we have $\mathbf{T} = \mathbf{T}_{1,1} = \mathbf{Z}_{1,1}$. It can be seen that if $u = \overline{\mathrm{d}}A$ and $v = \underline{\mathrm{d}}A$, then $\mathbf{T}_{u,v} \subseteq \mathbf{S}(A) \subseteq \mathbf{Z}_{u,v}$ (cf. [1]). The following properties of the density set have also been established in [1]:

(i) $\mathbf{S}(A)$ is a convex set;

(ii) $\mathbf{S}(A)$ is a closed set.

The convexity of $\mathbf{S}(A)$ shows that the line Γ_A joining the origin to the density point (A) is concave. Let $\gamma_A\colon [0, \overline{\mathrm{d}}A] \longrightarrow [0, \underline{\mathrm{d}}A]$ be the mapping $x \mapsto \gamma_A(x) = \max\{y \geqslant 0\colon (x,y) \in \mathbf{S}(A)\}$, whose graph is Γ_A. The density set $\mathbf{S}(A)$ is characterized by γ_A. Furthermore, γ_A gives in huge part the asymptotic behaviour of the set A.

In the sequel, we will say that a subset $\mathbf{S}$ of $\mathbf{T}$ is *admissible* if there exists a set of integers A such that $\mathbf{S}(A) = \mathbf{S}$. It was proved in [1] that $\mathbf{S}$ is admissible if and only if $\mathbf{S}$ is convex, closed and there exist two real numbers $1 \geqslant u \geqslant v \geqslant 0$ such that $\mathbf{T}_{u,v} \subseteq \mathbf{S} \subseteq \mathbf{Z}_{u,v}$.

In [2] the authors were interested in showing the regularity for inclusion of the mapping $A \mapsto \mathbf{S}(A)$ from $\mathbb{N}$ onto the set of the admissible subsets of $\mathbf{T}$. More precisely, they have shown

Theorem 1. *Let $\mathbf{S}_0$ and $\mathbf{S}_1$ be two admissible subsets of $\mathbf{T}$ such that $\mathbf{S}_0 \supseteq \mathbf{S}_1$. Then there exist two sets of positive integers A_0 and A_1 such that $A_0 \supseteq A_1$, $\mathbf{S}(A_0) = \mathbf{S}_0$ and $\mathbf{S}(A_1) = \mathbf{S}_1$.*

In the construction of the sets A_0 and A_1 given in [2], we notice that A_0 really depends on both admissible sets $\mathbf{S}_0$ and $\mathbf{S}_1$. A quite natural question arises: is it true that for any given admissible subset $\mathbf{S}_0$ of $\mathbf{T}$, it is possible to find a set A_0 with density set $\mathbf{S}_0$ such that for any admissible set $\mathbf{S}_1 \subseteq \mathbf{S}_0$, there exists $A_1 \subseteq A_0$ with $\mathbf{S}(A_1) = \mathbf{S}_1$? This question was asked by Professor Stylianos K. Pichorides (see problem session of this issue).

This statement, if true, would imply that the set A_0 in Theorem 1 can be chosen uniformly in $\mathbf{S}_1$. The aim of this note is to prove this result.

Theorem 2. *Let $\mathbf{S}_0$ be an admissible subset of $\mathbf{T}$. Then there exists a set A_0 of positive integers such that:*

(i) $\mathbf{S}(A_0) = \mathbf{S}_0$;

(ii) *for any admissible subset $\mathbf{S}_1$ of $\mathbf{S}_0$, there exists $A_1 \subseteq A_0$ with $\mathbf{S}(A_1) = \mathbf{S}_1$.*

This theorem may be generalized in the following way:

Theorem 3. *Let $k \geqslant 1$. For any decreasing sequence $\mathbf{S}_0 \supseteq \mathbf{S}_1 \supseteq \cdots \supseteq \mathbf{S}_k$ of admissible subsets of $\mathbf{T}$, there exists a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k$ of subsets of $\mathbb{N}$ such that for each $j = 0, 1, \ldots, k$, $\mathbf{S}(A_j) = \mathbf{S}_j$ and the set*

$A_j = A_j(\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_j)$ *depends only on* $\mathbf{S}_0, \mathbf{S}_1, \dots$ *and* $\mathbf{S}_j$.

In fact, we will prove the following sharper result stating that the previous one is uniform in k.

Theorem 4. *For any decreasing sequence* $\mathbf{S}_0 \supseteq \mathbf{S}_1 \supseteq \cdots \supseteq \mathbf{S}_k \supseteq \dots$ *of admissible subsets of* $\mathbf{T}$, *there exists a decreasing sequence* $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k \supseteq \dots$ *of subsets of* $\mathbb{N}$ *such that for each* $j \geqslant 0$, $\mathbf{S}(A_j) = \mathbf{S}_j$ *and the set* $A_j = A_j(\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_j)$ *depends only on* $\mathbf{S}_0, \mathbf{S}_1, \dots$ *and* $\mathbf{S}_j$.

2. Preparing the proof

For any given positive real number $u \leqslant 1$, we define

$$\mathbb{N}(u) = \left\{ \left\lfloor \frac{n}{u} \right\rfloor : n \geqslant 1 \right\},$$

that is the image of $\mathbb{N}$ by the map $\theta_u : n \mapsto \left\lfloor \frac{n}{u} \right\rfloor$. When $u > 1$, let θ_u be the identical map of $\mathbb{N}$, so that we simply have $\mathbb{N}(u) = \mathbb{N}$. Let $(u_j)_{1 \leqslant j \leqslant k}$ be a sequence of positive real numbers. We define for $k \geqslant 2$

$$\mathbb{N}(u_1; u_2; \dots; u_k) = \begin{cases} \theta_{u_1}\big(\mathbb{N}(u_2/u_1; \dots; u_k/u_1)\big) & \text{if } u_1 \leqslant 1, \\ \mathbb{N}(u_2; \dots; u_k) & \text{if } u_1 \geqslant 1. \end{cases}$$

Hence if $u = \min_{1 \leqslant j \leqslant k} u_j \geqslant 1$, then $\mathbb{N}(u_1; u_2; \dots; u_k) = \mathbb{N}$. Otherwise we have

$$\mathbb{N}(u_1; u_2; \dots; u_k) = \mathbb{N}(v_1; v_2; \dots; v_\ell), \tag{1}$$

where $(v_j)_{1 \leqslant j \leqslant \ell}$ is the decreasing subsequence of $(u_j)_{1 \leqslant j \leqslant k}$ constructed as follows:

Let $j_1 = \min\{i \geqslant 1 \text{ such that } u_i < 1\}$ and $v_1 = u_{j_1}$. If $v_1 = u$, we stop. Otherwise as long as it is possible, let $j_h = \min\{i > j_{h-1} \text{ such that } u_i < v_{h-1}\}$ and $v_h = u_{j_h}$, $h \geqslant 2$. The process stops when we get $v_\ell = u$ for some integer $\ell \leqslant k$.

It is an easy matter to show that for $0 < u \leqslant 1$, we have for any real $X \geqslant 0$

$$uX - 1 \leqslant \big|\mathbb{N}(u) \cap (0, X]\big| \leqslant uX + 1, \tag{2}$$

thus

$$uY - 2 \leqslant \big|\mathbb{N}(u) \cap (X, X+Y]\big| \leqslant uY + 2,$$

for any $X, Y \geqslant 0$. More generally, we have the following lemma:

Lemma 5. *Let $(u_j)_{1\leqslant j\leqslant k}$ be a sequence of $k\geqslant 1$ positive real numbers. Let $u=\min_{1\leqslant j\leqslant k} u_j$ and assume that $u\leqslant 1$. Then we have*

$$uY-k-1\leqslant \left|\mathbb{N}(u_1;u_2;\ldots;u_k)\cap(X,X+Y]\right|\leqslant uY+k+1\,,$$

for any nonnegative real numbers X and Y.

Proof. The result being obvious for $u=1$, we may assume that $u<1$. Let $(v_j)_{1\leqslant j\leqslant \ell}$ be the decreasing subsequence of $(u_j)_{1\leqslant j\leqslant k}$ such that (1) holds. It suffices to show that for any $X\geqslant 0$,

$$uX-1\leqslant \left|\mathbb{N}(v_1;v_2;\ldots;v_\ell)\cap(0,X]\right|\leqslant uX+\ell\,.$$

We argue by induction on ℓ. Equation (2) gives the result for $\ell=1$. Let $\ell\geqslant 2$. Hence $v_1>u$ and $u=\min_{2\leqslant j\leqslant \ell} v_j$, implying $u/v_1=\min_{2\leqslant j\leqslant \ell} v_j/v_1$. By induction we thus may assume that for any $X\geqslant 0$

$$\frac{u}{v_1}X-1\leqslant \left|\mathbb{N}\left(\frac{v_2}{v_1};\ldots;\frac{v_\ell}{v_1}\right)\cap(0,X]\right|\leqslant \frac{u}{v_1}X+\ell-1\,. \tag{3}$$

We have

$$\mathbb{N}(v_1;v_2;\ldots;v_\ell)\cap(0,X]=\left\{n\in\mathbb{N}\left(\frac{v_2}{v_1};\ldots;\frac{v_\ell}{v_1}\right):\ \left\lfloor\frac{n}{v_1}\right\rfloor\leqslant X\right\},$$

giving

$$\begin{aligned}\mathbb{N}\left(\frac{v_2}{v_1};\ldots;\frac{v_\ell}{v_1}\right)\cap(0,v_1X]&\subseteq \mathbb{N}(v_1;v_2;\ldots;v_\ell)\cap(0,X]\\&\subseteq \mathbb{N}\left(\frac{v_2}{v_1};\ldots;\frac{v_\ell}{v_1}\right)\cap(0,v_1X+1]\,.\end{aligned}$$

The result then follows from the induction hypothesis (3). □

3. Proof of Theorem 4

Let $\mathbf{S}$ be an admissible subset of $\mathbf{T}$. We denote by (u_0,v_0) the upper point of $\mathbf{S}$, that is $u_0=\max\{u\colon \exists v \text{ such that } (u,v)\in\mathbf{S}\}$ and $v_0=\max\{v\colon \exists u \text{ such that } (u,v)\in\mathbf{S}\}$. To $\mathbf{S}$ there corresponds its boundary function, that is the concave non decreasing function f from $[0,u_0]$ onto $[0,v_0]$ defined by $f(x)=\max\{v\colon (x,v)\in\mathbf{S}\}$. In particular we have $f(0)=0$ and $f(u_0)=v_0$. Moreover the right derivative $f'(x)$ of f at any point $x\in[0,u_0)$ exists and is less than or equal to 1 since it is non increasing by the concavity of f and

$f'(0) \leqslant 1$. It implies that f is a 1-Lipschitzian function: for any x and y in $[0,u_0]$ we have $\bigl|f(x)-f(y)\bigr| \leqslant |x-y|$.

For any $i \geqslant 0$, let f_i be the boundary function of $\mathbf{S}_i$. We also denote $u_{i,0} = \max\bigl\{u\colon \exists v \text{ such that } (u,v) \in \mathbf{S}_i\bigr\}$ and $v_{i,0} = f(u_{i,0})$. We define for any $k \geqslant 1$ the non increasing sequences

$$\begin{cases} u_{i,j}^{(k)} = \frac{k-j}{k} u_{i,0}\,, \\ v_{i,j}^{(k)} = f_i(u_{i,j}^{(k)})\,, \end{cases} \qquad j = 0,1,\dots,k\,,$$

and let $(\mu_{i,j}^{(k)})_{0\leqslant j\leqslant k}$ be the decreasing sequence of real numbers defined by

$$\mu_{i,0}^{(k)} = 0\,, \quad \mu_{i,j}^{(k)} = \frac{v_{i,j}^{(k)} - v_{i,j-1}^{(k)}}{u_{i,j}^{(k)} - u_{i,j-1}^{(k)}}, \qquad j = 1,\dots,k\,.$$

Let $\mathcal{T} = \{t_i = (i+1)(i+2)/2\colon i \geqslant 0\}$ be the set of the positive triangular numbers. For any $\ell \geqslant 0$, we denote $\mathcal{T}_\ell = (\mathcal{T}+\ell)\cap[t_\ell+\ell,+\infty) = \{t_i+\ell\colon i \geqslant \ell\}$.

Let $X_0 = 2$ and $X_{k+1} = X_k^2 + X_k$ for $k \geqslant 0$. We will use the fact that for any $k \geqslant 1$,

$$\frac{k}{2} \leqslant \log\log X_k \leqslant k\,. \tag{4}$$

We first define the following subset of $\mathbb{N}$

$$A_0 = A_0(\mathbf{S}_0) = \bigcup_{\substack{k\geqslant 1\\ k\in\mathcal{T}}} \left(\bigcup_{j=0}^{k-1} \Bigl(X_k + \mu_{0,j}^{(k)} X_k^2,\, X_k + \mu_{0,j+1}^{(k)} X_k^2\Bigr] \cap \mathbb{N}\bigl(u_{0,j}^{(k)}\bigr) \right) \cup \bigcup_{\substack{k\geqslant 1\\ k\notin\mathcal{T}}} (X_k, X_{k+1}] \cap \mathbb{N}(u_{0,0})\,,$$

and for any $i \geqslant 1$, we put

$$A_i = A_i(\mathbf{S}_0,\mathbf{S}_1,\dots,\mathbf{S}_i) = \bigcup_{\ell=0}^{i} \bigcup_{k\in\mathcal{T}_\ell} \left(\bigcup_{j=0}^{k-1} \Bigl(X_k + \mu_{\ell,j}^{(k)} X_k^2,\, X_k + \mu_{\ell,j+1}^{(k)} X_k^2\Bigr] \cap \mathbb{N}\bigl(u_{0,0}; u_{1,0}; \dots; u_{\ell-1,0}; u_{\ell,j}^{(k)}; u_{\ell+1,0}; \dots; u_{i,0}\bigr) \right) \cup \bigcup_{\substack{k\geqslant 1\\ k\notin \bigcup_{\ell=0}^{i} \mathcal{T}_\ell}} (X_k; X_{k+1}] \cap \mathbb{N}\bigl(u_{0,0}; u_{1,0}; \dots; u_{i,0}\bigr)\,.$$

Let $i \geqslant 0$. Our aim is to show that $\mathbf{S}(A_i) = \mathbf{S}_i$, or equivalently that (i) $B \subseteq A_i$ implies $D(B) \in \mathbf{S}_i$; and (ii) for any $0 \leqslant x \leqslant u_{i,0}$, there exists $B \subseteq A_i$ such that $D(B) = \bigl(x, f_i(x)\bigr)$.

(i) Let $B \subseteq A_i$ and assume that $\overline{\mathrm{d}}B \leqslant x$ where $0 < x \leqslant u_{i,0}$. We must show that $\underline{\mathrm{d}}B \leqslant f_i(x)$. Let $\varepsilon > 0$. For any sufficiently large integer $k \in \mathcal{T}_i$, there exists j such that

$$u_{i,j+1}^{(k)} < x \leqslant u_{i,j}^{(k)},$$

and

$$B\big(X_k + \mu_{i,j+1}^{(k)} X_k^2\big) \leqslant (x+\varepsilon)\mu_{i,j+1}^{(k)} X_k^2 + X_k\,.$$

By Lemma 5 we get

$$\begin{aligned} B(X_{k+1}) &\leqslant (x+\varepsilon)\mu_{i,j+1}^{(k)} X_k^2 + X_k + A(X_{k+1}) - A\big(X_k + \mu_{i,j+1}^{(k)} X_k^2\big) \\ &\leqslant (x+\varepsilon)\mu_{i,j+1}^{(k)} X_k^2 + X_k + \sum_{q=j+1}^{k-1} \Big(u_{i,q}^{(k)} \Big(\mu_{i,q+1}^{(k)} - \mu_{i,q}^{(k)}\Big) X_k^2 + i + 2 \Big)\,. \end{aligned}$$

Let $x = t u_{i,j}^{(k)} + (1-t) u_{i,j+1}^{(k)}$ where $0 < t \leqslant 1$. Thus

$$\begin{aligned} B(X_{k+1}) \leqslant \varepsilon X_{k+1} + X_k &+ t\left(u_{i,j}^{(k)} \mu_{i,j+1}^{(k)} + \sum_{q=j+1}^{k-1} u_{i,q}^{(k)} \Big(\mu_{i,q+1}^{(k)} - \mu_{i,q}^{(k)}\Big)\right) X_k^2 \\ &+ (1-t)\left(u_{i,j+1}^{(k)} \mu_{i,j+1}^{(k)} + \sum_{q=j+1}^{k-1} u_{i,q}^{(k)} \Big(\mu_{i,q+1}^{(k)} - \mu_{i,q}^{(k)}\Big)\right) X_k^2 + (i+2)k\,, \end{aligned}$$

yielding

$$B(X_{k+1}) \leqslant \varepsilon X_{k+1} + X_k + \Big(t v_{i,j}^{(k)} + (1-t) v_{i,j+1}^{(k)} \Big) X_k^2 + (i+2)k\,.$$

Since $X_{k+1} \geqslant X_k^2$ and $i^2 \leqslant 2k \leqslant 4 \log\log X_k$, we obtain

$$\begin{aligned} \frac{B(X_{k+1})}{X_{k+1}} &\leqslant \varepsilon + \frac{1}{X_k} + \Big(t v_{i,j}^{(k)} + (1-t) v_{i,j+1}^{(k)} \Big) + \frac{(i+2)k}{X_k^2} \\ &\leqslant \varepsilon + \Big(t v_{i,j}^{(k)} + (1-t) v_{i,j+1}^{(k)} \Big) + O\big(X_k^{-1}\big)\,, \end{aligned}$$

and by the concavity of f_i, we get

$$\frac{B(X_{k+1})}{X_{k+1}} \leqslant \varepsilon + f_i(x) + O\big(X_k^{-1}\big)\,.$$

Letting the integer $k \in \mathcal{T}_i$ to tend to infinity, we get $\underline{\mathrm{d}}B \leqslant \varepsilon + f_i(x)$. This being true for any $\varepsilon > 0$, we finally obtain $\underline{\mathrm{d}}B \leqslant f_i(x)$.

(ii) Let $0 < x \leqslant u_{i,0}$ and define

$$B = \bigcup_{\ell=0}^{i} \bigcup_{k\in\mathcal{T}_\ell} \left(\bigcup_{j=0}^{k-1} \Big(X_k + \mu_{\ell,j}^{(k)} X_k^2,\, X_k + \mu_{\ell,j+1}^{(k)} X_k^2 \Big) \right.$$
$$\left. \cap\, \mathbb{N}\big(u_{0,0}; u_{1,0}; \dots; u_{\ell-1,0}; u_{\ell,j}^{(k)}; u_{\ell+1,0}; \dots; u_{i,0}; x\big) \right)$$
$$\cup \bigcup_{\substack{k\geqslant 1\\ k\notin \bigcup_{\ell=0}^{i} \mathcal{T}_\ell}} \big(X_k, X_{k+1}\big] \cap \mathbb{N}\big(u_{0,0}; u_{1,0}; \dots; u_{i,0}; x\big)\,.$$

Let $k \geqslant 1$ be an integer. For any real number such that $X_k < X \leqslant X_{k+1}$, we plainly deduce from the upper bound in Lemma 5 that $B(X) - B(X_k) \leqslant x(X - X_k) + (i+3)k$, hence $B(X) \leqslant xX + O(ik^2)$. Since $i^2 \leqslant 2k \leqslant 4\log\log X_k$, we get $B(X) \leqslant xX + O\big((\log\log X)^{5/2}\big)$ for any $X > 0$. This gives $\overline{\mathrm{d}}B \leqslant x$.

Furthermore, for any positive integer $k \notin \bigcup_{\ell=0}^{i} \mathcal{T}_\ell$ and any $X_k < X \leqslant X_{k+1}$, we have by Lemma 5

$$B(X) - B(X_k) \geqslant x(X - X_k) - (i+3)\,, \tag{5}$$

hence in particular, putting $X = X_{k+1}$, dividing both sides by X_{k+1} and letting k to tend to infinity, we get $\overline{\mathrm{d}}B \geqslant x$, thus $\overline{\mathrm{d}}B = x$.

Now, let X be a large real number such that $X_k < X \leqslant X_{k+1}$ for some $k \in \bigcup_{\ell=0}^{i} \mathcal{T}_\ell$. Let ℓ be the unique integer $0 \leqslant \ell \leqslant i$ such that $k \in \mathcal{T}_\ell$. We denote by m the unique integer such that $0 \leqslant m \leqslant k-1$ and

$$X_k + \mu_{\ell,m}^{(k)} X_k^2 < X \leqslant X_k + \mu_{\ell,m+1}^{(k)} X_k^2\,.$$

We put

$$\Delta_\ell^{(k)}(x) = f_\ell(x) - \left(tf\Big(u_{\ell,j}^{(k)}\Big) + (1-t) f\Big(u_{\ell,j+1}^{(k)}\Big) \right) = f_\ell(x) - \Big(t\, v_{\ell,j}^{(k)} + (1-t)\, v_{\ell,j+1}^{(k)} \Big)\,,$$

where $x = t\,u_{\ell,j}^{(k)} + (1-t)u_{\ell,j+1}^{(k)}$, for some $0 \leqslant j \leqslant k-1$ and $0 < t \leqslant 1$. Since f is concave, we have $\Delta_\ell^{(k)}(x) \geqslant 0$ and since f_ℓ is a 1-Lipschitzian function, we have

$$\Delta_\ell^{(k)}(x) \leqslant t\Big(u_{\ell,j}^{(k)} - x\Big) + (1-t)\Big(x - u_{\ell,j+1}^{(k)}\Big) = 2t(1-t)\Big(u_{\ell,j}^{(k)} - u_{\ell,j+1}^{(k)}\Big) \leqslant \frac{1}{2k}\,. \tag{6}$$

If $m \leqslant j$, then $x \leqslant u_{\ell,m}^{(k)}$, thus by Lemma 5

$$\begin{aligned} B(X) - B(X_k) \geqslant \sum_{q=0}^{m-1} &\left(x\left(\mu_{\ell,q+1}^{(k)} - \mu_{\ell,q}^{(k)}\right) X_k^2 - (i+3) \right) \\ &+ x\left(X - \mu_{\ell,m}^{(k)} X_k^2 - X_k\right) - (i+3) \\ \geqslant\ & x(X - X_k) - (i+3)(m+1). \end{aligned} \tag{7}$$

If $m \geqslant j+1$, then by Lemma 5 again

$$\begin{aligned} B(X) - B(X_k) \geqslant\ & \sum_{q=0}^{j} \left(x\left(\mu_{\ell,q+1}^{(k)} - \mu_{\ell,q}^{(k)}\right) X_k^2 - (i+3) \right) \\ &+ \sum_{q=j+1}^{m-1} \left(u_{\ell,q}^{(k)} \left(\mu_{\ell,q+1}^{(k)} - \mu_{\ell,q}^{(k)}\right) X_k^2 - (i+3) \right) \\ &+ u_{\ell,m}^{(k)}\left(X - \mu_{\ell,m}^{(k)} X_k^2 - X_k\right) - (i+3), \end{aligned}$$

thus

$$\begin{aligned} B(X) - B(X_k) \geqslant & \, t\left(u_{\ell,j}^{(k)} \mu_{\ell,j+1}^{(k)} + \sum_{q=j+1}^{m-1} u_{\ell,q}^{(k)} \left(\mu_{\ell,q+1}^{(k)} - \mu_{\ell,q}^{(k)}\right) - u_{\ell,m}^{(k)} \mu_{\ell,m}^{(k)} \right) X_k^2 \\ & + (1-t)\left(u_{\ell,j+1}^{(k)} \mu_{\ell,j+1}^{(k)} + \sum_{q=j+1}^{m-1} u_{\ell,q}^{(k)} \left(\mu_{\ell,q+1}^{(k)} - \mu_{\ell,q}^{(k)}\right) - u_{\ell,m}^{(k)} \mu_{\ell,m}^{(k)} \right) X_k^2 \\ & + u_{\ell,m}^{(k)}(X - X_k) - (i+3)(m+1) \\ = & \left(t v_{\ell,j}^{(k)} + (1-t) v_{\ell,j+1}^{(k)} \right) X_k^2 - v_{\ell,m}^{(k)} X_k^2 \\ & + u_{\ell,m}^{(k)}(X - X_k) - (i+3)(m+1) \\ \geqslant & \left(t v_{\ell,j}^{(k)} + (1-t) v_{\ell,j+1}^{(k)} \right) X_k^2 - v_{\ell,m}^{(k)}(X_{k+1} - X) - (i+3)(m+1), \end{aligned}$$

since $u_{\ell,m}^{(k)} \geqslant f_\ell(u_{\ell,m}^{(k)}) = v_{\ell,m}^{(k)}$. Since $v_{\ell,m}^{(k)} \leqslant v_{\ell,j+1}^{(k)} \leqslant t v_{\ell,j}^{(k)} + (1-t) v_{\ell,j+1}^{(k)}$, we get

$$\begin{aligned} B(X) - B(X_k) \geqslant & \left(t v_{\ell,j}^{(k)} + (1-t) v_{\ell,j+1}^{(k)} \right)(X - X_k) - (i+3)(m+1) \\ & + \left(t v_{\ell,j}^{(k)} + (1-t) v_{\ell,j+1}^{(k)} - v_{\ell,m}^{(k)} \right)(X_{k+1} - X) \\ \geqslant & \left(t v_{\ell,j}^{(k)} + (1-t) v_{\ell,j+1}^{(k)} \right)(X - X_k) - (i+3)(m+1) \\ = & \left(f_\ell(x) - \Delta_\ell^{(k)}(x) \right)(X - X_k) - (i+3)(m+1) \\ \geqslant & \left(f_\ell(x) - \frac{1}{2k} \right)(X - X_k) - (i+3)(m+1), \end{aligned}$$

using the upper bound (6).

In view of (7) and the fact that $x \geqslant f_\ell(x)$, the same bound holds when $m \leqslant j$. This yields

$$B(X) - B(X_k) \geqslant \left(f_\ell(x) - \frac{1}{2k}\right)(X - X_k) - (i+3)k\,,$$

for any $X_k < X \leqslant X_{k+1}$ whenever k belongs to $\mathcal{T}_\ell$.

We now observe that $\mathbf{S}_{\ell+1} \subseteq \mathbf{S}_\ell$ implies $f_\ell \geqslant f_{\ell+1}$ for any $\ell \geqslant 0$, thus with (5), this yields

$$B(X) - B(X_k) \geqslant \left(f_i(x) - \frac{1}{2k}\right)(X - X_k) - (i+3)k\,,$$

for any $X_k < X \leqslant X_{k+1}$ and any $k \geqslant 1$. It follows that

$$B(X) \geqslant B(X_{k-1}) + \left(f_i(x) - \frac{1}{2(k-1)}\right)(X - X_{k-1}) - 2(i+3)k\,,$$

for any $X_k < X \leqslant X_{k+1}$ and any $k \geqslant 2$. We finally obtain for any large real number X

$$B(X) \geqslant f_i(x)X - X^{1/2} + O\left(\frac{X}{\log\log X}\right) + O\big((\log\log X)^{3/2}\big)\,,$$

using the bounds (4), $X_k \geqslant X_{k-1}^2$ and $i^2 \leqslant 2k$. We thus have $\underline{\mathrm{d}}B \geqslant f_i(x)$. Since $B \subseteq A_i$ and $\overline{\mathrm{d}}B = x$, we get from part (i) that $\underline{\mathrm{d}}B \leqslant f_i(x)$, yielding $D(B) = \big(x, f_i(x)\big)$. This ends the proof of Theorem 4.

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