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A GEOMETRIC CONDITION FOR DIFFERENTIABILITY

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ABSTRACT. A criterion for a differentiability of a mapping $f : X \to Y$ in terms of a contingent of its graph $G(f)$ is given. The paper also contains a geometric condition for a mapping to be locally Lipschitz.

1. Notations and definitions

Let X, Y be real normed spaces. We use the following notations:

- p_X, p_Y are natural projections from $X \times Y$ onto X and Y, respectively;
- $G(f)$ denotes the graph of a mapping f;
- $\mathrm{cl}\, A$ and ∂A stand for the closure and boundary of A respectively;
- $B(x_0, r)$ is an open ball centered at x_0 and of radius r;
- by a vertical vector we mean each vector of the form $(0, \alpha) \in X \times Y$ with $\alpha \neq 0$.

The differentiability is understood in the Fréchet sense.

Definition 1.1. Let Z be a real normed space. Let $\emptyset \neq M \subset Z$ and $z \in \mathrm{cl}\, M$. The set

$$\left\{v \in Z : \exists \{z_n\}_{n\in\mathbb{N}},\ z_n \in M,\ z_n \to z,\ \exists \{\lambda_n\}_{n\in\mathbb{N}},\ \lambda_n > 0 : \lim_{n\to\infty} \lambda_n(z_n - z) = v\right\}$$

is called a *contingent* of M at the point z (or a tangent cone of M at z) and will be denoted as $\mathrm{Tan}(M, z)$. Elements of $\mathrm{Tan}(M, z)$ are called *tangent vectors* to M at the point z.

Definition 1.1 of a contingent is adopted from [5] and will be used throughout the text. For other definitions, see, e.g., [1], [2], [4].

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2. A contingent and differentiability

Let X, Y be real normed spaces. The following proposition is well known.

Theorem 2.1. ([5]) *If $f\colon X \to Y$ is differentiable at $x_0 \in X$ then*

$$\operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big) = G\big(f'(x_0)\big)\,.$$

Note that $\operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big)$ here is obviously a linear subspace of $X \times Y$ containing no vertical vectors.

The following lemma will be used in the proof of the next proposition.

Lemma 2.1. *Let a mapping $g\colon U \to Y$ be given, where U is an open subset of X, and let a point $\xi_0 \in U$ be fixed. Assume that each sequence $\{\xi_n\colon n \in \mathbb{N}\}$, $\xi_n \in U$, $\xi_n \to \xi_0$, contains a subsequence $\{\xi_{n_k}\colon k \in \mathbb{N}\}$ such that $\lim\limits_{k\to\infty} g(\xi_{n_k}) = 0$. Then for each sequence $\{\xi_n\colon n \in \mathbb{N}\}$, $\xi_n \to \xi_0$, we have $\lim\limits_{n\to\infty} g(\xi_n) = 0$.*

We omit an easy proof.

The next result may be regarded as a converse to Theorem 2.1 in the case of finite dimensional spaces.

Theorem 2.2. *Let X, Y be finite dimensional normed spaces. Let U be an open subset of X. Let a mapping $f\colon U \to Y$ be given. Let f be continuous at a point $x_0 \in U$. Assume that $T = \operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big)$ is a linear subspace of $X \times Y$ which does not contain vertical vectors. Then*

$$f \text{ is differentiable at } x_0\,; \tag{1}$$

$$f'(x_0) = p_Y \circ (p_X|T)^{-1}; \tag{2}$$

$$\big\|f'(x_0)\big\| = \sup\big\{\|\beta\|\colon (\alpha,\beta) \in T,\ \|\alpha\| = 1\big\}\,. \tag{3}$$

P r o o f. First, we will show that, given a sequence $\{x_n\colon n \in \mathbb{N}\}$, $x_n \in U$, $x_n \to x_0$, the sequence

$$\left\{\frac{f(x_n) - f(x_0)}{\|x_n - x_0\|}\colon n \in \mathbb{N}\right\} \tag{4}$$

is bounded. Assume this is not the case. Then there is a sequence $\{x_n\colon n \in \mathbb{N}\}$, $x_n \in U$, such that $\|x_n - x_0\| < \frac{1}{n}$ and $\frac{\|f(x_n) - f(x_0)\|}{\|x_n - x_0\|} > n$ for $n \in \mathbb{N}$.

Consider the sequence $\{\lambda_n : n \in \mathbb{N}\}$, $\lambda_n = \|f(x_n) - f(x_0)\|^{-1}$. Since $\lambda_n(f(x_n) - f(x_0)) \in \mathcal{S}_Y = \{y \in Y : \|y\| = 1\}$ and $\dim Y < \infty$, we may assume without loss of generality (choosing a subsequence, if necessary) that there exists

$$\lim_{n\to\infty} \lambda_n\big(f(x_n) - f(x_0)\big) = \gamma \in \mathcal{S}_Y .$$

Therefore $\lim_{n\to\infty} \lambda_n\big(x_n - x_0,\, f(x_n) - f(x_0)\big) = (0,\gamma) \in T$ which is impossible, since by our assumption, T does not contain vertical vectors. Thus the sequence (4) is bounded for each sequence $\{x_n : n \in \mathbb{N}\}$, $x_n \in U$, $x_n \to x_0$.

Next, we will show that

$$p_X(T) = X . \tag{5}$$

Assume that (5) does not hold. Take a vector $\xi \in B(x_0, r) \subset U$, $r > 0$, such that $\xi - x_0 \notin p_X(T)$. Consider the sequence $\{\xi_n : n \in \mathbb{N}\}$, $\xi_n = x_0 + \frac{1}{n}(\xi - x_0)$. By the above argument, the sequence

$$\left\{ \frac{f(\xi_n) - f(x_0)}{\|\xi_n - x_0\|} : n \in \mathbb{N} \right\} \tag{6}$$

is bounded and as $\dim Y < \infty$, it contains a convergent subsequence. To simplify the notations, we may assume without loss of generality that (6) is convergent. Let

$$\beta = \lim_{n\to\infty} \frac{f(\xi_n) - f(x_0)}{\|\xi_n - x_0\|} .$$

Then we get

$$\lim_{n\to\infty} \frac{1}{\|\xi_n - x_0\|}\big(\xi_n - x_0,\, f(\xi_n) - f(x_0)\big) = (\alpha, \beta) \in T ,$$

where $\alpha = \frac{\xi - x_0}{\|\xi - x_0\|} \notin p_X(T)$, a contradiction. This proves (5) whence it follows that $p_X|T : T \to X$ is a linear isomorphism. Therefore the operator $p_Y \circ (p_X|T)^{-1} : X \to Y$ is well defined.

Now, we will prove that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - \big(p_Y \circ (p_X|T)^{-1}\big)(x - x_0)}{\|x - x_0\|} = 0 . \tag{7}$$

Take any sequence $\{x_n : n \in \mathbb{N}\}$, $x_n \in U$, $x_n \to x_0$. It has been already shown above that the sequence (4) is bounded. Since $\dim Y < \infty$, it contains a convergent subsequence $\left\{ \frac{f(x_{n_k}) - f(x_0)}{\|x_{n_k} - x_0\|} : k \in \mathbb{N} \right\}$. Let

$$\beta = \lim_{k\to\infty} \frac{f(x_{n_k}) - f(x_0)}{\|x_{n_k} - x_0\|} . \tag{8}$$

Since $\dim X < \infty$, the sequence $\left\{\frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|} : k \in \mathbb{N}\right\}$ contains a convergent subsequence. Again, to simplify the notations, we may assume, as previously, that this sequence is already convergent. Let

$$\alpha = \lim_{k\to\infty} \frac{x_{n_k}-x_0}{\|x_{n_k}-x_0\|}\,. \tag{9}$$

Then we get

$$\lim_{k\to\infty} \frac{1}{\|x_{n_k}-x_0\|}\big(x_{n_k}-x_0,\, f(x_{n_k})-f(x_0)\big) = (\alpha,\beta) \in T,$$

and, clearly,

$$\big(p_Y \circ (p_X|T)^{-1}\big)(\alpha) = p_Y(\alpha,\beta) = \beta\,. \tag{10}$$

Since the operator $p_Y \circ (p_X|T)^{-1}$ is continuous, from (8)–(10) we obtain

$$\lim_{k\to\infty} \frac{f(x_{n_k})-f(x_0)-\big(p_Y\circ(p_X|T)^{-1}\big)(x_{n_k}-x_0)}{\|x_{n_k}-x_0\|} = 0\,. \tag{11}$$

Thus we have shown that, given a sequence $\{x_n : n \in \mathbb{N}\}$, $x_n \in U$, $x_n \to x_0$, there exists a subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ such that (11) holds. By Lemma 2.1,

$$\lim_{n\to\infty} \frac{f(x_n)-f(x_0)-\big(p_Y\circ(p_X|T)^{-1}\big)(x_n-x_0)}{\|x_n-x_0\|} = 0$$

for each sequence $\{x_n : n \in \mathbb{N}\}$, $x_n \in U$, $x_n \to x_0$, whence (2) and (7) follow.

Finally, observe that by Theorem 2.1 the contingent T is equal to the graph of $f'(x_0) = p_Y \circ (p_X|T)^{-1}$, whence the relation (3) is straightforward. □

Remark 2.1. Claim (1) of Theorem 2.2 can be deduced from Proposition (4.3.3) in [3]. Our proof, however, is quite independent and is furnished in the form suitable for getting relations (2), (3).

Remark 2.2. Theorem 2.2, in view of Theorem 2.1, is in fact a geometric criterion for the differentiability of a mapping at a point in the case of finite dimensional spaces.

The following examples show that Theorem 2.2 fails if at least one of the spaces X, Y is of infinite dimension.

Let l^2 be a real Hilbert space and $\{e_n : n \in \mathbb{N}\}$ the standard orthonormal system in l^2.

Example 2.1. Define $f : \mathbb{R} \to l^2$ letting for $t \geq 0$

$$f(t) = \begin{cases} te_1 & \text{for } t = \frac{1}{k}\,,\ k \in \mathbb{N}\,, \\ te_k & \text{for } \frac{1}{k+1} < t < \frac{1}{k}\,,\ k \in \mathbb{N}\,, \\ 0 & \text{for } t = 0 \text{ and for } t > 1\,, \end{cases}$$

and $f(t) = -f(-t)$ for $t < 0$.

The mapping f is continuous at $t = 0$ and $\mathrm{Tan}\big(G(f), (0,0)\big) = \{\lambda(1, e_1) : \lambda \in \mathbb{R}\}$. Take the sequence $\{\tau_n : n \in \mathbb{N}\}$, $\tau_n = \frac{2n+1}{2n(n+1)}$. Then the limit of the ratio $\frac{f(\tau_n) - f(0)}{\tau_n - 0} = e_n$ obviously does not exist as $n \to \infty$, so f is not differentiable at $t = 0$.

Observe that f is not continuous at the points $t = \pm 1/k$, $k \in \mathbb{N}$.

Remark 2.3. Example 2.1 is a modification of the F l e t t's Example [3] (4.3.3). Note that if we replace e_1 in the above definition of f, by any fixed $y \in l^2$, we obtain a mapping $f\colon \mathbb{R} \to l^2$ not differentiable at $t = 0$ and with $\mathrm{Tan}\big(G(f), (0,0)\big) = \{\lambda(1, y) : \lambda \in \mathbb{R}\}$.

In the next example f will be continuous everywhere.

Example 2.2. Consider two sequences: $\{a_n : n \in \mathbb{N}\}$, $a_n = \big(\frac{1}{n+1}, \frac{1}{(n+1)^2}\big) \in \mathbb{R}^2$, and $\{r_n : n \in \mathbb{N}\}$, where $r_n = n^{-5}$. Denote $B_n = B(a_n, r_n)$. Then $B_n \cap B_m = \emptyset$ for $n \neq m$. Define $f\colon \mathbb{R}^2 \to l^2$ letting

$$f(z) = \begin{cases} 0 & \text{for } \ z \notin \bigcup\limits_{n \in \mathbb{N}} B_n\,, \\ \dfrac{\mathrm{dist}(z, \partial B_n)\|a_n\| e_n}{r_n} & \text{for } \ z \in B_n\,,\ n \in \mathbb{N}\,, \end{cases}$$

where $\mathrm{dist}(z, \partial B_n)$ is the distance of z from the boundary of B_n. Denote $z_0 = (0,0) \in \mathbb{R}^2$. Then $\mathrm{Tan}\big(G(f), (z_0, 0)\big) = \mathbb{R}^2 \times \{0\}$, $(0 \in l^2)$, but f is not differentiable at z_0, because the limit of the ratio $\frac{f(a_n) - f(0)}{\|a_n - 0\|} = e_n$ does not exist.

Example 2.3. Consider the sequences: $\big\{\frac{e_n}{n+1} : n \in \mathbb{N}\big\}$, $\{r_n : n \in \mathbb{N}\}$, where $r_n = n^{-5}$ and denote $B_n = B\big(\frac{e_n}{n+1}, r_n\big)$. Then $B_n \cap B_m = \emptyset$ for $n \neq m$.

Define $f\colon l^2 \to \mathbb{R}$ letting

$$f(x) = \begin{cases} 0 & \text{for } \ x \notin \bigcup\limits_{n \in \mathbb{N}} B_n\,, \\ \dfrac{2 + (-1)^n}{n r_n} \mathrm{dist}(x, \partial B_n) & \text{for } \ x \in B_n\,,\ n \in \mathbb{N}\,. \end{cases}$$

Then $\mathrm{Tan}\big(G(f), (0,0)\big) = l^2 \times \{0\}$, but f is not differentiable at $x = 0$, because

$$\lim_{n \to \infty} \frac{f\big(\frac{e_n}{n+1}\big) - f(0)}{\big\|\frac{e_n}{n+1} - 0\big\|} = \lim_{n \to \infty} \frac{(2 + (-1)^n)(n+1)}{n}$$

does not exist.

3. Lipschitz mappings

Let X, Y be real normed spaces.

Proposition 3.1. *Let $U \subset X$ be an open set and $f\colon U \to Y$ a locally Lipschitz mapping with the constant $K > 0$. Then*

$$\operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big) \subset \big\{(\xi, \eta) \in X \times Y \colon \|\eta\| \le K\|\xi\|\big\}$$

for each $x_0 \in U$.

P r o o f. Let $x_0 \in U$ and $(\alpha, \beta) \in \operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big)$, $(\alpha, \beta) \neq 0$. There are two sequences $\{x_n \colon n \in \mathbb{N}\}$, $x_n \in U$, and $\{\lambda_n \colon n \in \mathbb{N}\}$, $\lambda_n > 0$, such that

$$\lim_{n\to\infty} x_n = x_0\,, \quad \lim_{n\to\infty} \lambda_n(x_n - x_0) = \alpha\,, \quad \lim_{n\to\infty} \lambda_n\big(f(x_n) - f(x_0)\big) = \beta\,.$$

Since f is locally Lipschitz, there is $N \in \mathbb{N}$ such that

$$\forall n \ge N \colon \frac{\|\lambda_n\big(f(x_n) - f(x_0)\big)\|}{\|\lambda_n(x_n - x_0)\|} = \frac{\|f(x_n) - f(x_0)\|}{\|x_n - x_0\|} \le K$$

which yields, letting $n \to \infty$, that $\frac{\|\beta\|}{\|\alpha\|} \le K$.

Therefore $(\alpha, \beta) \in \big\{(\xi, \eta) \colon \|\eta\| \le K\|\xi\|\big\}$, what was to be shown. □

Note that X, Y in Proposition 3.1 could be of infinite dimension.

Lemma 3.2. *Let X, Y be finite dimensional normed spaces, $U \subset X$ an open set and a mapping $f\colon U \to Y$ be continuous at $x_0 \in U$. Assume that*

$$\exists K_0 > 0 \colon \operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big) \subset \big\{(\alpha, \beta) \in X \times Y \colon \|\beta\| \le K_0\|\alpha\|\big\}\,. \tag{12}$$

Then

$$\forall K > K_0 \ \exists r_K > 0 \ \forall x \in B(x_0, r_K) \colon \big\|f(x_0) - f(x)\big\| \le K\|x_0 - x\|\,. \tag{13}$$

P r o o f. Assume that (13) fails. Then there is $K > K_0$ and a sequence $\{x_n \colon n \in \mathbb{N}\}$, $x_n \in U$, $x_n \to x_0$, such that

$$\big\|f(x_n) - f(x_0)\big\| > K\|x_n - x_0\|\,, \qquad n \in \mathbb{N}\,. \tag{14}$$

We claim that the sequence

$$\left\{\frac{f(x_n) - f(x_0)}{\|x_n - x_0\|} \colon n \in \mathbb{N}\right\} \tag{15}$$

is bounded. If not, there is a subsequence $\{x_{n_i}: i \in \mathbb{N}\}$ such that

$$\lim_{i\to\infty} \frac{x_{n_i} - x_0}{\|f(x_{n_i}) - f(x_0)\|} = 0\,. \tag{16}$$

Since $\dim Y < \infty$, by (16) and the compactness of the sphere $\mathcal{S}_Y = \{y \in Y: \|y\| = 1\}$ we conclude that the sequence

$$\left\{\frac{\big(x_{n_i} - x_0,\, f(x_{n_i}) - f(x_0)\big)}{\|f(x_{n_i}) - f(x_0)\|}: i \in \mathbb{N}\right\}$$

contains a subsequence convergent to a vertical vector $(0,\beta) \in \operatorname{Tan}\Big(G(f), \big(x_0, f(x_0)\big)\Big)$, $\|\beta\| = 1$, which contradicts the assumption (12). Thus (15) is bounded and therefore it contains a convergent subsequence.

Since $\dim Y < \infty$, $\dim X < \infty$, we may assume, without loss of generality (just like in our previous arguments), that the sequences (15) and $\left\{\frac{x_n - x_o}{\|x_n - x_o\|}: n \in \mathbb{N}\right\}$ are convergent.

Denote $\alpha = \lim\limits_{n\to\infty} \frac{x_n - x_0}{\|x_n - x_0\|}$, $\beta = \lim\limits_{n\to\infty} \frac{f(x_n) - f(x_0)}{\|x_n - x_0\|}$. Letting $\lambda_n = \frac{1}{\|x_n - x_0\|}$ we then obtain that $\lim\limits_{n\to\infty} \lambda_n\big(x_n - x_o,\, f(x_n) - f(x_o)\big) = (\alpha,\beta) \in \operatorname{Tan}\Big(G(f), \big(x_0,\, f(x_0)\big)\Big)$, and therefore by (12) we have $\|\beta\| \le K_0\|\alpha\| = K_0$. On the other hand, it follows from (14) that $\|\beta\| \ge K > K_0$, a contradiction. This completes the proof of (13). $\square$

Theorem 3.3. *Let X, Y be finite dimensional normed spaces, U an open subset of X. Assume that for a continuous mapping $f: U \to Y$ there is a constant $K > 0$ such that*

$$\forall x \in U:\ \operatorname{Tan}\Big(G(f), \big(x, f(x)\big)\Big) \subset \big\{(\alpha,\beta) \in X \times Y: \|\beta\| \le K\|\alpha\|\big\}\,.$$

Then f is locally Lipschitz with the constant K, i.e.,

$$\forall z \in U\ \exists r > 0\ \forall x, y \in B(z,r):\ \big\|f(x) - f(y)\big\| \le K\|x - y\|\,.$$

P r o o f. We will show that for any couple of points $x_1, x_2 \in U$, such that $[x_1, x_2] := \big\{x_1 + t(x_2 - x_1): t \in [0,1]\big\} \subset U$, we have $\big\|f(x_1) - f(x_2)\big\| \le K\|x_1 - x_2\|$.

Let $x_1, x_2 \in U$ be such that $[x_1, x_2] \subset U$ and let $\varepsilon > 0$ be fixed.

Define $h: [0,1] \to [x_1, x_2]$ letting $h(t) = x_1 + t(x_2 - x_1)$ and consider the set

$$\begin{aligned} E &= \big\{t \in [0,1]: \|(f\circ h)(t) - (f\circ h)(0)\| \le (K+\varepsilon)\|h(t) - h(0)\|\big\} \\ &= \big\{t \in [0,1]: \|(f\circ h)(t) - f(x_1)\| \le (K+\varepsilon)t\|x_2 - x_1\|\big\}\,. \end{aligned}$$

Clearly, E is nonempty since $0 \in E$, and closed since f and h are continuous. We will show that $\sup E = 1$. Assume this is not the case. Then we have

$$\sup E = c < 1\,, \tag{17}$$

$c \in E$ and $\|(f \circ h)(c) - (f \circ h)(0)\| \leq (K+\varepsilon)\|h(c) - h(0)\|$.

By Lemma 3.2 (cf. (13)), there exists $r > 0$ such that for each $x \in B\big(h(c), r\big)$

$$\|f(x) - f\big(h(c)\big)\| \leq (K+\varepsilon)\|x - h(c)\|\,. \tag{18}$$

It is easy to check that for $t \in [0,1] \cap \Big(c - \frac{r}{\|x_2 - x_1\|},\, c + \frac{r}{\|x_2 - x_1\|}\Big)$ we have $h(t) \in B\big(h(c), r\big)$. Take for instance $t_0 = c + \frac{1}{2}\min\Big\{1 - c, \frac{r}{2\|x_2 - x_1\|}\Big\} > c$. Then $h(t_0) \in B\big(h(c), r\big)$ whence by (18) we get

$$\begin{aligned}\|(f \circ h)(t_0) - (f \circ h)(0)\| &\leq \|(f \circ h)(t_0) - (f \circ h)(c)\| + \|(f \circ h)(c) - (f \circ h)(0)\| \\ &\leq (K+\varepsilon)\big(\|h(t_0) - h(c)\| + \|h(c) - h(0)\|\big) \\ &= (K+\varepsilon)\big((t_0 - c)\|x_2 - x_1\| + c\|x_2 - x_1\|\big) \\ &= (K+\varepsilon)\,t_0\|x_2 - x_1\|\,.\end{aligned}$$

Thus $c < t_0 \in E$ which contradicts (17). Therefore $\sup E = 1$ which yields

$$\|f(x_2) - f(x_1)\| = \|(f \circ h)(1) - (f \circ h)(0)\| \leq (K+\varepsilon)\|x_2 - x_1\|\,.$$

This inequality being true for each $\varepsilon > 0$, we conclude that

$$\|f(x_2) - f(x_1)\| \leq K\|x_2 - x_1\|$$

holds, what was to be shown. □

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