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ON THE EXISTENCE OF NONSINGULAR BILINEAR FORMS ON PROJECTIVE MODULES

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ABSTRACT. We prove that in a domain R an ideal of R admits a nonsingular bilinear form if and only if its square is a principal ideal. We also show that the isometry classes of bilinear forms on an ideal correspond bijectively to the square classes of the group of units of R. When R is one-dimensional noetherian domain, each finitely generated projective R-module is isomorphic to $M = I \oplus R^n$ and it admits a nonsingular bilinear form if and only if I^2 is a principal ideal. When R is Dedekind domain of characteristic not two in which 2 is not a unit, we show that for each invertible nonprincipal ideal I the direct sum $M = I \oplus I^{-1}$ admits a nonsingular bilinear form β which makes M into an indecomposable inner product space.

1. Introduction

For a finitely generated (f.g.) free module M over a commutative ring R there is a very simple criterion for the nonsingularity of a symmetric bilinear form β on M. The bilinear form β is nonsingular if and only if its matrix in any basis of M is invertible. Hence it is easy to equip any f.g. free R-module with a nonsingular bilinear form. When we consider the Witt ring $W(R)$ of the ring R, we deal with nonsingular bilinear forms on f.g. *projective* modules over R and so we should be able to construct nonsingular bilinear forms on f.g. projective R-modules. This is not always possible. We will demonstrate this in Section 2 on ideals of a domain R. An ideal I of R viewed as R-module is projective if and only if it is invertible. We shall prove that an invertible ideal I of R admits a nonsingular bilinear form if and only if its square I^2 is a principal ideal. Thus, for example, if R is the maximal order of an algebraic number field with odd class number, then all nonprincipal ideals in R are projective but none of them admits a nonsingular bilinear form. Our motivation for studying

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bilinear forms on projective modules comes from Witt ring theory and we are particularly interested in Witt rings of orders in Dedekind domains. These are one-dimensional noetherian domains and so we need a description of projective modules over rings of that class. It is known that for a one-dimensional noetherian domain R, each finitely generated projective R-module is isomorphic to $M = I \oplus R^n$ for some invertible ideal I and nonnegative integer n. We show that M admits a nonsingular bilinear form if and only if I^2 is a principal ideal.

In Section 3 we classify the nonsingular bilinear forms on an ideal of a domain R up to isometry and show that the nonisometric bilinear forms on an ideal correspond bijectively to the square classes of the group of units of R. When R is a Dedekind domain we show that nonsingular bilinear forms on an ideal I of R are similar (that is, belong to the same Witt class) if and only if they are isometric.

In Section 4 we study decompositions of bilinear spaces (M, β) into orthogonal direct sums. For a wide class of Dedekind domains (including maximal orders in algebraic number fields with class number greater than 1) we show explicit examples of nonsingular bilinear forms on the direct sum of modules which are not the *orthogonal* direct sums of nonsingular bilinear forms.

Throughout the paper all bilinear forms are assumed to be symmetric. Recall that a bilinear form $\beta\colon M \times M \to R$ on an R-module M is nonsingular if the adjoint homomorphism $\widehat{\beta}\colon M \to M^* = \mathrm{Hom}_R(M, R)$ defined by

$$\widehat{\beta}(m)(n) = \beta(m, n)$$

is an isomorphism of R-modules. Actually, if M is projective and $\widehat{\beta}$ is surjective, then it is also injective, hence an isomorphism. For suppose $\widehat{\beta}$ is surjective and $\widehat{\beta}(m) = 0$ for some $m \in M$. Then $\beta(m, n) = \widehat{\beta}(m)(n) = 0$ for all $n \in M$. By symmetry, $\beta(n, m) = 0$ for all $n \in M$, that is, $\widehat{\beta}(n)(m) = 0$ for all $n \in M$. Since $\widehat{\beta}$ is surjective, it follows that all linear functionals on M vanish on the element m. Since M is projective, the dual bases lemma (see [5, Prop. 1.14, p. 25]) implies that $m = 0$. This proves the injectivity of $\widehat{\beta}$.

2. Projective modules over domains

First we collect some known properties of invertible ideals in a domain R and of projective R-modules ([7, Chapter I]).

Lemma 2.1. *Let R be a domain and let I be an invertible ideal in R.*

(1) $I^{-1} \cong I^*$.

(2) $I \otimes_R J \cong IJ$ *for any ideal J of R.*

(3) $I \otimes_R I^* \cong R$.

Lemma 2.2. *Let R be a domain and let I be an ideal in R.*

(1) *If I is invertible, then it is a projective R-module of rank 1 and conversely, each projective R-module of rank 1 is isomorphic with some invertible ideal of R.*

(2) *If R is a one-dimensional noetherian domain and M is a finitely generated projective R-module of rank $n+1>1$, then there is an invertible ideal I in R such that*

$$M \cong I \oplus R^n \,.$$

We shall point out a necessary and sufficient condition for the existence of a nonsingular bilinear form on any invertible ideal I of a domain R. In [1] it is mentioned that when R is a Dedekind domain then a nonzero ideal I of R admits a nonsingular bilinear form if and only if I^2 is a principal ideal. We show that the same condition works for any domain R and any invertible ideal I in R. We also show how to find all nonsingular bilinear forms on I and how to classify them with respect to isometry.

Lemma 2.3. *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K and let $\varphi \in I^*$ be a linear functional on the R-module I. Then there is an element $c \in K$ such that*

$$\varphi(x) = cx$$

for all $x \in I$. If I is invertible, then $c \in I^{-1}$.

Proof. Since I is a fractional ideal in K, there is a nonzero element $d \in R$ such that $dI \subseteq R$. Then for all $x, y \in I$ we have $dx, dy \in R$ and

$$dx\varphi(y) = \varphi(dxy) = dy\varphi(x)\,.$$

Hence there is an element $c \in K$ satisfying $\varphi(x) = cx$ for all $x \in I$. For invertible ideal I, from $cx \in R$ for all $x \in I$, it follows that $c \in I^{-1}$. $\square$

Observe that Lemma 2.3 proves the first assertion in Lemma 2.1. Indeed, $I^{-1} \to I^*$, $c \mapsto \varphi$ is a module isomorphism.

Theorem 2.4. *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K such that $I^2 = (a)$ for some $a \in K$, $a \neq 0$. Then the map $\beta\colon I \times I \to R$ defined by*

$$\beta(x,y) = \frac{xy}{a}$$

is a nonsingular bilinear form on the ideal I.

Proof. For $x, y \in I$ we have $xy \in I^2 = (a) = aR$ so that $\frac{xy}{a} \in R$. Clearly, the map β is a bilinear form on I. We show that the adjoint homomorphism

$\widehat{\beta}\colon I \to I^*$ is an isomorphism. It is sufficient to show that $\widehat{\beta}$ is surjective.

Let $\varphi \in I^*$. Then, by Lemma 2.3, there is $c \in K$ such that $\varphi(x) = cx$ for all $x \in I$. In particular, $\frac{\varphi(a)}{a} = c$. Since $I^2 = (a)$, we have $a = \sum x_i y_i$ for some $x_i, y_i \in I$. Since I is a fractional ideal in K, there is an element $d \in R$ such that $dI \subseteq R$. Thus $dx_i \in R$ and since also $\varphi(y_i) \in R$, we have

$$d\varphi(a) = \varphi(ad) = \sum \varphi(dx_i y_i) = \sum dx_i \varphi(y_i) = d \sum x_i \varphi(y_i) \in dI\,.$$

Hence $\varphi(a) \in I$ and we have

$$\widehat{\beta}\big(\varphi(a)\big)(x) = \beta\big(\varphi(a), x\big) = \frac{\varphi(a)}{a}\, x = \varphi(x)$$

for all $x \in I$. Thus $\widehat{\beta}\big(\varphi(a)\big) = \varphi$ and $\widehat{\beta}$ is surjective. □

THEOREM 2.5. *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K. The ideal I admits a nonsingular bilinear form if and only if I^2 is a principal ideal.*

P r o o f. If I^2 is a principal ideal, then I admits a nonsingular bilinear form by Theorem 2.4. On the other hand, if β is a nonsingular bilinear form on I, then the adjoint homomorphism $\widehat{\beta}\colon I \to I^*$ is an isomorphism of R-modules, hence, using Lemma 2.1, we get

$$I^2 = I \cdot I \cong I \otimes I \cong I \otimes I^* \cong R\,.$$

Thus I^2 is a free R-module of rank 1, hence a principal ideal in R. □

We end this section with a definitive result about the existence of nonsingular bilinear forms on f.g. projective modules over a one-dimensional noetherian domain R. According to Lemma 2.2 each f.g. projective module M over R is of the form $M \cong I \oplus R^n$, where I is an invertible ideal in R. Here we prove a necessary and sufficient condition for the existence of a nonsingular bilinear form on such a module. In the proof below (and in § 4) we use the following theorem of Steinitz good for any domain R with the field of fractions K (see [3, I.1.6]):

> *If $\mathfrak{a}_i, \mathfrak{b}_j$ are nonzero ideals of R and the R-modules $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ and $\mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_k$ are isomorphic, then there is an element $c \in K$ such that*
>
> $$\mathfrak{a}_1 \cdots \mathfrak{a}_n = c\,\mathfrak{b}_1 \cdots \mathfrak{b}_k\,.$$

THEOREM 2.6. *Let R be a noetherian domain of dimension 1 and let $M \cong I \oplus R^n$, where I is an invertible ideal in R. The module M admits a nonsingular bilinear form β if and only if I^2 is a principal ideal.*

Proof. Assume first that (M, β) is a nonsingular bilinear space. Hence the adjoint homomorphism $\widehat{\beta}\colon M \to M^*$ is an isomorphism of R-modules,

$$I \oplus R^n \cong (I \oplus R^n)^* .$$

Using Lemma 2.1, we get

$$I \oplus R^n \cong (I \oplus R^n)^* \cong I^* \oplus R^n \cong I^{-1} \oplus R^n .$$

Hence the R-modules I and I^{-1} are stably isomorphic. By Steinitz's theorem, there is $c \in K$ such that $I \cong cI^{-1}$. It follows that the R-modules I and I^{-1} are isomorphic and so I^2 is a principal ideal.

Suppose now $I^2 = (a)$ for an $a \in R$, $a \neq 0$. By Theorem 2.4, the map

$$\beta\colon I \times I \to R\,, \qquad \beta(x,y) = \frac{xy}{a}$$

is a nonsingular bilinear form on I. We take any nonsingular bilinear form γ on the free R-module R^n and then the orthogonal direct sum (see beginning of § 4) $\beta \oplus \gamma$ on $I \oplus R^n$ is a nonsingular bilinear form on M. $\square$

3. Classification

In Theorem 2.4 we have defined a certain nonsingular bilinear form on an ideal I (provided I^2 is a principal ideal) and now we describe *all* nonsingular bilinear forms on the ideal I. Then we classify the bilinear forms on I with respect to isometry, and when R is a Dedekind domain, also with respect to similarity. Below $R^\times$ denotes the group of invertible elements in R.

Theorem 3.1. *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K and assume $I^2 = (a)$ for some $a \in K$, $a \neq 0$. If γ is a nonsingular bilinear form on I, there exists a unique element $t \in R^\times$ such that for all $x, y \in I$ we have*

$$\gamma(x,y) = t\,\frac{xy}{a}\,. \tag{3.1}$$

Conversely, if $t \in R^\times$, then the map $\gamma\colon I \times I \to R$ defined by (3.1) *is a nonsingular bilinear form on I.*

Proof. First assume that γ is a nonsingular bilinear form on the ideal I. Then the adjoint homomorphism $\widehat{\gamma}\colon I \to I^*$ is an isomorphism of R-modules. By Lemma 2.3, for each $\varphi \in I^*$ there is a unique element c in the field K such that

$$\varphi(z) = cz$$

for all $z \in I$. Hence for each $x \in I$ there is an element $c_x \in K$ such that

$$\widehat{\gamma}(x)(z) = c_x z$$

for all $z \in I$. Take $x, y \in I$ and the corresponding elements $c_x, c_y \in K$. Then

$$c_x y = \widehat{\gamma}(x)(y) = \gamma(x, y) = \gamma(y, x) = \widehat{\gamma}(y)(x) = c_y x\,.$$

Hence for all $x, y \in I \setminus \{0\}$ the ratio

$$\frac{c_x}{x} = \frac{c_y}{y} =: c \in K$$

is a constant. Hence $c_x = cx$ for all $x \in I$ and

$$cxy = \gamma(x, y) \in R \qquad \text{for all } x, y \in I\,.$$

Since the products xy generate I^2, we have $cI^2 \subset R$ and $t := ca$ belongs to R. Hence for all $x, y \in I$ we have

$$\gamma(x, y) = t\,\frac{xy}{a}\,.$$

The uniqueness of t follows from the cancellation property in a domain. It remains to show that t is invertible in R.

Since $\widehat{\gamma}$ is surjective, for each $\varphi \in I^*$ there is a unique element $x \in I$ such that $\widehat{\gamma}(x) = \varphi$, that is, $t\,\frac{xy}{a} = \varphi(y)$ for all $y \in I$. In particular, for $y = a$ we get

$$tx = t\,\frac{xa}{a} = \varphi(a)\,.$$

Since I is invertible, it is a projective R-module. Hence, by the dual bases lemma (see [5, Prop. 1.14, p. 25]), there are $x_1, \ldots, x_n \in I$, $\varphi_1, \ldots, \varphi_n \in I^*$ such that

$$a = \sum x_i\,\varphi_i(a)\,.$$

On the other hand, for each $i \in \{1, \ldots, n\}$ there is a unique element $y_i \in I$ such that $ty_i = \varphi_i(a)$. Hence, since $x_i y_i \in I^2$, we get

$$a = \sum t\,x_i y_i = t \sum x_i y_i = tax$$

for some $x \in R$. Hence $tx = 1$ and so $t \in R^\times$, as desired.

To prove the last statement it suffices to notice that

$$\gamma(x, y) = t\,\frac{xy}{a} = \frac{xy}{t^{-1}a}$$

and $t^{-1}a$ also generates the ideal $I^2 = (a)$. The nonsingularity of γ follows from Theorem 2.4. □

Remark 3.2. In the notation of Theorem 3.1 we have $I^2 = (a) = (t^{-1}a)$. Hence the result can be also stated as follows. If γ is a nonsingular bilinear form on I, then I^2 is a principal ideal and one can choose a generator a for I^2 so that $\gamma(x, y) = \frac{xy}{a}$ for all $x, y \in I$.

Now we classify all nonsingular bilinear forms on an ideal up to isometry.

Theorem 3.3. *Let R be a domain and K its field of fractions. Let I be a fractional ideal in K and assume $I^2 = (a)$ for some $a \in K$, $a \neq 0$. Let $t_1, t_2 \in R^\times$ and let γ_1, γ_2 be nonsingular forms on I given by*

$$\gamma_1(x,y) = t_1 \frac{xy}{a}, \quad \gamma_2(x,y) = t_2 \frac{xy}{a},$$

for all $x, y \in I$. Then

$$(I, \gamma_1) \cong (I, \gamma_2) \quad \Longleftrightarrow \quad t_1 \in t_2 R^{\times 2}.$$

Proof. Let $\varphi \colon I \to I$ be an isometry between (I, γ_1) and (I, γ_2). Then for all $x, y \in I$ we have

$$\gamma_1(x,y) = \gamma_2\big(\varphi(x), \varphi(y)\big). \tag{3.2}$$

If $dI \subseteq R$ for some nonzero $d \in R$, then $d\varphi \colon I \to I$ assumes the values in R since $(d\varphi)(x) = d \cdot \varphi(x) \in R$, and so $d\varphi \in I^*$. By Lemma 2.3, there is a unique $c \in K$ satisfying

$$d\varphi(x) = cx$$

for all $x \in I$. Combining this with (3.2) we get

$$t_1 \frac{xy}{a} = t_2 \frac{cd^{-1}xcd^{-1}y}{a}$$

for all $x, y \in I$. Hence $t_1 = t_2(cd^{-1})^2$ for some $c \in K$ such that $(cd^{-1})^2 \in R^\times$. It remains to show that $cd^{-1} \in R^\times$. Since φ is surjective, $cd^{-1}I = \varphi(I) = I$, and hence

$$(a) = I^2 = cd^{-1}I \cdot I = cd^{-1}I^2 = (cd^{-1}a).$$

Thus there is $r \in R^\times$ for which $a = cd^{-1}a \cdot r$. Hence $cd^{-1}r = 1$ and $cd^{-1} \in R^\times$, as required.

Conversely, assume that $t_1 = t_2\varepsilon^2$ for some $\varepsilon \in R^\times$. Consider the homomorphism $\varphi \colon I \to I$ defined by

$$\varphi(x) = \varepsilon x$$

for all $x \in I$. This is an isomorphism of R-modules. Moreover, for all $x, y \in I$ we have

$$\gamma_2\big(\varphi(x), \varphi(y)\big) = t_2 \frac{\varphi(x)\varphi(y)}{a} = t_2 \varepsilon^2 \frac{xy}{a} = \gamma_1(x,y).$$

Hence φ is an isometry between (I, γ_1) and (I, γ_2). $\square$

Remark 3.4. Using Theorem 3.3 we can transport the group structure from $R^\times / R^{\times 2}$ onto the set of isometry classes of bilinear forms on a fractional ideal I. Then the group operation on isometry classes of nonsingular bilinear forms on the ideal I (induced by multiplication in $R^\times / R^{\times 2}$) can be viewed as a composition of bilinear forms on I.

We will switch now from emphasizing bilinear forms to considering the bilinear spaces (M, β) where M is a f.g. projective module and β is a bilinear form on M. When β is nonsingular the bilinear space (M, β) is said to be nonsingular or an *inner product space* (see [4]). When there is a direct sum decomposition $M = M_1 \oplus M_2$ for which $\beta(M_1, M_2) = 0$, then we say that the bilinear space (M, β) is an orthogonal direct sum (ODS, for short) of bilinear spaces (M_i, β_i) and write

$$(M, \beta) = (M_1, \beta_1) \perp (M_2, \beta_2).$$

(M, β) is also said to be *decomposable* (into ODS). For a bilinear space (M, β) and for a submodule N of M we consider the *orthogonal complement* of N in M. This is the submodule

$$N^{\perp} := \{m \in M : \beta(m, N) = 0\}.$$

The submodule N is said to be *totally isotropic* if $\beta(N, N) = 0$. For this it is necessary and sufficient that $N \subseteq N^{\perp}$. An inner product space (M, β) is said to be *metabolic* if there exists a direct sum representation

$$M = N \oplus S \quad \text{and} \quad N = N^{\perp}. \tag{3.3}$$

We say that inner product spaces (X_1, γ_1) and (X_2, γ_2) are *similar* (or, in the same Witt class) if there exist metabolic spaces (M_1, β_1) and (M_2, β_2) so that the ODSs $(X_1, \gamma_1) \perp (M_1, \beta_1)$ and $(X_2, \gamma_2) \perp (M_2, \beta_2)$ are isometric.

Theorem 3.5. *Let R be a Dedekind domain and let I be a fractional ideal with I^2 a principal ideal. Let γ_1, γ_2 be nonsingular bilinear forms on the ideal I. Then the inner product spaces (I, γ_1) and (I, γ_2) are similar if and only if they are isometric.*

Proof. We will use the fact that (I, γ_1) and (I, γ_2) are similar if and only if the corresponding Witt classes, that is elements of the Witt ring $W(R)$, are equal: $\langle I, \gamma_1 \rangle = \langle I, \gamma_2 \rangle$. If K is the field of fractions of R, then by Knebusch's theorem the natural homomorphism $W(R) \to W(K)$ is injective ([4, p. 93]) and so the spaces (I, γ_1) and (I, γ_2) are similar if and only if their images in $W(K)$ are equal, $\langle I, \gamma_1 \rangle_K = \langle I, \gamma_2 \rangle_K$.

If $I^2 = (a)$, then according to Theorem 3.1 there are units t_1, t_2 in R such that

$$\gamma_1(x, y) = t_1 \frac{xy}{a}, \quad \gamma_2(x, y) = t_2 \frac{xy}{a},$$

for all $x, y \in I$. Then $\langle I, \gamma_i \rangle_K = \langle I \otimes K, \gamma_i \otimes K \rangle$ and the latter is a one-dimensional space with diagonalization $\langle a t_i \rangle_K$. Then we have

$$\langle a t_1 \rangle_K = \langle a t_2 \rangle_K \iff \langle t_1 \rangle_K = \langle t_2 \rangle_K \iff t_1 t_2 \in K^{\times 2}.$$

Since R is integrally closed, the latter is equivalent to $t_1 t_2 \in R^{\times 2}$, that is to $t_1 \in t_2 R^{\times 2}$. By Theorem 3.3 this is the case if and only if the spaces (I, γ_1) and (I, γ_2) are isometric. □

4. Indecomposable inner product spaces

For a commutative ring R and R-modules M_1, M_2 with bilinear forms β_1, β_2 the direct sum $M = M_1 \oplus M_2$ can be equipped with the bilinear form $\beta = \beta_1 \oplus \beta_2$ defined by setting

$$\beta\big((m_1, m_2), (m_1', m_2')\big) = \beta_1(m_1, m_1') + \beta_2(m_2, m_2').$$

β is then the orthogonal direct sum (ODS) of β_1 and β_2 since $\beta(M_1, M_2) = 0$. The form β is nonsingular if and only if β_1 and β_2 are nonsingular. On the other hand, when $M = M_1 \oplus M_2$ and a bilinear form β is given on M, then β need not be the ODS of bilinear forms on M_1 and M_2. We give a simple example showing that even if we assume that β is nonsingular we cannot expect β to be an ODS of bilinear forms defined on direct summands of M.

Example 4.1. Let I be an invertible ideal in the domain R and consider the R-module $M = I \oplus I^{-1}$. We define a bilinear form β on M by setting

$$\beta\big((x, u), (y, v)\big) = xv + uy.$$

It is clear that β is not the ODS of its restrictions to I and I^{-1}, since these submodules of M are not orthogonal. For $\beta\big((x, 0), (0, v)\big) = xv$. We now give a direct proof that β is nonsingular. To this end we show that $\widehat{\beta}$ is surjective. Take any $\varphi \in M^* = I^* \oplus I^{-1*}$. Then $\varphi(x, u) = \varphi_1(x) + \varphi_2(u)$, where φ_1, φ_2 are linear functionals on I and I^{-1}, respectively. By Lemma 2.3 there are c_1, c_2 in the field of fractions of R such that

$$\varphi_1(x) = c_1 x, \quad \varphi_2(u) = c_2 u,$$

for all $x \in I$ and $u \in I^{-1}$. Observe that $c_1 x, c_2 u \in R$ so that $c_1 \in I^{-1}$ and $c_2 \in (I^{-1})^{-1} = I$. Hence

$$\varphi(x, u) = c_1 x + c_2 u = \beta\big((c_2, c_1), (x, u)\big) = \widehat{\beta}(c_2, c_1)(x, u).$$

This shows that $\varphi = \widehat{\beta}(c_2, c_1)$ and proves that $\widehat{\beta}$ is surjective.

There is another explanation of the fact noticed in Example 4.1 using the concept of metabolic forms (or spaces). First notice that, if $M \neq 0$ is metabolic and (3.3) holds, then N is necessarily a nonzero module. For otherwise, $0 = N = N^\perp = M$, a contradiction. Observe that the direct sum representation (3.3) is never an ODS representation. For $N = N^\perp$ implies $\beta(N, N) = 0$ and so the restriction β_1 of β to N is the zero form on a nonzero module N. Hence $(M, \beta) = (N, \beta_1) \perp (S, \beta_2)$ is impossible since (M, β) is nonsingular and (N, β_1) is not.

Now looking back at the Example 4.1 we see that the inner product space $(I \oplus I^{-1}, \beta)$ is metabolic because $I = I^\perp$. Indeed, $\beta\big((x, 0), (y, 0)\big) = 0$ implies

$I \subseteq I^{\perp}$, and $\beta\big((x,u),I\big) = 0$ implies $u = 0$ and so $I^{\perp} \subseteq I$. Hence $M = I \oplus I^{-1}$ is not an ODS representation.

The remainder of this section is devoted to a proof of the deeper result saying that, under some mild conditions on the Dedekind ring R and the ideal I, the space $(I \oplus I^{-1}, \beta)$ is actually *indecomposable*, that is, it does not have the structure of an ODS at all.

Recall that in many cases metabolic spaces are decomposable. For example, if (M,β) is any bilinear space, then the ODS $(M,\beta) \perp (M,-\beta)$ is a metabolic space (see [4, Lemma 6.2, p. 13]). Further, if the ground ring is a field, then every metabolic space is an ODS of subspaces (metabolic space over a field of any characteristic has an orthogonal basis, see [6, Theorems 11.1.1 and 11.1.2, pp. 132–133, and p. 140]). Further examples are possible when we recall the terminology used in the more advanced literature (going back to M. Knebusch [2]).

For any bilinear space (P,ρ) we consider the associated bilinear space

$$\big(\mathbb{M}(P),\, \mu\big) := (P \oplus P^{*}, \mu)\,, \qquad \mu\big((p,f),(p',f')\big) = f(p') + f'(p) + \rho(p,p')\,,$$

for all $p, p' \in P$, $f, f' \in P^{*}$. One can prove that for any bilinear space (P,ρ) the bilinear space $(\mathbb{M}(P),\mu)$ is an inner product space and, in fact, a metabolic space. Actually, an inner product space (S,β) is metabolic if and only if there exists a bilinear space (P,ρ) so that (S,β) is isometric to $\big(\mathbb{M}(P),\, \mu\big)$. Now we can recall yet another example of a decomposable metabolic space. If a f.g. projective module P has a direct sum representation $P = P_1 \oplus P_2$, then $\mathbb{M}(P) \cong \mathbb{M}(P_1) \perp \mathbb{M}(P_2)$ (see [2, Satz 3.1.1, p. 17]).

We also consider a special type of metabolic spaces $\mathbb{M}(P)$ with the finitely generated projective module P equipped with the zero bilinear form $\rho = 0$. In that case we write $\mathbb{H}(P)$ instead of $\mathbb{M}(P)$ and call $\mathbb{H}(P)$ a hyperbolic space. Thus

$$\big(\mathbb{H}(P), \mu\big) := (P \oplus P^{*}, \mu)\,, \qquad \mu\big((p,f),(p',f')\big) = f(p') + f'(p)\,,$$

for all $p, p' \in P$, $f, f' \in P^{*}$. It is easy to check that P and P^{*} are maximal totally isotropic submodules of $\mathbb{H}(P)$. Hence each hyperbolic space — as a module — is the direct sum of two maximal totally isotropic submodules.

As an example, take $P = I$, an invertible ideal of a domain R. By Lemma 2.1 we have $I^{*} \cong I^{-1}$, hence the metabolic space $(I \oplus I^{-1}, \beta)$ considered in the Example 4.1 is just the hyperbolic space $\mathbb{H}(I)$.

Theorem 4.2. *Let R be a Dedekind domain of characteristic not 2 and let 2 be a nonunit in R. Let I be an invertible ideal in R and let $M = I \oplus I^{-1}$ be the R-module equipped with the bilinear form β,*

$$\beta\big((x,u),(y,v)\big) = xv + yu\,.$$

Then the hyperbolic space $\mathbb{H}(I) = (M, \beta)$ is indecomposable into orthogonal direct sum of subspaces.

P r o o f . Suppose there exists an ODS decomposition

$$M = M_1 \oplus M_2 \,, \quad M_1 \neq 0 \,, \quad M_2 \neq 0 \,, \quad \beta(M_1, M_2) = 0 \,. \tag{4.1}$$

We show in several steps that this leads to a contradiction.

Step 1. *There exists a fractional ideal J in the field of fractions K of R such that $J^2 = (a)$ for some $a \in K$ and*

$$(M, \beta) \cong (J \oplus J^{-1}, \gamma) \,, \qquad \text{where} \quad \gamma\big((x,u),(y,v)\big) = \frac{xy}{a} - auv \tag{4.2}$$

for all $x, y \in J \,,\; u, v \in J^{-1}$.

There is a $d \in R$ such that $dI^{-1} \subseteq R$. Then $dM = dI \oplus dI^{-1}$ is a direct sum of two integral ideals. Then also $dM = dM_1 \oplus dM_2$. Here dM_1 and dM_2 are f.g. projective modules over a Dedekind domain, hence direct sums of ideals of R. By Steinitz's theorem dM_1 and dM_2 are isomorphic to some integral ideals J_1 and J_2 and there is an element $c \in K$ such that

$$cd^2 R = c \cdot dI \cdot dI^{-1} = J_1 \cdot J_2 \,.$$

It follows that $J_2 = cd^2 J_1^{-1}$ and so $M_1 \cong d^{-1} J_1$ and $M_2 \cong d^{-1} J_2 = cdJ_1^{-1}$. Hence putting $J := d^{-1} J_1$ we get

$$M = I \oplus I^{-1} \cong J \oplus cJ^{-1} \,.$$

We write β again for the induced bilinear form on $J \oplus cJ^{-1}$, and then, by (4.1), $\beta(J, cJ^{-1}) = 0$. Thus

$$M \cong (J, \beta_1) \perp (cJ^{-1}, \beta_2) \,,$$

where β_1, β_2 are restrictions of β to J and cJ^{-1}, respectively. It follows that (J, β_1) is an inner product space. Hence, by Theorem 2.5, J^2 is a principal ideal, $J^2 = (a)$, say, with some $a \in K$. Then also $(cJ^{-1})^2 = (c^2 a^{-1})$. By Remark 3.2 and by Theorem 3.1 we have

$$\beta_1(x, y) = \frac{xy}{a} \,, \qquad \beta_2(cu, cv) = t\frac{cu \cdot cv}{c^2 a^{-1}} = atuv \,,$$

for all $x, y \in J \,,\; u, v \in J^{-1}$ and for some unit $t \in R^{\times}$. Since M contains (a lot of) isotropic elements, and since J and cJ^{-1} are anisotropic submodules, there are nonzero $x \in J$ and $u \in J^{-1}$ such that

$$0 = \beta\big((x,u),(x,u)\big) = \frac{x^2}{a} + atu^2 \,.$$

From this it follows that $-t \in R^\times$ is a square in the field K, and so, since R is integrally closed in K, $-t = \varepsilon^2 \in R^{\times 2}$. Thus $\beta_2(cu, cv) = -a\varepsilon^2 uv$ and so the map $cJ^{-1} \to J^{-1}$, $cu \mapsto \varepsilon u$ is an isomorphism of R-modules and

$$\beta_2(cu, cv) = -a\varepsilon^2 uv = \beta_2'(\varepsilon u, \varepsilon v),$$

where β_2' is the bilinear form on J^{-1} given by $\beta_2'(u,v) = -auv$. Hence $(cJ^{-1}, \beta_2) \cong (J^{-1}, \beta_2')$ and we have proved the claim with $\gamma = \beta_1 \oplus \beta_2'$.

Step 2. *The inner product space $(J \oplus J^{-1}, \gamma)$ has exactly two maximal totally isotropic subspaces, and these are*

$$S_+ = \{(au, u)\colon u \in J^{-1}\} \quad \text{and} \quad S_- = \{(-au, u)\colon u \in J^{-1}\}.$$

Clearly S_+ and S_- are submodules of $J \oplus J^{-1}$ and they are totally isotropic since

$$\gamma\big((\pm au, u), (\pm au, u)\big) = \frac{\pm au \cdot \pm av}{a} - auv = 0.$$

If $(x, u) \in J \oplus J^{-1}$ is any isotropic element, then $0 = \gamma\big((x,u),(x,u)\big) = \frac{x^2}{a} - au^2$, hence $(x + au)(x - au) = 0$ and it follows that $(x, u) = (\pm au, u) \in S_+ \cup S_-$. Now if S is a totally isotropic submodule containing sharply S_+, then it follows that some element $(-au, u)$ lies in S and $u \neq 0$. Then, however, also $(au, u) + (-au, u) = (0, 2u) \in S$ and so $0 = \gamma\big((0,2u),(0,2u)\big) = -4au^2$, a contradiction. Hence S_+ is a maximal totally isotropic submodule, and a similar argument applies to S_-.

Step 3. *The inner product space $(J \oplus J^{-1}, \gamma)$ is a hyperbolic space if and only if 2 is a unit in R.*

Since $S_\pm$ are the only maximal totally isotropic submodules, the space $J \oplus J^{-1}$ is hyperbolic if and only if we have the direct sum decomposition

$$J \oplus J^{-1} = S_+ \oplus S_-. \tag{4.3}$$

This is equivalent to saying that each element $(x, u) \in J \oplus J^{-1}$ can be written uniquely in the form

$$(x, u) = (aU, U) + (-aV, V), \qquad U, V \in J^{-1}.$$

Solving for U, V gives

$$2U = u + \frac{x}{a}, \qquad 2V = u - \frac{x}{a}. \tag{4.4}$$

Here $\frac{x}{a} \in J^{-1}$ since from $J^2 = (a)$ it follows that $x \in J = aJ^{-1}$. If $2 \in R^\times$, then U, V are uniquely determined elements of J^{-1} and we get the decomposition (4.3). On the other hand if U, V can be determined from (4.4), then in particular $2U = u$ is solvable for U and arbitrary $u \in J^{-1}$. Hence $\frac{1}{2}J^{-1} \subseteq J^{-1}$ and it follows $R \subseteq 2R$, and so $2 \in R^\times$.

Step 4. Combining Steps 1 and 3 we conclude that the hypothesis (4.1) implies that the hyperbolic space M is isometric to the nonhyperbolic space $(J \oplus J^{-1}, \gamma)$, a contradiction. $\square$

COROLLARY 4.3. *In the notation of Theorem 4.2, the inner product space* $\mathbb{H}(I) = (I \oplus I^{-1}, \beta)$ *does not contain any proper inner product subspaces.*

Proof. If S is a nonsingular subspace of $\mathbb{H}(I)$, then by the Orthogonal Decomposition Lemma (see [4, (3.1), p. 5]) we have the ODS decomposition $\mathbb{H}(I) = S \perp S^{\perp}$. Thus $\mathbb{H}(I)$ would be decomposable, contrary to Theorem 4.2. □

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