Tatra Mt. Math. Publ. **28** (2004), 241–250

Tatra Mountains
Mathematical Publications

ON RANDOM FOURIER SERIES

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ABSTRACT. Some random trigonometric series in the context of the homogeneous and semihomogeneous Banach spaces of functions on the circle group are investigated.

1. Introduction

In this paper we consider random trigonometric series

$$\sum_{n=-\infty}^{\infty} Z_n e^{int}, \tag{1}$$

where Z_n are complex random variables on a probability (Ω, A, P), independent and symmetric (that is, Z_n and $-Z_n$ have the same distribution) and we investigate some properties of these series such as a.s. (almost sure) Cesàro summability, convergence a.s. and so on. It is well-known that a number of properties of (1) have the same probability. Let us mention, e.g., the Paley–Zygmund theorem [3, Ch. V] dealing with the properties "(1) is a Fourier-Stieltjes series", "(1) represents a function in L^p", (1) $\leq p < \infty$, "(1) converges almost everywhere", and the Billard theorem [3, Ch. V] dealing with the properties "(1) represents a bounded function", "(1) represents a continuous function", "(1) converges everywhere".

We establish and survey some results concerning similar properties which cover some of the results given, e.g., in [3, Ch. V]. In order to do this we consider so-called homogeneous Banach function spaces, B, in particular L^p spaces, $1 \leq p < \infty$, continuous periodic functions and the dual spaces, B', of homogeneous Banach spaces, in particular, bounded functions, measures and generalized functions—distributions of Schwartz [4, 7]. We show, e.g., that sets of elementary events "(1) $\in B'$" are random events (that (1) represents an element

2000 Mathematics Subject Classification: 60G, 60H.
Keywords: homogeneous Banach space, random Fourier series, convergence almost sure.
Supported by grant VEGA 4137/24.

in B, B', respectively), each of them has probability zero or one, moreover $(1) \in B$ a.s. if and only if (1) converges a.s. in B, $(1) \in B'$ a.s. if and only if (1) is bounded a.s. in B' (i.e., (1) has bounded a.s. partial sums in B').

2. Preliminaries

Throughout this paper, T or the circle means the group of real numbers defined modulo 2π, $T = \mathbb{R}/2\pi\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}$ being reals and integers, respectively, t is a point on the circle, all measures and functions are defined on the circle, and all integrals are taken over the circle T. Basic reference is [3 and 4].

Remark. Although in [3] the only reference given in connection with homogeneous spaces is Katznelson's course in 1964–5, homogeneous spaces were considered many years ago by Shilov [5, 6 and 7] and later also by Wang [8]. This paper was written taking this remark into consideration.

Given an arbitrary ordinary trigonometric two-way series with z_n real or complex numbers

$$\sum_{n=-\infty}^{\infty} z_n e^{int} \tag{2}$$

we write $(2) \in M(T)$ if (2) is a Fourier-Stieltjes series, i.e., there exists a (complex) measure $d\mu(t)$ such that

$$z_n = \int e^{-int} d\mu(t)\,, \qquad n = 0, \pm 1, \pm 2, \dots$$

We write $(2) \in L^p(T)$ if (2) is the Fourier-Lebesgue series of a function $f \in L^p(T)$, $1 \le p \le \infty$, and $(2) \in C(T)$ if (2) is a Fourier-Lebesgue series of a continuous function f, i.e.,

$$z_n = \frac{1}{2\pi} \int e^{-int} f(t)\, dt\,, \qquad n = 0, \pm 1, \pm 2, \dots$$

We say also that (2) represents $d\mu$ or f. Recall also that by definition the Fejér sums of (2) are the trigonometric polynomials

$$\sigma_N(z;t) = \sigma_N(t) = \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right) z_n e^{int}, \qquad N = 1, 2, \dots$$

It is well-known that if (2) converges, i.e.,

$$\lim_{N\to\infty} \sum_{-N}^{N} z_n e^{int}$$

exists, then $\sigma_N(z;t)$ converges for $N \to \infty$, but the converse does not hold. We write $\sigma_N(z;t) = \sigma_N(f;t)$ or $\sigma_N(\mu;t)$ if (2) is a Fourier series of f or μ, respectively.

We consider $M(T)$, $L^p(T)$, $C(T)$ as Banach spaces on the circle; $L^1(T)$ as a closed subspace of M and C as a closed subspace of L^∞. We shall write $\|\cdot\|_p$ for the norm in L^p, $1 \le p \le \infty$.

Recall the following important results ([9, pp. 136, 145]).

Proposition. *For the series* (2) *we have*

(i) $(2) \in L^p$ *if and only if* (σ_N) *converges in* L^p, $1 \le p < \infty$;

(ii) $(2) \in C$ *if and only if* (σ_N) *converges in* C;

(iii) $(2) \in M$ *if and only if* $\sup\limits_N \|\sigma_N\|_1 < \infty$;

(iv) $(2) \in L^p$ *if and only if* $\sup\limits_N \|\sigma_N\|_p < \infty$, $1 < p \le \infty$.

As a consequence we have that "$(1) \in L^p$", "$(1) \in C$" and "$(1) \in M$" are random events. Since the zero-one law applies, each of them has probability zero or one [3, Ch. V]. Moreover, we may apply Theorem 1 of [3, p. 11] and obtain [3, p. 40] the following

$(1) \in M$ a.s. if and only if (1) is a.s. bounded in M;

$(1) \in L^p$ a.s. if and only if (1) converges a.s. in L^p, $1 \le p < \infty$;

$(1) \in L^p$ a.s. if and only if (1) is a.s. bounded in L^p, $1 < p \le \infty$;

$(1) \in C$ a.s. if and only if (1) converges a.s. in C.

[(1) is a.s. bounded means that its partial sums are a.s. bounded].

Let us recall also the meaning of the general Paley–Zygmund Theorem ([3, Th. 1, p. 45]): either (1) is a.s. divergent almost everywhere and not a Fourier–Stieltjes series or else it converges a.s. almost everywhere to a function which is nearly bounded (that is to a function F such that $\exp(rF^2) \in L^p$ for all $r > 0$ a.s.).

As for the Billard Theorem ([3, Th. 3, p. 49]) it says that the following statements are equivalent: a) (1) represents a.s. a bounded function; b) (1) represents a.s. a continuous function; c) (1) converges uniformly a.s.; d) (1) converges everywhere a.s.

3. Homogeneous Banach spaces

Now we shall show that the results mentioned for the spaces L^p, $1 \le p < \infty$, and C are valid for function spaces $B(T)$ other than L^p, $1 \le p < \infty$, and C including at least so-called homogeneous Banach spaces over T. These are, by definition, the linear subspaces of $L^1(T)$ endowed with a norm $\| \ \|_B$ under which we have a Banach space such that

(i) $\|f\|_1 \le \|f\|_B$ for all $f \in B(T)$;

(ii) if $f \in B(T)$ and $s \in T$, then $f_s \in B(T)$ and $\|f_s\|_B = \|f\|_B$ $[f_s(t) = f(t-s)]$;

(iii) $\lim\limits_{s\to 0} \|f_s - f\|_B = 0$ for all $f \in B(T)$.

(See [1; 4; 5; 6; 7].)

We shall make use of the following result valid for the homogeneous Banach spaces; note that the proof requires the integration of $B(T)$-valued functions on T [5; 6; 7].

PROPOSITION 1. *Let $B(T)$ be a homogeneous Banach space over T. Then the following conditions are equivalent:*

a) *$(2) \in B(T)$;*

b) *(σ_N) converges in $B(T)$ for $N \to \infty$ (i.e., (2) is summable in $B(T)$).*

Proof. That a) implies b) was, in fact, proved by Šilov [5; 6; 7, VII. § 1, Th. 2] because the Šilov definition of the homogeneous Banach space differs only little from the Katznelson definition (used here) of the homogeneous Banach space for which the similar result is proved in [4, I. Th. 2.12].

If (σ_N), for $N \to \infty$, converges in $B(T)$, it converges to some function f in $B(T)$, and hence (σ_N) converges to f in $L^1(T)$, from which we deduce that $z_n = \frac{1}{2\pi}\int e^{-int} f(t)\,dt$, i.e., $(2) \in B(T)$, hence b) implies a) and the proof is complete. □

We are now in state to establish the following result.

THEOREM 1. *Let $B(T)$ be a homogeneous Banach space over T. Then "$(1) \in B(T)$" is a random event which has probability zero or one. Moreover $(1) \in B(T)$ a.s. if and only if (1) is a.s. convergent in $B(T)$.*

Proof. That "$(1) \in B(T)$" is a random event follows from the fact that $(2) \in B(T)$ if and only if (σ_n), for $N \to \infty$, converges in $B(T)$ (see Proposition 1). Moreover, since "$(1) \in B(T)$" does not depend on the values of any finite number of Z_n (i.e., "$(1) \in B(T)$" is an asymptotic property of Z_n) the zero-one law applies ([3, I. 6]) and so "$(1) \in B(T)$" has probability zero or one. Since Z_n are independent symmetric random variables $(1) \in B(T)$ if and only if (1) converges a.s. in $B(T)$ because a.s. summability implies a.s. convergence ([3, Th. 1, p. 11]). □

In order to show a significance of the Theorem 1 we will furnish some homogeneous Banach spaces.

1. All spaces $L^p(T)$, $1 \le p < \infty$.

2. $C(T)$—the space of all continuous 2π-periodic functions; the norm $\|f\| = \max_t |f(t)|$.

3. $C^n(T)$—the subspace of $C(T)$ of all n-times continuously differentiable functions; the norm

$$\|f\|_n = \sum_{j=0}^{n} \frac{1}{j!} \max_t \left| f^{(t)}(t) \right| .$$

4. $L^{(1)}(T)$—the Banach space of all functions f on T such that f is absolutely continuous; the norm

$$\|f\|_{L(1)} = \|f\|_1 + \|f'\|_1 .$$

Recall that the Banach spaces $B(T)$ in $L^1(T)$ satisfying the first two conditions for a homogeneous Banach space and not necessarily the third condition are called semihomogeneous Banach spaces over T. So the ("great") Lipschitz spaces $\mathrm{Lip}_\alpha(T)$, $0 < \alpha \le 1$, of all continuous functions f on T for which

$$\sup_{\substack{t \\ h\neq 0}} \frac{\left|f(t+h) - f(t)\right|}{|h|^\alpha} < \infty$$

with the norm

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup_t \left|f(t)\right| + \sup_{\substack{t \\ h\neq 0}} \frac{\left|f(t+h) - f(t)\right|}{|h|^\alpha}$$

are the Banach subspaces of $L^1(T)$ satisfying the first two conditions for a homogeneous Banach space, the third condition is not satisfied, hence they are semihomogeneous Banach spaces. Clearly every homogeneous Banach space is a semihomogeneous Banach space.

5. The Banach spaces $\mathrm{lip}_\alpha(T)$, $0 < \alpha < 1$, of all functions f in $\mathrm{Lip}_\alpha(T)$ for which

$$\lim_{h\to\infty} \sup_t \frac{\left|f(t+h) - f(t)\right|}{|h|^\alpha} = 0$$

with the norm $\|f\|_{\mathrm{Lip}_\alpha}$ and $\mathrm{lip}_1(t)$, putting $\mathrm{lip}_1(T) = C^1(T)$, are homogeneous Banach spaces.

Let us give some examples of semihomogeneous Banach spaces.

(a) Lipschitz spaces $\mathrm{Lip}_\alpha(T)$, $0 < \alpha \le 1$. Recall that $\mathrm{lip}_\alpha(T)$, $0 < \alpha < 1$, $\mathrm{lip}_1(T) = C^1(T)$ are the maximal homogeneous Banach spaces in $\mathrm{Lip}_\alpha(T)$, $0 < \alpha < 1$, $\mathrm{Lip}_1(T)$, respectively.

(b) $L^\infty(T)$—the space of all essentially bounded functions in $L^1(T)$ with the norm $\|f\|_\infty = \operatorname{ess\,sup}_t |f(t)|$. The maximal homogeneous Banach space in $L^\infty(T)$ is the space $C(T)$.

(c) $\mathrm{CBV}(T)$— the space of all continuous functions on T with bounded total variation $V_0^{2\pi} f$ with the norm $\|f\|_V = \|f\|_1 + V_0^{2\pi} f$. The maximal homogeneous Banach space in $\mathrm{CBV}(T)$ is $L^{(1)}(T)$.

(d) $\mathrm{BV}(T)$—the space of all functions on T with bounded total variation $V_o^{2\pi} f$ with the norm $\|f\|_V = \|f\|_1 + V_0^{2\pi} f$.

4. Duals of homogeneous Banach spaces

Let now $B(T)$ be a homogeneous Banach space and $B'(T)$ its dual space (the space of all continuous linear forms on $B(T)$). If $u \in B'(T)$, the complex number defined by $z_n = u(e^{-int})$ is called the nth Fourier coefficient of u.

Remark. If $B(T)$ is a homogeneous Banach space over T, then for every $f \in B(T)$ and $n \in \mathbb{Z}$ the function $t \to \hat{f}(n)e^{int}$ belongs to $B(T)$ ([4, p. 6. Lemma 1.9 and p. 17, Ex. 13]). It follows that if the Fourier-Lebesgue coefficient $\hat{f}(n)$ of f is not zero, then the exponential $t \to e^{int}$ belongs to the space $B(T)$ for this $n \in \mathbb{Z}$. So we may say that the exponential $t \to e^{int}$ belongs to the space $B(T)$ if there exists a function $f \in B(T)$ such that $\hat{f}(n)$ is not zero. With regard to this fact we will suppose tacitly for the sake of simplicity that $B(T)$ is a homogeneous Banach space containing all exponentials $t \to e^{int}$, $n \in \mathbb{Z}$.

We shall write $(2) \in B'(T)$ if $z_n = u(e^{-int})$, $n \in \mathbb{Z}$, for some $u \in B'(T)$, i.e., if z_n is the nth Fourier coefficient of some $u \in B'(T)$, in such a case

$$\begin{aligned} \sigma_n(z,t) &= \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right) z_n e^{int} \\ &= \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{u}(n) e^{int} = \sigma_N(u,t)\,, \quad N = 1, 2, \dots \end{aligned}$$

By means of the functions $\sigma_N(z,t)$ or $\sigma_N(u,t)$, a linear form, denoted by $\sigma_N(z)$, belonging to $B'(T)$ is defined, namely

$$\langle f, \sigma_N(z) \rangle = \frac{1}{2\pi} \int f(t)\, \sigma_N(z,t)\, dt$$

for all $f \in B(T)$. In particular, for $u \in B'(T)$, the linear forms $\sigma_n(u)$ are elements from $B'(T)$ and

$$\begin{aligned}\langle f, \sigma_n(u)\rangle &= \frac{1}{2\pi}\int f(t)\,\sigma_N(u,t)\,dt \\ &= \sum_{-N}^{N}\left(1-\frac{|n|}{N+1}\right)\hat{f}(-n)\,\hat{u}(n)\end{aligned}$$

for all $f \in B(T)$. We remark that for every $u \in B'(T)$ the sequence from $B'(T)$, $\sigma_N(u)$ converges weak star to u, i.e.,

$$\lim_{N\to\infty}\langle f, \sigma_N(u)\rangle = \langle f, u\rangle$$

for all $f \in B(T)$ ([4, p. 35, Parseval's formula]). We denote the norm of $\sigma_N(z)$ as that of an element of $B'(T)$ by $\|\sigma_N(z)\|_{B'(T)}$. We are now ready to state the following result.

Proposition 2. *Let $B(T)$ be a homogeneous Banach space over T ad $B'(T)$ be its dual space. Then the following conditions are equivalent:*

a) $(2) \in B'(T)$;

b) $\sup\limits_N \|\sigma_N(z)\|_{B'(T)} < \infty$.

Proof of this proposition requires only small modification of the arguments in [4, p. 36–37].

Theorem 2. *Let $B(T)$ be a homogeneous Banach space over T and $B'(T)$ be its dual space. Then "$(1) \in B'(T)$" is a random event which has probability zero or one. Moreover $(1) \in B'(T)$ a.s. if and only if (1) is a.s. bounded in $B'(T)$.*

Proof. "$(1) \in B'(T)$" is a random event, since $(2) \in B'(T)$ if and only if the sequence $\sigma_n(z)$ is bounded in $B'(T)$ (see Proposition 2). Since "$(1) \in B'(T)$" does not depend on the values of any finite number of Z_n, ("$(1) \in B'(T)$" is an asymptotic property of Z_n, similarly as "$(1) \in B(T)$" was) the zero-one law applies ([3, I. 6]) and so "$(1) \in B'(T)$" has probability zero or one. Since Z_n are independent symmetric random variables we have $(1) \in B'(T)$ a.s. if and only if (1) is a.s. bounded in $B'(T)$, because a.s. summability boundedness implies a.s. boundedness ([3, p. 11, Th. 1]). $\square$

Let us recall that for $B(T) = C(T)$ the dual space $B'(T)$ is identified with the space $M(T)$ by $\langle f, u\rangle = \int f\,d\mu$; for $B(T) = L^p(T)$, $1 \le p < \infty$, the dual space $B'(T)$ is the space $L^q(T)$, $1 < q \le \infty$, with $\frac{1}{p}+\frac{1}{q}=1$. As for the space $L^1(T)$ it is well-known that it is not the dual space of any Banach space, but it is the dual space of $L^\infty(T)$ if $L^\infty(T)$ is endowed with weak star topology $\sigma(L^\infty, L^1)$. Let us remark that the condition $\sup\limits_N \left\|\sigma_n(z)\right\|_1 < \infty$ is not, in general, sufficient for $(2) \in L^1(T)$ (only for $(2) \in M(T)$) and in fact, $(2) \in L^1(T)$ if and only if the sequence $\left(\sigma_N(z,t)\right)$ is weakly compact, i.e., some subsequence $\left(\sigma_{N(k)}(z,t)\right)$ converges (weakly) in $L^1(T)$ or equivalently for some subsequence $\left(\sigma_{N(k)}(z,t)\right)$ the set functions

$$E \to \int \sigma_{N(k)}(z,t)\,dt\,, \quad k = 1,2,\dots\,,$$

are uniformly (or equi-) absolutely continuous ([2, II. 12.2.9]).

As for $B(T) = C^m(T)$ recall that the elements of the dual space $B'(T)$ are called (Schwartz) distributions or generalized functions of order m, $B'(T) = D^m(T)$. Since $C^{m+1}(T) \subset C^m(T)$ we have $D^m(T) \subset D^{m+1}(T)$, and we write $D(T) = \bigcup\limits_m D^m(T)$ for the space of all distributions.

We may now state the following corollary.

Corollary. *"$(1) \in L^p$", $1 < p \le \infty$, "$(1) \in M$" "$(1) \in D^m$", "$(1) \in D$" are random events which have probability zero or one. Further*

$(1) \in L^p$ *a.s., $1 < p \le \infty$, if and only if (1) is a.s. bounded in L^p, $1 < p \le \infty$;*

$(2) \in M$ *a.s. if and only if (1) is a.s. bounded in M.*

5. Semihomogeneous spaces

We have seen that Theorem 1 and Theorem 2 deal with the homogeneous Banach spaces of functions and the duals of homogeneous Banach spaces of functions, respectively. We have also seen that there are important Banach spaces $B(T)$ of functions which are semihomogeneous but not homogeneous. Many of them are, however, convolutable ([8]), i.e., for every $f \in L^1(T)$ and $g \in B(T)$ the convolution function $f*g$, $(f*g)(t) = \int g(t-s)f(s)\,ds$, belongs to $B(T)$ and $\|f*g\|_B \le K\|g\|_B$, (K—a positive finite number). For example, all the spaces $\mathrm{BV}(T)$, $\mathrm{CBV}(T)$, $\mathrm{Lip}_\alpha(T)$, $L^\infty(T)$ (and, of course, all the homogeneous Banach spaces) are convolutable.

In the following $(2) \in \mathrm{BV}(T)$, $(2) \in \mathrm{CBV}(T)$ mean that z_n are the Fourier–Lebesgue coefficients of a function f in $\mathrm{BV}(T)$ or in $\mathrm{CBV}(T)$, respectively. From the preceeding we obtain the following (cf. [9], p. 138).

Proposition 3.

1. For the space $\mathrm{BV}(T)$ the following conditions are equivalent:

(a) $(2) \in \mathrm{BV}(T)$;

(*b*) $\sup\limits_N \|\sigma_N\|_V < \infty$

2. For the space $\mathrm{CBV}(T)$ *the following conditions are equivalent:*

(a) $(2) \in \mathrm{CBV}(T)$*;*

(*b*) $\sup\limits_N \|\sigma_N\|_V < \infty$ *and* (σ_N) *converges uniformly in* T.

P r o o f.

1. follows from the correspondence between $M(T)$ and $\mathrm{BV}(T)$, taking in account that $\mathrm{BV}(T)$ is convolutable, that $\sup\limits_N \|\sigma_N\|_V$ means $\sup\limits_N \|\sigma'_N\|_{L'}$ and $\sup\limits_N \|\sigma'_N\|_{L'} < \infty$ (using of course Proposition 2).

2. follows from 1. and Proposition 1 applied to $C(T)$.

Note that $L^\infty(T)$ is contained also in Proposition 2. From the preceeding we could derive the following:

"$(1) \in \mathrm{BV}(T)$" is a random event which has probability zero or one; $(1) \in \mathrm{BV}(T)$ a.s. if and only if (1) is a.s. bounded in $\mathrm{BV}(T)$. Further "$(1) \in \mathrm{CBV}(T)$" is a random event which has probability zero or one; $(1) \in \mathrm{CBV}(T)$ a.s. if and only if (1) is a.s. bounded in $\mathrm{CBV}(T)$ and convergent in $C(T)$. It is interesting to note that by the Billard Theorem [3, p. 52] $(1) \in \mathrm{BV}(T)$ a.s. implies $(1) \in L^{(1)}$ a.s., and similarly $(1) \in \mathrm{Lip}_1(T)$ a.s. implies $(1) \in C^1$ a.s. □

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Received July 23, 2003

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