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AN APPLICATION OF ASYMPTOTIC DENSITY TO CHARACTERIZING VOTING RULES

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ABSTRACT. In this paper, we outline an application of asymptotic density in the field of mathematical social science. The topic we analyze is the axiomatic characterization of voting rules. Specifically, we extend the existing characterizations of majority and supermajority rules to the case of an infinite electorate. After presenting the classic results for the finite setting, we use asymptotic density to define density q-rules, which are characterized by neutrality, monotonicity, and bounded anonymity on almost all sets. We give an example illustrating the limits of this characterization and close by analyzing density majority rule.

1. Introduction

Social choice theory is a part of mathematical social science concerned with the abstract properties of various methods for collective decision making. By collective decision making, we mean making a "social choice" as a function of the individual preferences in a society.

The purpose of this paper is to illustrate the use of asymptotic density in social choice theory in the context of characterizing voting rules. There are many possible methods for making social choices such as majority rule and supermajority rule, plurality rule, runoff rule, voting by agenda, Borda count and many more. Some, but not all, of these methods have been axiomatically characterized.[1]

After providing some general notation, we review two classic results characterizing majority and supermajority rules with a finite electorate. Then we modify the setting by considering an infinite number of voters and show how the

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[1]Characterizations have been found for specific rules, such as majority rule [12], Borda count rule [16], and plurality rule [13, 3], as well as broad classes of rules such as scoring rules [14, 17].

notion of asymptotic density allows us to extend the classic results to this new setting.[2] In a nutshell, our approach is to employ asymptotic density to compare the sizes of the sets of voters that favour each of the two alternatives.

2. Aggregation rules for the finite case

Suppose there are n individuals choosing between two alternatives, denoted $\{-1,+1\}$. Here we require that n be finite. The choice for the entire group should respect the desires of the individuals as much as possible. The *preference* of individual i is given by $R_i \in \{-1,+1\}$, for $i = 1,\dots,n$. In making the social choice, there are three possible *outcomes*: -1 is selected, $+1$ is selected, or the two choices tie. Thus, an *aggregation rule* is a function $f\colon \{-1,+1\}^n \to \{-1,0,+1\}$, with 0 representing a tie. In other words, the procedure f assigns an outcome to every possible arrangements of preferences. As an example of an aggregation rule, we can define the following: f is majority rule if

$$f(R_1,\dots,R_n) = \begin{cases} +1 & \text{if } \sum\limits_i R_i > 0\,, \\ 0 & \text{if } \sum\limits_i R_i = 0\,, \\ -1 & \text{if } \sum\limits_i R_i < 0\,. \end{cases}$$

Thus, majority rule chooses the alternative from $\{-1,+1\}$ that more individuals prefer and chooses 0 if the two alternatives are preferred by equal size groups.

As our goal is to give characterizations of various aggregation rules, we next describe several desirable properties that are standard in the literature. Specifically, an aggregation rule f satisfies

- *neutrality* if, for all $(R_1,\dots,R_n)$,
  $$f(-R_1,\dots,-R_n) = -f(R_1,\dots,R_n)\,,$$
- *anonymity* if, for any permutation π and for all $(R_1,\dots,R_n)$,
  $$f(R_{\pi(1)},\dots,R_{\pi(n)}) = f(R_1,\dots,R_n)\,,$$
- *monotonicity* if $(R'_1,\dots,R'_n) \geqslant (R_1,\dots,R_n)$ implies $f(R'_1,\dots,R'_n) \geqslant f(R_1,\dots,R_n)$,
- *fragility of ties* if $f(R_1,\dots,R_n) = 0$ and $(R'_1,\dots,R'_n) \geqslant (R_1,\dots,R_n)$ with $R'_j > R_j$ for some j implies $f(R'_1,\dots,R'_n) = +1$, and
- *positive responsiveness* if it satisfies monotonicity and fragility of ties,

[2] Other work that uses asymptotic density to deal with infinite sets of individuals includes Lauwers [8, 9, 10], Lauwers, Van Liedekerke [11]. In a different context, Campbell and Kelly [2] use asymptotic density to study social choice trade-offs with an infinite set of alternatives.

where $(R'_1, \dots, R'_n) \geqslant (R_1, \dots, R_n)$ if $R'_i \geqslant R_i$ for all $i = 1, \dots, n$.[3] In other words, anonymity is invariance under permutations of individuals and neutrality is invariance under permutations of the alternatives. A rule is monotonic if increasing the support for an alternative does not hurt the standing of that alternative. Finally, an aggregation rule satisfies fragility of ties if a single voter changing her vote will break a tie.

3. Results for the finite case

Majority rule is the predominant method of social choice from two alternatives in the real world (and in mathematical models). It is not difficult to see that majority rule satisfies anonymity, neutrality, and positive responsiveness. In an important contribution to the development of social choice theory, Kenneth May [12] showed that majority rule is the only procedure that possesses these three properties, with a finite number of voters. This characterization is illustrated in Figure 1, which summarizes our first theorem.

anonymity neutrality positive responsiveness	$\Longleftrightarrow$	majority rule

FIGURE 1. May's characterization of majority rule.

THEOREM 1. (May 1952) *An aggregation rule f satisfies neutrality, anonymity, and positive responsiveness if and only if f is majority rule.*

Proof. It is easy to verify that majority rule satisfies the three properties. For the reverse, let $n(v)$ be the number of individuals with $R_i = v$, for $v \in \{-1, +1\}$. By anonymity, f can only depend on the values of $n(-1)$ and $n(+1)$.

We first show that $n(+1) = n(-1)$ implies $f = 0$. Suppose that $f = +1$ in this case. But if $(R_1, \dots, R_n)$ satisfies $n(+1) = n(-1)$ then so does $(-R_1, \dots, \dots, -R_n)$. Thus $f(-R_1, \dots, -R_n) = f(R_1, \dots, R_n) = +1$, which violates neutrality.

By positive responsiveness, $n(+1) = n(-1) + 1$ implies $f = 1$. Repeated application of this condition yields $n(+1) > n(-1) \Rightarrow f = +1$. By neutrality, this implies $n(-1) > n(+1) \Rightarrow f = -1$. □

[3] For clarity, monotonicity and fragility of ties are defined with respect to alternative $+1$, but, in combination with the neutrality axiom, the corresponding conditions apply to alternative -1.

Next, suppose we weaken the positive responsiveness condition in May's Theorem to only require monotonicity. In this case, the case of majority rule generalizes to supermajority rules, which we term q-rules. Formally, f is a q-rule for $q \in [1/2, 1]$ if

$$f(R_1, \ldots, R_n) = \begin{cases} +1 & \text{if } \sum_i R_i > qn\,, \\ 0 & \text{if } 1 - qn \leqslant \sum_i R_i \leqslant qn\,, \\ -1 & \text{if } \sum_i R_i < 1 - qn\,. \end{cases}$$

Note that majority rule is included in this definition, as a q-rule for $q = 1/2$ is just majority rule. An easy generalization of May's Theorem provides an axiomatization of q-rules.

THEOREM 2. *An aggregation rule f satisfies neutrality, anonymity, and positive responsiveness if and only if f is q-rule.*

P r o o f . It is easy to verify that q-rules satisfy the three properties. For the reverse, let $n(v)$ be the number of individuals with $R_i = v$, for $v \in \{-1, +1\}$. By anonymity, f can only depend on the values of $n(-1)$ and $n(+1)$.

Now define the set of decisive sizes, D, as the set of all integers η such that $\sum_i R_i = \eta$ implies $f(R_1, \ldots, R_n) = +1$. If this set is empty, this corresponds to a q-rule with $q = 1$. So suppose D is nonempty and let $d = \min D$. By monotonicity, $D = \{d, d+1, \ldots, n\}$. By neutrality, $\eta \in D$ implies $n - \eta \notin D$ and thus $n - d < d$, so $d > n/2$. From this, we conclude that f is a q-rule with $q = d/n$ and that $q \in [1/2, 1]$. □

This theorem is summarized in Figure 2.

anonymity neutrality monotonicity	$\Longleftrightarrow$	q-rules

FIGURE 2. A characterization of q-rules.

4. Preliminaries for the infinite case

In this section we consider aggregation rules with an infinite electorate. There are two pressing issues that confront us in this setting. First, the well known fact that all countably infinite sets have the same cardinality significantly complicates the comparison of the "size" of the sets of voters who prefer one alternative to the other. Second, the standard anonymity axiom presented above requires that the outcome is invariant to any permutation of individuals. But it is easy to see that any countably infinite set can be permuted to any other such set and thus all such sets would be assigned the same outcome. Clearly, then, this version of anonymity is too strong.

The following notation is used in the remainder of the paper. Let $\mathbb{N} = \{1, 2, \dots\}$ be a countably infinite society and denote the power set of $\mathbb{N}$ by $\mathcal{P}$. A preference profile for the whole society is $v\colon \mathbb{N} \to \{-1, +1\}$ and we define $V = v^{-1}(+1)$, the set of voters who prefer alternative $+1$ to alternative -1. Let V^c denote the complement of V. We say that V is cofinite if V^c is finite. Let $\mathcal{I} \subset \mathcal{P}$ denote the collection of infinite sets with infinite complement. An aggregation rule f is a function $f\colon \mathcal{P} \to \{-1, 0, +1\}$.

Next, we modify the axioms presented above to handle an infinite electorate as follows: An aggregation rule f satisfies

- *neutrality* if, for all $A \in \mathcal{P}$, $f(A^c) = -f(A)$,
- *monotonicity* if, for all $A, B \in \mathcal{P}$, $A \subseteq B$ implies $f(A) \leqslant f(B)$,
- *positive responsiveness* if, for all $A, B \in \mathcal{P}$, $A \subsetneq B$ and $f(A) \geqslant 0$ imply $f(B) = 1$.

A permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ is a bijective mapping and we denote the group of all permutations on $\mathbb{N}$ by $\mathbb{G}$. In addition, let πA denote the image of $A \in \mathcal{P}$ under π. The Lévy group $\mathcal{G}$ consists of $\pi \in \mathbb{G}$ for which

$$\lim_{n\to\infty} \frac{\bigl|\{k\colon k \leqslant n < \pi(k)\}\bigr|}{n} = 0\,.$$

A permutation $\pi \in \mathcal{G}$ is called a *bounded* permutation.

Now we can define our modification of the anonymity axiom. An aggregation rule f satisfies *bounded anonymity* if, for all $A \in \mathcal{P}$ and all $\pi \in \mathcal{G}$, $f(\pi A) = f(A)$. That is, bounded anonymity specifies invariance under bounded permutations.

5. Asymptotic density and aggregation rules

For a set $A \in \mathcal{P}$, let $A(n) = \bigl|A \cap \{1, \dots, n\}\bigr|$. The *asymptotic density* of A is

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}\,,$$

if this limit exists. Asymptotic density provides us with the "intuitively correct" size of some infinite sets. The *lower asymptotic density* of A is $\underline{d}(A) = \liminf\limits_{n\to\infty} \frac{A(n)}{n}$ and the *upper asymptotic density* of A is $\overline{d}(A) = \limsup\limits_{n\to\infty} \frac{A(n)}{n}$.

Obviously, $d(A)$ exists if and only if $\underline{d}(A) = d(A) = \overline{d}(A)$. Unfortunately, it is well known that there are subsets of $\mathbb{N}$ for which asymptotic density does not exist. Therefore, we define $\mathcal{D} \subset \mathcal{P}$ to be the collection of sets that have asymptotic density.

Although asymptotic density does not always exist, we can consider extensions of asymptotic density to all subsets of $\mathbb{N}$. We call such extensions *density measures*. Formally, μ is a density measure if it is a finitely additive measure on $\mathbb{N}$ such that $\mu(A) = d(A)$ whenever $d(A)$ exists. One widely used method of extension [1, 7, 15, 10] is to use the generalized limit operator $\lim\limits_{\mathcal{U}}$, for some free ultrafilter $\mathcal{U}$, and define

$$\mu(A) = \lim_{\mathcal{U}} \frac{A(n)}{n} . \tag{1}$$

Let $\mathcal{M}$ denote the set of all such density measures. It is easy to see that $\underline{d}(A) \leqslant \mu(A) \leqslant \overline{d}(A)$ for each density measure $\mu \in \mathcal{M}$.

We can use the concept of asymptotic density to define the family of *density q-rules* in an analogous fashion to q-rules in a finite population. A drawback of these rules is that they can only assign an outcome to sets in $\mathcal{D}$. To overcome this deficiency, we can use density measures to define a family of *density measure q-rules*. These rules are defined as follows. An aggregation rule f is

(1) an open density measure q-rule (open density q-rule) for $q \in [1/2, 1]$ if there is a density measure $\mu \in \mathcal{M}$ ($\mu = d$) such that, for all $V \in \mathcal{P}$ ($V \in \mathcal{D}$),

$$f(V) = \begin{cases} -1 & \text{if } \mu(V) < 1-q\,, \\ 0 & \text{if } \mu(V) \in [1-q, q]\,, \\ +1 & \text{if } \mu(V) > q\,, \end{cases}$$

(2) a closed density measure q-rule (closed density q-rule) for $q \in (1/2, 1]$ if there is a density measure $\mu \in \mathcal{M}$ ($\mu = d$) such that, for all $V \in \mathcal{P}$ ($V \in \mathcal{D}$),

$$f(V) = \begin{cases} -1 & \text{if } \mu(V) \leqslant 1-q\,, \\ 0 & \text{if } \mu(V) \in (1-q, q)\,, \\ +1 & \text{if } \mu(V) \geqslant q\,, \end{cases}$$

(3) a density measure q-rule if it is either an open density measure q-rule or a closed density measure q-rule.

(4) a density q-rule if it is either an open density q-rule or a closed density q-rule.

Next, we define a second type of rule that will play a role in our results. For $n \in \mathbb{N}_0 \cup \{\infty\}$, an aggregation rule f is a n-rule if

$$f(V) = \begin{cases} -1 & \text{if } |V| < n\,, \\ \ \ 0 & \text{if } |V| \geqslant n \ \text{ and } |V^c| \geqslant n\,, \\ +1 & \text{if } |V^c| < n \end{cases}$$

for all $V \in \mathcal{P}$. In other words, an n-rule chooses -1 or $+1$ if fewer than n voters disagree with the choice, and otherwise chooses 0.

6. Results for the infinite case

In this section, we present our main results. Details of the proofs for these results can be found in F e y [4]. The first result furnishes two important classes of rules that satisfy our three axioms of interest.

Theorem 3. *Each n-rule and each density measure q-rule satisfies neutrality, monotonicity, and bounded anonymity.*

An aggregation rule f *agrees with* an aggregation rule f' if f assigns the same outcome as f' to all sets $V \in \mathcal{D}$. Our second theorem shows that the converse of Theorem 3 holds in the sense that a rule satisfying the given conditions must agree with an n-rule or a density q-rule.

Theorem 4. *If an aggregation rule f satisfies neutrality, monotonicity, and bounded anonymity, then f agrees with an n-rule or a density q-rule.*

These two theorems establish a full characterization for sets with density, but they do not provide a complete answer for those sets without density. As a result, we make the following two points. First, the strong law of large numbers implies that almost all subsets of $\mathbb{N}$ have asymptotic density equal to $1/2$. Thus, our characterization is complete for almost all sets. The strong law of large numbers also yields the following immediate result:

Corollary 1. *Suppose an aggregation rule f satisfies neutrality, monotonicity, and bounded anonymity. Then $f(V) = 1/2$ for almost all V.*

Our second point is that it is possible to show that the converse of Theorem 3 does not hold for the class of density measures $\mathcal{M}$. Specifically, we provide a counterexample consisting of an aggregation rule f that satisfies neutrality, monotonicity, and bounded anonymity but that is not a density measure q-rule for any q. Here we summarize the details of the counterexample. To begin, let f assign values in $\mathcal{D}$ according to an open density $2/3$-rule. A set $A \in \mathcal{P}$ with

$A \notin \mathcal{D}$ is *grounded by* $D \in \mathcal{D}$ if $D \subseteq A$ and $f(D) = 1$. If A or A^c is grounded by some $D \in \mathcal{D}$, then both A and A^c are assigned to $\mathcal{K}$. In the first case, we assign $f(A) = 1$ and in the second, $f(A) = 0$. The remaining sets in $\mathcal{P}$ comprise $\mathcal{F}$ and are assigned $f = 1/2$. That $\mathcal{F}$ is nonempty follows from a result of Grekos [5].

This aggregation rule satisfies neutrality and monotonicity by construction. The only potential violation of bounded anonymity is $\pi \in \mathcal{G}$ and $A \in \mathcal{M}$ and $B \in \mathcal{F}$ such that $\pi A = B$. Suppose A is grounded by a set $D \in \mathcal{D}$. But $d(\pi D) = d(D)$, so $f(\pi D) = 1$ and $\pi D \subseteq \pi A = B$, which contradicts $B \in \mathcal{F}$. We conclude by showing that f is not representable as a density measure q-rule. By the assignment for sets in $\mathcal{D}$, we are restricted to density measure 2/3-rules. A useful result of Grekos [5] states that for every real α and β, $0 \leqslant \alpha < \beta \leqslant 1$, there exists a set $A \in \mathcal{P}$ with $\underline{d}(A) = \alpha$ and $\overline{d}(A) = \beta$ such that any $B \subset A$ with $B \in \mathcal{D}$ must have $d(B) = 0$.[4] This result implies that there exists a set $A \in \mathcal{F}$ with $2/3 < \underline{d}(A) < \overline{d}(A)$. Because $\underline{d}(A) \leqslant \mu(A) \leqslant \overline{d}(A)$ for every density measure $\mu \in \mathcal{M}$, no density measure 2/3-rule can assign $f(A) = 1/2$, as required. Therefore, f is not a density measure q-rule.

The results presented so far have characterized the infinite equivalents of supermajority rules. We conclude by giving a characterization of majority rule in this setting. It is obvious that May's positive responsiveness cannot be satisfied with an infinite number of voters. So we now define a version of positive responsiveness that employs asymptotic density. An aggregation rule f satisfies *density positive responsiveness* if f satisfies monotonicity and, for all $A \in \mathcal{D}$ with $f(A) = 1/2$ and all $D \in \mathcal{D}$ with $A \cap D = \emptyset$ and $d(D) > 0$, $f(A \cup D) = 1$. In addition, we say f is a density majority rule if

$$f(V) = \begin{cases} -1 & \text{if } d(D) < 1/2\,, \\ 0 & \text{if } d(D) = 1/2\,, \\ +1 & \text{if } d(D) > 1/2\,, \end{cases}$$

for all $D \in \mathcal{D}$. It is clear that density majority rule satisfies neutrality, density positive responsiveness, and bounded anonymity. In the following corollary, we show that the converse holds in the sense that any rule satisfying these three conditions must agree with density majority rule.

COROLLARY 2. *If an aggregation rule f satisfies neutrality, density positive responsiveness, and bounded anonymity, then f agrees with density majority rule.*

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[4] See also Grekos and Volkmann [6].

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