Tatra Mt. Math. Publ. **31** (2005), 39–54

TATRA MOUNTAINS
Mathematical Publications

SOME GENERALIZED VERSIONS OF THE CONDORCET JURY THEOREM

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ABSTRACT. In the context of binary choice voting procedures, we prove some generalized version of the classical Condorcet Jury Theorem, concerning the behaviour of the probability of choosing the right alternative when the number of voters increases to ∞.

1. Introduction

Consider the following situation: a team of n voters has to make a binary choice; assume that one (and only one) of the alternatives is the correct one, and that the kth member of the team has a probability v_k $(0 \leqslant v_k \leqslant 1)$ of identifying it (in technical language such a probability is also called *competence.*)

The team of voters takes its collective decision according to the majority rule: they choose the alternative which has received more than $n/2$ preferences.

Let $P(n)$ denote the probability that the whole team chooses the right alternative. In the specialized literature (see [4] for instance), the *Condorcet Jury Theorem* is stated as follows: assume that $v_k = p$ for all k; then, if $p > 1/2$ and $n \geqslant 3$,

1. $P(n) > p$, and
2. $\lim_{n\to\infty} P(n) = 1$.

In the current paper we shall be concerned in discussing point 2 above. We observe that the classical setting is somehow too restrictive in the following two senses:

(i) there are situations in which the individual competencies of the voters are not completely known, so that it is more realistic to consider them as random

2000 Mathematics Subject Classification: 91B12, 60F05, 40A05.
Keywords: Condorcet Jury Theorem, weighted majority rule, Lyapunov's Theorem, statistical convergence.

This paper is partially supported by MIUR, Italy.

variables V_k (in the sequel we shall denote random variables with capital letters, and real numbers with small letters); see [1] for some examples;

(ii) the standard majority rule assigns the same importance to each voter; as a matter of fact, there are situations in which the individual votes do not "weigh" equally one to another: for instance the so called "expert rule" gives importance to one voter only (the expert of the team), but there are also intermediate cases: see [3] for an account on this topic.

Hence, in the general case a fruitful point of view is to assume that the members of the team adopt a "weighted majority rule" as their collective decision rule. This will mean that to the kth juror's individual vote a weight W_k is attached and the alternative is chosen for which the sum of the obtained weights is greater than the analogous sum for the other one (according to this point of view, the classical majority rule gives unitary (=1) weight to each individual vote).

Note that the weight W_k is indicated with a capital letter since in many cases it will be considered as a random variable; we refer to our Example (2.6) below for more details on this point.

The first part of the present paper (Sections 1–4) deals with the behaviour of $P(n)$ (the probability of making the right choice) in these more general settings, and is organized as follows: Section 2 contains the statement of our main result (Theorem (2.3)) for the case of random weights, along with some examples; in Section 3 we prove Theorem (2.3); Section 4 deals with the case of deterministic weights; the main results of Section 4 are Theorem (4.2) and Corollary (4.7).

The second part of the paper (Section 5) contains a more detailed study of the case of the standard majority rule. Here our results will be stated in terms of the concept of *statistical convergence*, recently introduced and studied in a series of paper (see [9] for an account). The use of statistical convergence is justified in Remark (5.6).

Acknowledgement. The author wishes to thank V. Baláž for suggesting the use of the notion of statistical convergence in the context of mathematical voting problems.

2. The random weighted Condorcet Jury Theorem

We adopt the following stochastic model: the decision of the kth voter is represented by the random variable

$$Z_k = \begin{cases} 1 & \text{if the } k\text{th voter chooses the right decision}, \\ 0 & \text{in the opposite case}. \end{cases}$$

We assume that the competence of the kth voter is a random variable V_k (with $P(0 < V_k < 1) = 1$, for every k); this means that, for each k, the conditional law of Z_k, given $V_k = v$ has a Bernoulli density with parameter v; in formulas

$$P(Z_k = 1 | V_k = v) = v\,, \qquad P(Z_k = 0 | V_k = v) = 1 - v\,. \tag{2.1}$$

The above imply easily that each Z_k is Bernoulli with parameter $\mathbf{E}[V_k]$.

To the kth individual vote a random weight W_k is attached; we assume that there exists a number $\delta > 0$ such that, for every k, the random variable W_k has finite absolute $(2+\delta)$th moment (i.e., $\mathbf{E}\big[|W_k|^{2+\delta}\big] < +\infty$).

We assume moreover that (Z_k, V_k, W_k) form a sequence of independent three-dimensional random vectors. In particular, the fact that (Z_k) is a sequence of independent r. v.'s obviously means that none of the voters is influenced by or can influence other members of the team.

With these notations, the right (resp. wrong) alternative receives a cumulative vote equal to

$$\sum_{k=1}^{n} W_k Z_k \quad \left(\text{resp. } \sum_{k=1}^{n} W_k (1 - Z_k)\right),$$

and the right alternative is chosen if

$$\sum_{k=1}^{n} W_k Z_k > \sum_{k=1}^{n} W_k (1 - Z_k)\,,$$

or, equivalently, if

$$\sum_{k=1}^{n} W_k (2Z_k - 1) > 0\,.$$

Hence, we are led to the formula

$$P(n) = P\left(\sum_{k=1}^{n} W_k (2Z_k - 1) > 0\right).$$

For technical reasons, our results will be stated in terms of the complementary probability

$$Q(n) = P\left(\sum_{k=1}^{n} W_k (2Z_k - 1) \leqslant 0\right).$$

Put, for every integer k

$$\mu_k = \mathbf{E}\big[W_k (2Z_k - 1)\big]\,,$$

$$\sigma_k^2 = \mathbf{Var}\big(W_k (2Z_k - 1)\big)\,,$$

and, for every n,

$$x_n = -\frac{\sum\limits_{k=1}^{n} \mu_k}{\sqrt{\sum\limits_{k=1}^{n} \sigma_k^2}}\,. \tag{2.2}$$

In what follows the symbol Φ will indicate the probability distribution function of the standard gaussian law:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)\, dt\,.$$

The random weighted Condorcet Theorem is stated as follows:

(**2.3**) **Theorem.** *Assume that at least one of the variables $W_k(2Z_k-1)$ and at least one of the variables $|W_k|$ has non-degenerate distribution; assume moreover that the following condition holds*

$$\sum_{k=1}^{n} \mathbf{E}\Big[|W_k|^{2+\delta}\Big] = o\left(\left(\sum_{k=1}^{n} \mathbf{Var}\big(|W_k|\big)\right)^{1+\delta/2}\right). \tag{2.4}$$

Then we have

$$\lim_{n\to\infty} \big|Q(n) - \Phi(x_n)\big| = 0\,.$$

(**2.5**) **Remark.** Assumption (2.4) is for instance verified if

$$\inf_k \mathbf{Var}\big(|W_k|\big) > 0 \quad \text{and} \quad \sup_k \mathbf{E}\big[|W_k|^{2+\delta}\big] < \infty\,.$$

The proof of Theorem (2.3) is postponed to the next section. Here we are concerned with illustrating it with some examples and applications.

(2.6) Example. An important particular case of the situation studied in (2.3) is frequently considered in the applied literature (see for instance [1] and [3]). Assume that the weights (W_k) are of the form

$$W_k = \phi_k(V_k)\,,$$

where (ϕ_k) is any sequence of measurable functions $\phi_k : \mathbb{R} \to \mathbb{R}$. This has the intuitive (and reasonable) meaning that each voter has at disposal a weight which depends on his own competence about the matter.

The most frequently encountered situation is when $\phi_k = \phi$, for every k, and the (V_k) are identically distributed with a common law ν. Assume that both the functions $v \mapsto \phi(v)(2v-1)$ and $v \mapsto |\phi|(v)$ are not constant on $\operatorname{Im} V_1$ (i.e., the laws of $\phi(V_k)(2V_k - 1)$ and of $|\phi|(V_k)$ are non-degenerate); Assumption (2.4) is automatically satisfied: since $\mathbf{E}\big[|\phi|(V_1)^{2+\delta}\big] \doteq M < +\infty$, we have

$$\sum_{k=1}^{n} \mathbf{E}\Big[|W_k|^{2+\delta}\Big] = nM = o\big(n^{1+\delta/2}\big)\,.$$

The mean μ_k of $\mathbf{E}\big[W_k(2Z_k - 1)\big]$ in this case is given by

$$\begin{aligned}\mathbf{E}\big[\phi(V_k)(2Z_k - 1)\big] &= \mathbf{E}\Big[\mathbf{E}\big[\phi(V_k)(2Z_k - 1)|V_k\big]\Big] \\ &= \mathbf{E}\big[\phi(V_k)(2V_k - 1)\big] = \int \phi(v)(2v-1)\,\nu(dv) \doteq \mu\,.\end{aligned}$$

Put $\sigma^2 = \mathbf{Var}|\phi|(V_1) > 0$. From (2.2) we get easily

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} -\frac{\sqrt{n}\mu}{\sigma^2} = \begin{cases} -\infty & \text{if } \mu > 0\,, \\ 0 & \text{if } \mu = 0\,, \\ +\infty & \text{if } \mu < 0\,, \end{cases}$$

and, correspondingly, by applying Theorem (2.3) we find that

$$\lim_{n\to\infty} Q(n) = \begin{cases} 0 & \text{if } \mu > 0\,, \\ 1/2 & \text{if } \mu = 0\,, \\ 1 & \text{if } \mu < 0\,. \end{cases}$$

We so discover that in this case only three values are allowed as limit of $Q(n)$; moreover the value of such limit depends only on the sign of $\mu = \mathbf{E}\big[\phi(V_1)(2V_1 - 1)\big]$. In particular, point 2 of the classical Condorcet Theorem holds for every ϕ such that

$$\phi(v)(2v-1) > 0\,, \qquad \text{for } \nu\text{-almost every } v \in (0,1)\,. \tag{2.7}$$

(**2.8**) **Remark.** In particular, if the competences (V_k) are all greater than $1/2$ (which means that each voter is "competent" enough about the matter), (2.7) holds for every ϕ such that $\phi(x) > 0$ for ν-almost every $x \in (1/2, 1)$. This kind of situation is studied in [1], where the competences are assumed to be uniformly distributed on the interval $[1/2, 1]$. Other families of distributions are studied in [3] and [7].

(**2.9**) **Remark.** In the paper [1] the authors consider the random weights constructed through the function

$$\phi(v) = \log\frac{v}{1-v}\,, \qquad 0 < v < 1\,. \tag{2.10}$$

The proof of Lemma (4.13) of the present paper (see Remark (4.14)) suggests the use of another function, namely

$$\phi(v) = \frac{2v-1}{v(1-v)}\,, \qquad 0 < v < 1\,. \tag{2.11}$$

Note that both functions (2.10) and (2.11) verify

$$\phi(v)(2v-1) \geqslant 0\,, \qquad 0 < v < 1\,,$$

hence $\mu = \mathbf{E}\big[\phi(V_1)(2V_1-1)\big] \geqslant 0$; moreover $\mu > 0$ if $P(V_1 = 1/2) < 1$.

The above remarks and examples (from (2.5) to (2.11)) provide relevant generalizations of the usual settings of the CJT.

3. The proof of Theorem 2.3

Our proof relies on the following classical result (see [8] for instance):

(3.1) Theorem. (Lyapunov). *Let $(Y_k)_k$ be a sequence of centered independent random variables, at least one of which has non-degenerate distribution. Let $\mathbf{E}\big[|Y_k|^{2+\delta}\big] < \infty$ for some $\delta > 0$ $(k = 1, 2, \ldots)$. We put*

$$B_n = \sum_{k=1}^{n} \mathbf{Var}\, Y_k\,,$$

$$F_n(x) = P\left(B_n^{-1/2}\sum_{k=1}^{n} Y_k < x\right),$$

$$G_n(x) = P\left(B_n^{-1/2}\sum_{k=1}^{n} Y_k \leqslant x\right).$$

If

$$B_n^{-1-\delta/2}\sum_{k=1}^{n}\mathbf{E}|Y_k|^{2+\delta} \to 0\,, \tag{3.2}$$

then

$$\lim_{n\to\infty}\sup_x\big|G_n(x)-\Phi(x)\big| = \lim_{n\to\infty}\sup_x\big|F_n(x)-\Phi(x)\big| = 0\,.$$

Actually, in [8] the above Theorem is proved only for F_n, but it is not difficult to deduce the analogous result for G_n. We omit the details for simplicity.

We want to apply Theorem (3.1) to the sequence of variables

$$Y_k = W_k(2Z_k - 1) - \mu_k\,;$$

we have

$$Q(n) = P\left(\sum_{k=1}^{n} W_k(2Z_k - 1) \leqslant 0\right) = P\left(B_n^{-1/2}\sum_{k=1}^{n} Y_k \leqslant x_n\right) = G_n(x_n),$$

where x_n has been defined in (2.2). By Theorem (3.1), we have only to prove that (3.2) holds.

We first observe that, for every k,

$$|2Z_k - 1| = 1; \tag{3.3}$$

from this we get

$$\sigma_k^2 = \mathbf{Var}\big(W_k(2Z_k-1)\big) = \mathbf{E}\big[W_k^2(2Z_k-1)^2\big] - \mu_k^2 = \mathbf{E}\big[W_k^2\big] - \mu_k^2 \tag{3.4}$$

and

$$|\mu_k| \leqslant \mathbf{E}\big[|W_k||2Z_k-1|\big] = \mathbf{E}\big[|W_k|\big]. \tag{3.5}$$

From (3.4) and (3.5) we get

$$\sigma_k^2 \geqslant \mathbf{E}\big[W_k^2\big] - \mathbf{E}^2\big[|W_k|\big] = \mathbf{Var}\big(|W_k|\big). \tag{3.6}$$

From the inequality

$$\big(|a|+|b|\big)^{\alpha} \leqslant 2^{\alpha-1}\big(|a|^{\alpha}+|b|^{\alpha}\big), \qquad \alpha > 0,$$

we obtain (recalling (3.3) again)

$$\begin{aligned}\mathbf{E}\big[|W_k(2Z_k-1)-\mu_k|^{2+\delta}\big] &\leqslant 2^{1+\delta}\Big(\mathbf{E}\Big[|W_k|^{2+\delta}\Big] + \mathbf{E}^{2+\delta}\Big[|W_k|\Big]\Big)\\ &\leqslant 2^{2+\delta}\mathbf{E}\Big[|W_k|^{2+\delta}\Big],\end{aligned} \tag{3.7}$$

where we have used the classical inequality

$$\mathbf{E}^r\big[|X|\big] \leqslant \mathbf{E}\big[|X|^r\big].$$

From (3.6) we get

$$B_n^{-1-\delta/2} = \left(\sum_{k=1}^{n} \mathbf{Var} Y_k\right)^{-1-\delta/2} \leqslant \left(\sum_{k=1}^{n} \mathbf{Var}\big(|W_k|\big)\right)^{-1-\delta/2}; \tag{3.8}$$

from (3.7) we deduce

$$\sum_{k=1}^{n} \mathbf{E}\Big[|W_k(2Z_k-1)-\mu_k|^{2+\delta}\Big] \leqslant 2^{2+\delta}\sum_{k=1}^{n}\mathbf{E}\Big[|W_k|^{2+\delta}\Big]. \tag{3.9}$$

Assumption (2.4), along with (3.8) and (3.9), gives (3.2) immediately.

4. The deterministic weighted Condorcet Jury Theorem

Theorem (2.3) does not cover the practically quite relevant case of deterministic weights, i.e., the case in which $W_k = w_k$ (a constant for each integer k). In this section we shall consider such a situation. Here, the sequence (x_n) defined in (2.2) has the following form

$$x_n = \frac{\sum_{k=1}^{n} w_k\big(1 - 2\mathbf{E}[V_k]\big)}{\Big(\sum_{k=1}^{n} w_k^2 \mathbf{E}[V_k]\big(1 - \mathbf{E}[V_k]\big)\Big)^{1/2}} . \tag{4.1}$$

We shall look at the weights as n-dimensional vectors, and put

$$w = (w_1, w_2, \dots, w_n),$$

when necessary, in the sequel we shall stress the dependence of x_n on w by writing $x_n^{(w)}$ instead of x_n.

Our result is the following

(**4.2**) **Theorem.** *Let as usual $Q(n)$ be the probability of making the wrong choice, and assume that*

$$\sum_{k=1}^{\infty} \mathbf{E}[V_k]\big(1 - \mathbf{E}[V_k]\big) = \infty . \tag{4.3}$$

Then

$$\lim_{n\to\infty} \big|Q(n) - \Phi(x_n)\big| = 0 .$$

P r o o f . Put for simplicity

$$\alpha_k = \mathbf{E}[V_k]\big(1 - \mathbf{E}[V_k]\big) .$$

We apply Lyapunov's Theorem (3.1) to the random variables

$$Y_k = w_k(2Z_k - 1) - \mu_k$$

(where $\mu_k = \mathbf{E}[w_k(2Z_k - 1)]$). In this case we can take $\delta = 1$. We have

$$\begin{aligned}\mathbf{E}\big[|w_k(2Z_k - 1) - \mu_k|^3\big] &= 8|w_k|^3 \mathbf{E}\big[|Z_k - \mathbf{E}[Z_k]|^3\big] \\ &= 8|w_k|^3\Big(\big(1 - \mathbf{E}[Z_k]\big)^3 \mathbf{E}[Z_k] + \big(1 - \mathbf{E}[Z_k]\big)\mathbf{E}^3[Z_k]\Big) \\ &= 8|w_k|^3 \alpha_k(1 - 2\alpha_k),\end{aligned}$$

where we have used the fact that Z_k has Bernoulli law with parameter $\mathbf{E}[V_k]$.

Moreover

$$\sigma_k^2 = \mathbf{Var}\big(w_k(2Z_k - 1)\big) = 4w_k^2 \mathbf{Var} Z_k = 4w_k^2 \alpha_k\,;$$

the quantity in (3.2) becomes

$$\frac{\sum\limits_{k=1}^{n} |w_k|^3 \alpha_k (1 - 2\alpha_k)}{\left(\sum\limits_{k=1}^{n} w_k^2 \alpha_k\right)^{3/2}}\,. \tag{4.4}$$

Hence we have only to show that the above expression in (4.4) goes to 0 as $n \to \infty$. This follows from assumption (4.3) and the □

(**4.5**) **Lemma.** *Consider the fraction in* (4.4) *as a function of the n-dimensional vector $w = (w_1, w_2, \ldots, w_n)$. Then its maximum is less than*

$$\left(\sum_{k=1}^{n} \alpha_k\right)^{1/2}.$$

P r o o f. By the obvious inequality $0 \leqslant 1 - 2\alpha_k \leqslant 1$, we shall deal with the easier fraction

$$\frac{\sum\limits_{k=1}^{n} |w_k|^3 \alpha_k}{\left(\sum\limits_{k=1}^{n} w_k^2 \alpha_k\right)^{3/2}}\,,$$

and we shall find its maximum by means of elementary calculus. Our problem is clearly equivalent to searching the maximum of the function

$$\psi(w_1, w_2, \ldots, w_n) = \sum_{k=1}^{n} \alpha_k w_k^3$$

with the constraint

$$\mathcal{C} = \left\{ w \colon w_k \geqslant 0\ \forall k,\ \sum_{k=1}^{n} \alpha_k w_k^2 = 1 \right\}.$$

By applying the method of Lagrange multipliers we find that the maximum of ψ on $\mathcal{C}$ is attained in the vector

$$\overline{w} = \left(\left(\sum_{k=1}^{n} \alpha_k\right)^{-\frac{1}{2}}, \ldots, \left(\sum_{k=1}^{n} \alpha_k\right)^{-\frac{1}{2}} \right)$$

and its value is precisely

$$\left(\sum_{k=1}^{n} \alpha_k\right)^{-\frac{1}{2}}$$

□

We can also state a consequence of Theorem (4.2). Put, for every n,

$$c_n = \sum_{k=1}^{n} \frac{\beta_k^2}{\alpha_k}, \tag{4.6}$$

where $\beta_k = 1 - 2\mathbf{E}[V_k]$.

(**4.7**) **Corollary.** *Assume that*

$$c \doteq \sum_{k=1}^{\infty} \frac{\beta_k^2}{\alpha_k} < \infty. \tag{4.8}$$

Then

$$\Phi(-\sqrt{c}) \leqslant \liminf_{n\to\infty} Q(n) \leqslant \limsup_{n\to\infty} Q(n) \leqslant \Phi(\sqrt{c}).$$

P r o o f. Fix $\varepsilon > 0$. By Theorem (4.2) we have, for every sufficiently large n,

$$-\varepsilon + \Phi(x_n^{(w)}) \leqslant Q(n) \leqslant \Phi(x_n^{(w)}) + \varepsilon. \tag{4.9}$$

Put

$$M_n = \sup_{w\in\mathbb{R}^n} \Phi(x_n^{(w)}) = \Phi\left(\sup_{w\in\mathbb{R}^n} x_n^{(w)}\right), \tag{4.10}$$

$$m_n = \inf_{w\in\mathbb{R}^n} \Phi(x_n^{(w)}) = \Phi\left(\inf_{w\in\mathbb{R}^n} x_n^{(w)}\right), \tag{4.11}$$

where the last equalities in (4.10) and (4.11) hold since Φ is strictly increasing and continuous.

Then from (4.9) we get

$$m_n - \varepsilon \leqslant Q(n) \leqslant M_n + \varepsilon. \tag{4.12}$$

We prove an auxiliary result

(**4.13**) **Lemma.** *We have*

$$\max_{w\in\mathbb{R}^n} x_n^{(w)} = \sqrt{c_n}; \qquad \min_{w\in\mathbb{R}^n} x_n^{(w)} = -\sqrt{c_n},$$

(*where* c_n *is defined in* (4.6)).

Proof. We recall the notations $\alpha_k = \mathbf{E}[V_k](1-\mathbf{E}[V_k])$ and $\beta_k = 1-2\mathbf{E}[V_k]$. By the Cauchy–Schwartz inequality, we have

$$\left|x_n^{(w)}\right| \leqslant \frac{\sum_{k=1}^{n} \frac{|\beta_k|}{\sqrt{\alpha_k}} \sqrt{\alpha_k} w_k}{\left(\sum_{k=1}^{n} \alpha_k w_k^2\right)^{1/2}} \leqslant \left(\sum_{k=1}^{n} \frac{\beta_k^2}{\alpha_k}\right)^{1/2} = \sqrt{c_n}\,.$$

Moreover, for $w_k = \beta_k/\alpha_k$ (resp. $w_k = -\beta_k/\alpha_k$), $k = 1, \dots, n$, we have $x_n^{(w)} = \sqrt{c_n}$ (resp. $x_n^{(w)} = -\sqrt{c_n}$). This concludes the proof. ☐

We now get the statement of Corollary (4.7) by going to the limit in (4.12) and recalling (4.10) and (4.11). ☐

(**4.14**) **Remark.** We have

$$\frac{\beta_k}{\alpha_k} = \frac{1 - 2\mathbf{E}[V_k]}{\mathbf{E}[V_k](1 - \mathbf{E}[V_k])}\,.$$

For $V_k = v_k$ we can write

$$\frac{\beta_k}{\alpha_k} = -\phi(v_k)\,,$$

where ϕ is the function defined in (2.11).

(**4.15**) **Remark.** Conditions (4.3) and (4.8) are somehow better stated and interpreted by means of the sequence

$$y_k \doteq \frac{1}{2} - \mathbf{E}[V_k]$$

(note that y_k^2 measures the quadratic deviation of the mean kth competence $\mathbf{E}[V_k]$ from the "central" competence $1/2$). In fact, (4.3) is the same as

$$\lim_{n\to\infty} \frac{n}{4} - \sum_{k=1}^{n} y_k^2 = +\infty\,,$$

while (4.8) becomes

$$\sum_{k=1}^{\infty} \frac{y_k^2}{\frac{1}{4} - y_k^2} < \infty\,; \tag{4.16}$$

moreover, it is easy to see that (4.16) is in turn equivalent to the better looking condition

$$\sum_{k=1}^{\infty} y_k^2 < \infty\,.$$

The above considerations suggest that the behaviour of $Q(n)$ is mainly governed by the sequence (y_k^2) of the "quadratic" deviations, and more precisely by the behaviour of their partial sums $\sum_{k=1}^{n} y_k^2$.

5. Statistical convergence and the majority rule

THIS SECTION IS WRITTEN IN COLLABORATION WITH G. GREKOS[1]

In this section we treat in greater detail the case of majority rule, i.e., the case in which the weights w_k are equal to 1 for every k. With the usual notation (see Remark (4.15))

$$y_k \doteq \frac{1}{2} - \mathbf{E}[V_k],$$

the sequence (x_n) defined in (4.1) becomes

$$x_n = \frac{\sum_{k=1}^{n} y_k}{\left(\frac{n}{4} - \sum_{k=1}^{n} y_k^2\right)^{1/2}}. \tag{5.1}$$

Our aim is to find suitable conditions on the sequence (y_n) in order to assure that the sequence $Q(n)$ has a limit L as $n \to \infty$. We shall be only concerned with the case in which such a limit L belongs to the *open* interval $(0,1)$, since the two cases $L = 0$ and its symmetric $L = 1$ are extensively studied in the specialized literature: as to our knowledge, the most complete result for the case $L = 0$ (i.e., for the CJT) is contained in [2], where the authors prove that the CJT is equivalent to one of the following conditions:

1. $\lim_{n\to\infty} x_n = -\infty$;
2. for every sufficiently large n, $\mathrm{card}\Big(\big\{i\colon 1 \leqslant i \leqslant n,\ \mathbf{E}[V_i] = 1\big\}\Big) > n/2$.

Before stating our result (Theorem (5.5)), we recall some well known facts about weighted arithmetical densities and give the definition of statistically convergent sequence.

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(**5.2**) **Definition.** Let A be an infinite subset of $\mathbb{N} = \{1, 2, 3, \dots\}$. For a fixed real number $\alpha \geqslant -1$, define the *lower* and *upper* α*-densities* of A as

$$\underline{d}_\alpha A = \liminf_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)}, \qquad \overline{d}_\alpha A = \limsup_{n\to\infty} \frac{A_\alpha(n)}{\mathbb{N}_\alpha(n)},$$

where

$$A_\alpha(n) = \sum_{\substack{a \in A \\ a \leqslant n}} a^\alpha, \qquad \mathbb{N}_\alpha(n) = \sum_{k=1}^{n} k^\alpha.$$

We say that A possesses an α-density equal to $d_\alpha A$ if

$$\underline{d}_\alpha A = \overline{d}_\alpha A = d_\alpha A.$$

For $\alpha = 0$, the corresponding α-densities are also called *asymptotic densities.*

Concerning α-densities, we recall the following result:

(**5.3**) **Theorem.** *Let $\beta > -1$ be fixed. If the set A possesses a β-density equal to $d_\beta A$, then it has an α-density for every $\alpha \geqslant -1$. Moreover, for every $\alpha \geqslant -1$ we have*

$$d_\alpha A = d_\beta A.$$

Theorem (5.3) follows easily from Theorem 6.1, Example 6.8 and Example 6.10 of [6].

We can now give the definition of statistical convergence (see for instance [9]).

(**5.4**) **Definition.** The sequence of real numbers (a_n) is said to be *statistically convergent* to $x \in \mathbb{R}$ if, for any $\varepsilon > 0$, the set

$$A^{(\varepsilon)} = \{k \in \mathbb{N}\colon |a_k - x| > \varepsilon\}$$

has asymptotic density equal to 0.

We are ready to state the main result of this section

(**5.5**) **Theorem.** *Assume again that*

$$\sum_{k=1}^{\infty} \mathbf{E}[V_k]\big(1 - \mathbf{E}[V_k]\big) = \infty.$$

Assume moreover that the sequence $(\sqrt{n}\, y_n)$ is bounded and statistically convergent to $l \in \mathbb{R}$. Then

$$\lim_{n\to\infty} x_n = 4l,$$

and, by Theorem (4.2),

$$\lim_{n\to\infty} Q(n) = \Phi(4l).$$

(**5.6**) **Remark.** The use of statistical convergence instead of the usual concept of convergence allows us to take into account also practical situations in which some of the individuals' competencies become "arbitrarily random", i.e., some members of the team of voters behave "strangely" (in the sense that the deviation of their (mean) competence from 1/2 would force the sequence $(\sqrt{n}\, y_n)$ not to converge). In rough terms, our Theorem says that the team can have a regular behaviour "as a whole" despite the fact that some of its members are "too clever" or "too stupid".

P r o o f . By remembering (5.1), the statement will be proved if we show that (i)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} y_k}{\sqrt{n}} = 2l\,,$$

(ii)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} y_k^2}{\log n} = l\,.$$

Put

$$G_\alpha(n) = \begin{cases} n^{1+\alpha}/(1+\alpha) & \text{for } \alpha > -1\,, \\ \log n & \text{for } \alpha = -1\,, \end{cases}$$

and recall the relation

$$\mathbb{N}_\alpha(n) = \sum_{k=1}^{n} k^\alpha \sim G_\alpha(n)\,, \qquad n \to \infty\,. \tag{5.7}$$

Both (i) and (ii) can be deduced from the following Theorem (which can be viewed as a sort of "Cesaro's Theorem" for statistical convergence and generalizes previous results in [5]): □

(**5.8**) **Theorem.** *Let (z_k) be a real sequence and α a real number, with $\alpha \geqslant -1$. Assume moreover that the sequence $(k^{-\alpha} z_k)$ is bounded in modulus by a constant M and is statistically convergent to $l \in \mathbb{R}$. Then*

(i) *for $\alpha > -1$ we have*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} z_k}{n^{1+\alpha}} = \frac{l}{\alpha+1}\,,$$

(ii) *for $\alpha = -1$ we have*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} z_k}{\log n} = l\,.$$

P r o o f . By relation (5.7), both statements (i) and (ii) are equivalent to

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} z_k}{\mathbb{N}_\alpha(n)} = l\,,$$

which we are going to prove.

(i) We first consider the case $l = 0$.

Let ε and δ be two strictly positive numbers and put

$$A^{(\varepsilon)} = \left\{ n \in \mathbb{N}^* : \frac{\left|\sum\limits_{k=1}^{n} z_k\right|}{\mathbb{N}_\alpha(n)} > \varepsilon \right\},$$

$$E^{(\delta)} = \left\{ k \in \mathbb{N} \colon |z_k| k^{-\alpha} > \delta \right\}.$$

By hypothesis, we have $d_0 E^{(\delta)} = 0$.

For any $n \in A(\varepsilon)$, we can write

$$\begin{aligned} \varepsilon < \mathbb{N}_\alpha(n)^{-1} \sum_{k=1}^{n} |z_k| k^{-\alpha} k^{\alpha} &= \mathbb{N}_\alpha(n)^{-1} \left[\sum_{\substack{1 \leqslant k \leqslant n \\ k \in (E^{(\delta)})^c}} + \sum_{\substack{1 \leqslant k \leqslant n \\ k \in E^{(\delta)}}} \right] \\ &\leqslant \delta + M \mathbb{N}_\alpha(n)^{-1} \left(E^{(\delta)}\right)_\alpha(n)\,, \end{aligned} \tag{5.9}$$

where the sum has been split according to the indications.

Assume that $A(\varepsilon)$ is infinite; then, letting $n \in A(\varepsilon)$ go to ∞ in (5.9), we obtain

$$0 < \varepsilon \leqslant \delta + M \limsup_{\substack{n\to\infty \\ n \in A(\varepsilon)}} \frac{E_\alpha^{(\delta)}(n)}{\mathbb{N}_\alpha(n)} \leqslant \delta + M \overline{d}_\alpha E^{(\delta)}\,. \tag{5.10}$$

Since $d_0 E^{(\delta)} = 0$, Theorem (5.3) yields also $d_\alpha E^{(\delta)} = 0$. By passing to the limit as $\delta \to 0$ in (5.10) we get a contradiction, and we conclude that $A^{(\varepsilon)}$ is finite.

(ii) The case $l \neq 0$ can be easily reduced to the previous one by putting

$$\tilde{z}_k = z_k - l k^{\alpha}.$$

In fact the sequence $k^{-\alpha} \tilde{z}_k$ statistically converges to 0 and is bounded, hence, by case (i), we get

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} z_k}{\mathbb{N}_\alpha(n)} - l = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} \tilde{z}_k}{\mathbb{N}_\alpha(n)} = 0\,.$$

□

(5.11) Remark. In Theorem (5.8) it is possible to weaken the assumption of statistical convergence of the sequence $(k^{-\alpha} z_k)$; suppose that

$$\liminf_{\delta \to 0} \overline{d}_\alpha E^{(\delta)} = 0\,;$$

then we can pass to the limit as $\delta \to 0$ in (5.10), obtaining the same contradiction as before.

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Received November 11, 2004

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